



**Epistasis.
Estimation-of-Distribution Algorithms.**

Petr Pošík



Introduction to Epistasis



GA works well...

Problem f_1 :

- defined over 40-bit strings
- the quality of the worst solution: $f_1(x^{\text{worst}}) = 0$.
- the quality of the best solution: $f_1(x^{\text{opt}}) = 40$.
- the best solution: $x^{\text{opt}} = (1111 \dots 1)$.

Epistasis

- GA works well...
- GA fails...
- GA works again...
- Epistasis
- LI techniques

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

Scalability Analysis

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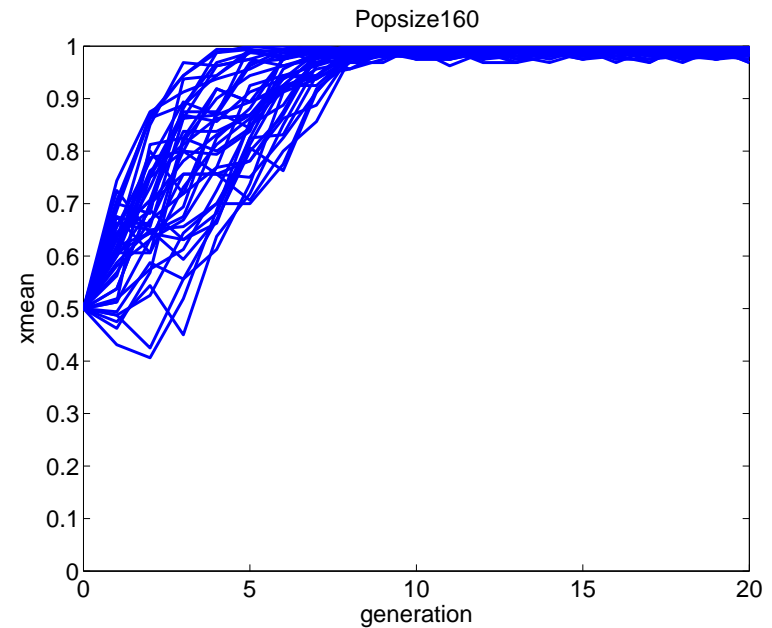
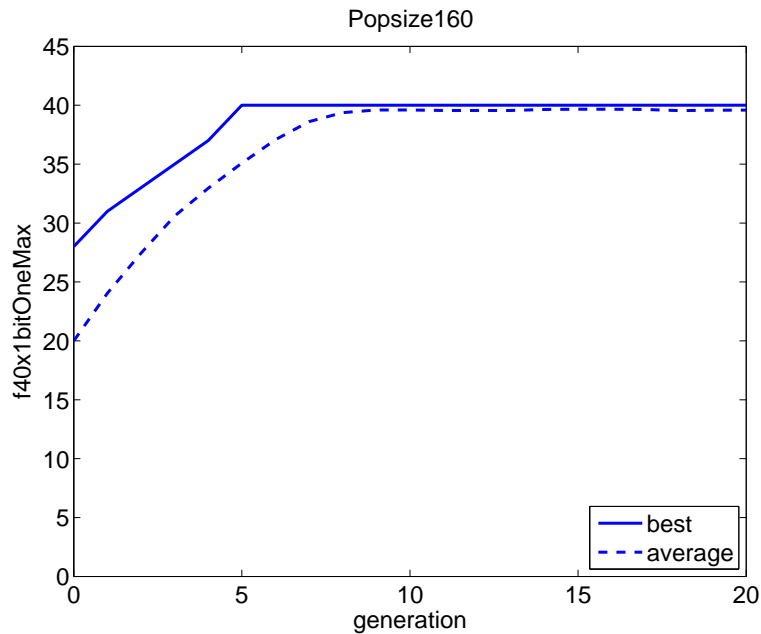
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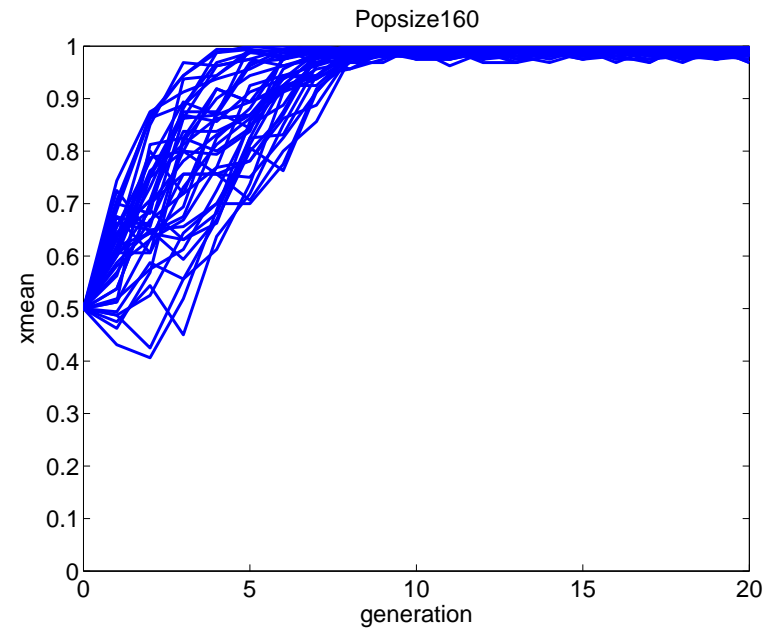
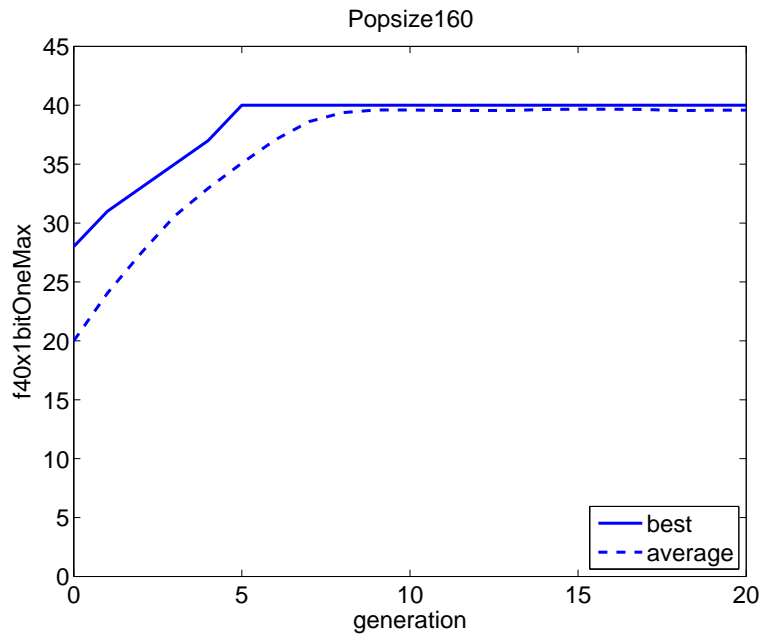
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Problem f_1 contains no epistatic interactions among design variables.



