As an example, suppose that $n = 7$ and

$$\pi = [3, 6, 2, 7, 5, 4, 1].$$

Then, after the first **while** loop, we have $i = 3$, since

$$2 < 7 > 5 > 4 > 1.$$

After the second **while** loop, we have $j = 6$ since $4 > 2$ and $1 < 2$. In the third step, we interchange π_3 and π_6, producing

$$[3, 6, 4, 7, 5, 2, 1].$$

Finally, we reverse the sublist

$$[7, 5, 2, 1],$$

producing the permutation

$$[3, 6, 4, 1, 2, 5, 7],$$

which is the successor of π.

It is now easy to generate all $n!$ permutations of $\{1, \ldots, n\}$. We can begin with the permutation $[1, 2, \ldots, n]$ (which is the first permutation lexicographically) and invoke Algorithm 2.14 a total of $n! - 1$ times.

We next turn to ranking and unranking permutations in lexicographic order. In the lexicographic ordering of permutations of $\{1, \ldots, n\}$, we first have the $(n-1)!$ permutations that begin with a "1", followed by the $(n - 1)!$ permutations that begin with a "2", etc. Hence, if π is a permutation of $\{1, \ldots, n\}$, it is clear that

$$(\pi[1] - 1)\,(n - 1)! \leq \mathsf{rank}(\pi) \leq \pi[1]\,(n - 1)! - 1.$$

Let r' denote the rank of π within the group of $(n-1)!$ permutations that begin with $\pi[i]$. Then r' is the rank of $[\pi[2], \ldots, \pi[n]]$ when it is considered as a permutation of $\{1, \ldots, n\} \backslash \{\pi[1]\}$. If we decrease every element of $[\pi[2], \ldots, \pi[n]]$ that is greater than $\pi[1]$ by one, then we obtain a permutation π' of $\{1, \ldots, n-1\}$ that also has rank r'.

This observation leads to a recursive formula for lexicographic rank of permutations of $\{1, \ldots, n\}$. For $n > 1$, we have

$$\mathsf{rank}(\pi, n) = (\pi[1] - 1)\,(n - 1)! + \mathsf{rank}(\pi', n - 1),$$

where

$$\pi'[i] = \begin{cases} \pi[i+1] - 1 & \text{if } \pi[i+1] > \pi[1] \\ \pi[i+1] & \text{if } \pi[i+1] < \pi[1]. \end{cases}$$

Initial conditions for this recurrence relation are given by

$$\mathsf{rank}([1], 1) = 0.$$

We work out a small example to illustrate:

$$\begin{aligned} \mathsf{rank}([2, 4, 1, 3], 4) &= 6 + \mathsf{rank}([3, 1, 2], 3) \\ &= 6 + 4 + \mathsf{rank}([1, 2], 2) \\ &= 6 + 4 + 0 + \mathsf{rank}([1], 1) \\ &= 6 + 4 + 0 + 0 \\ &= 10. \end{aligned}$$

It is easy to convert this recursive formula into a non-recursive algorithm, which we present as Algorithm 2.15.

Algorithm 2.15: PERMLEXRANK (n, π)

```
r ← 0
ρ ← π
for j ← 1 to n
  do { r ← r + (ρ[j] − 1)(n − j)!
       for i ← j + 1 to n
         do { if ρ[i] > ρ[j]
                then ρ[i] ← ρ[i] − 1
return (r)
```

Now suppose we want to unrank the integer r, where $0 \leq r \leq n! - 1$. Unranking can be done fairly easily if we first determine the *factorial representation* of r, by expressing r in the form

$$r = \sum_{i=1}^{n-1} (d_i \cdot i!),$$

where $0 \leq d_i \leq i$ for $i = 1, \ldots, n - 1$. (We leave it as an exercise to prove that any non-negative integer r such that $0 \leq r \leq n! - 1$ has a unique factorial representation of this form.)

Suppose that $\pi = \mathsf{unrank}(r)$ in the lexicographic ordering. It is easy to see that

$$\pi[1] = d_{n-1} + 1.$$

Thus the first element of π is determined immediately from the factorial representation of r. Now, denote

$$r' = r - d_{n-1} \cdot (n - 1)!,$$

and suppose that $\pi' = \mathsf{unrank}(r')$, where π' is a permutation of $\{1, \ldots, n - 1\}$. (This could be done recursively, for example.) Suppose we increment by one all elements of π' that are greater than d_{n-1}. Finally, define

$$\pi[i] = \pi'[i + 1]$$