GA fails...

Problem f_2 :

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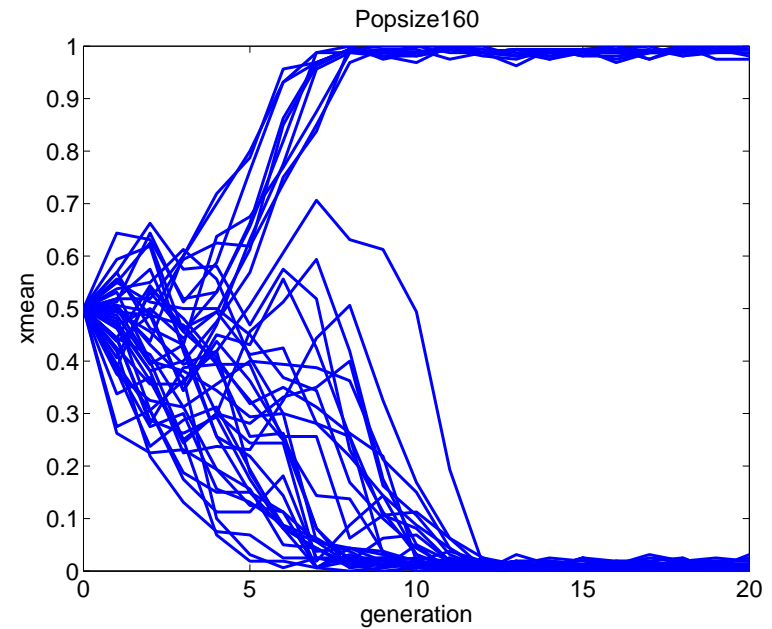
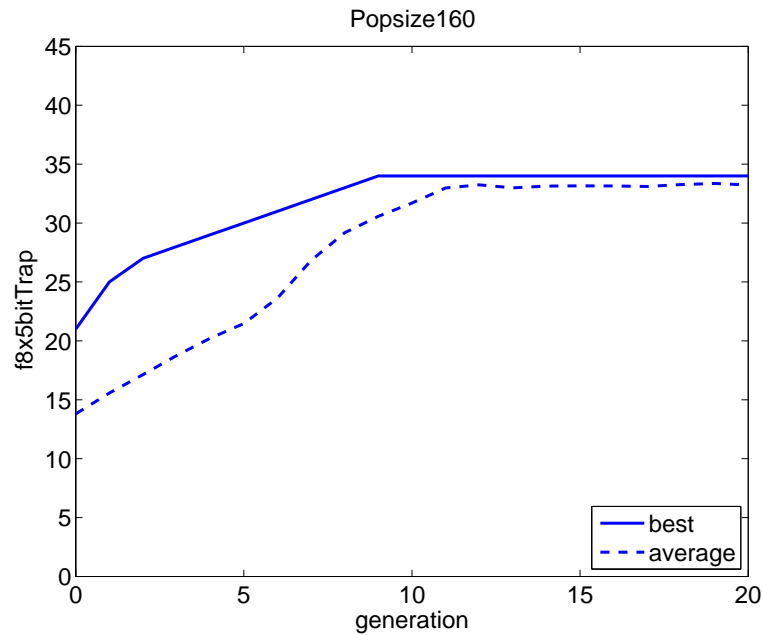
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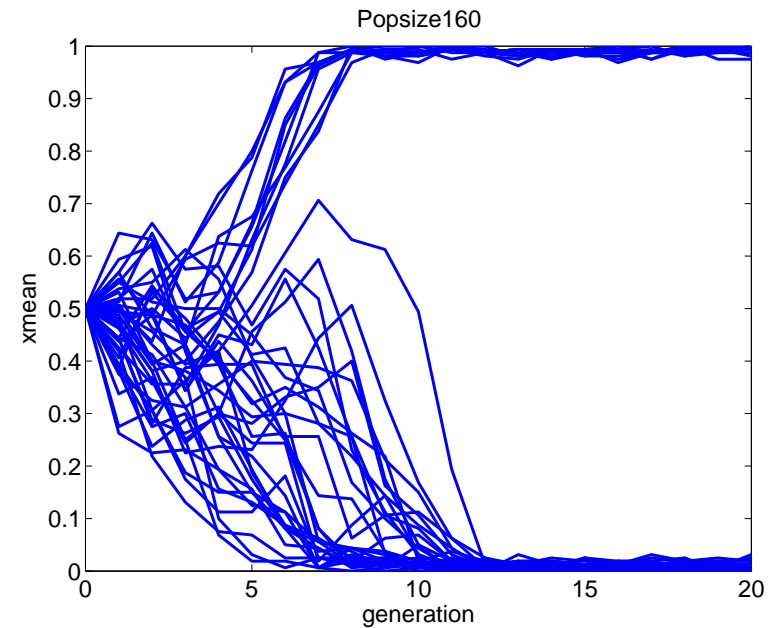
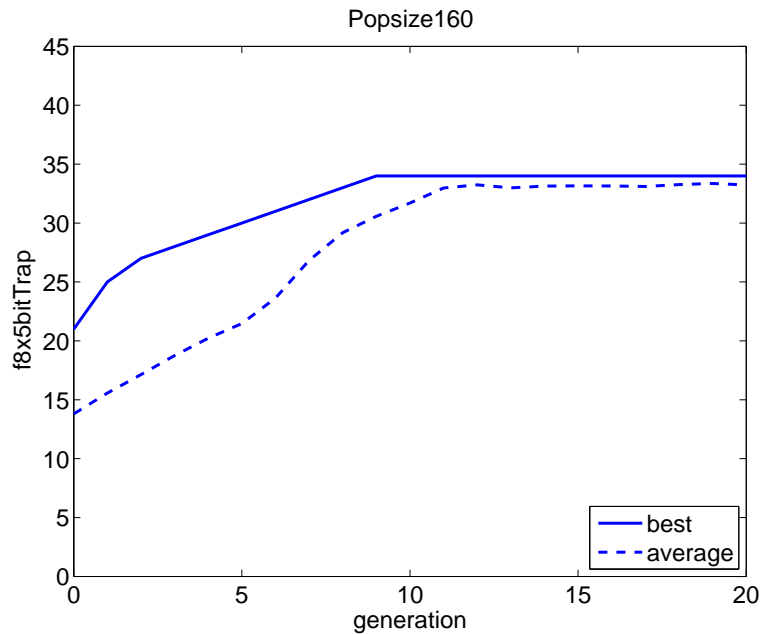
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Problem f_2 contains some interactions among variables, *GA is not aware of them* and works with the individual bits as if they were truly independent of each other.



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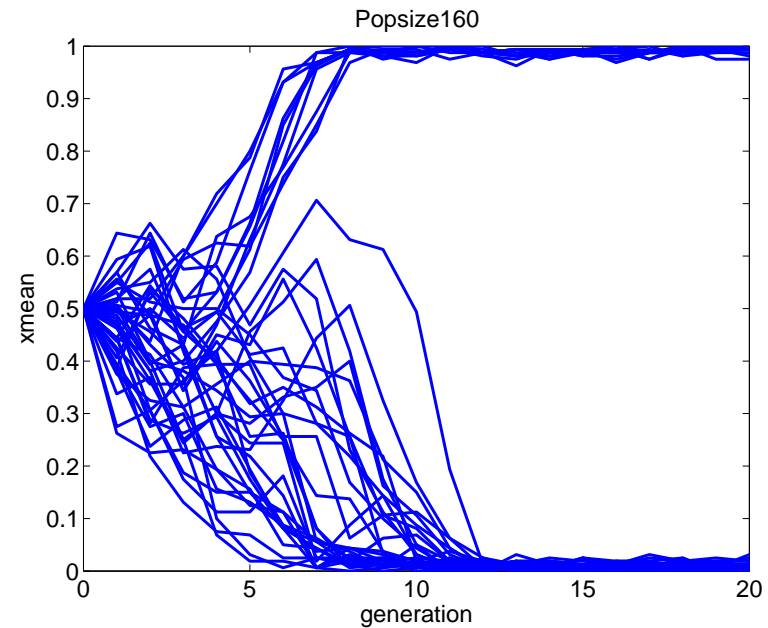
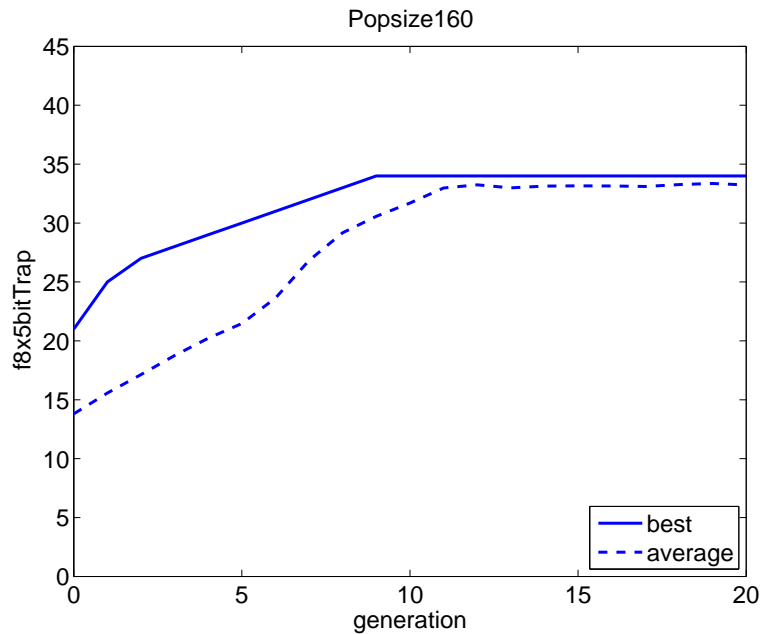
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Problem f_2 contains some interactions among variables, *GA is not aware of them* and works with the individual bits as if they were truly independent of each other.

None of the above mentioned problem characteristics is important to judge if the GA will work well!!!



GA works again...

Still solving f_2 :

- defined over 40-bit strings
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Instead of the uniform crossover,

- let us *allow the crossover only after each 5th bit*.

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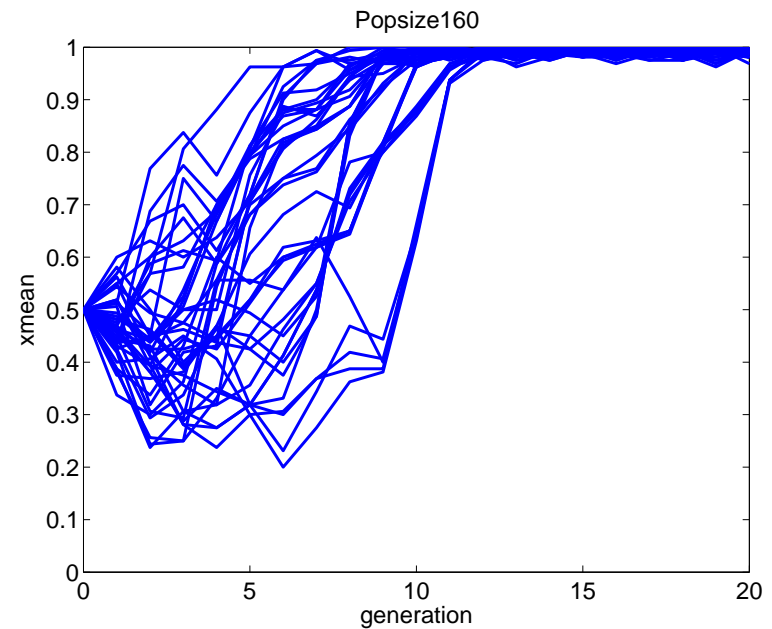
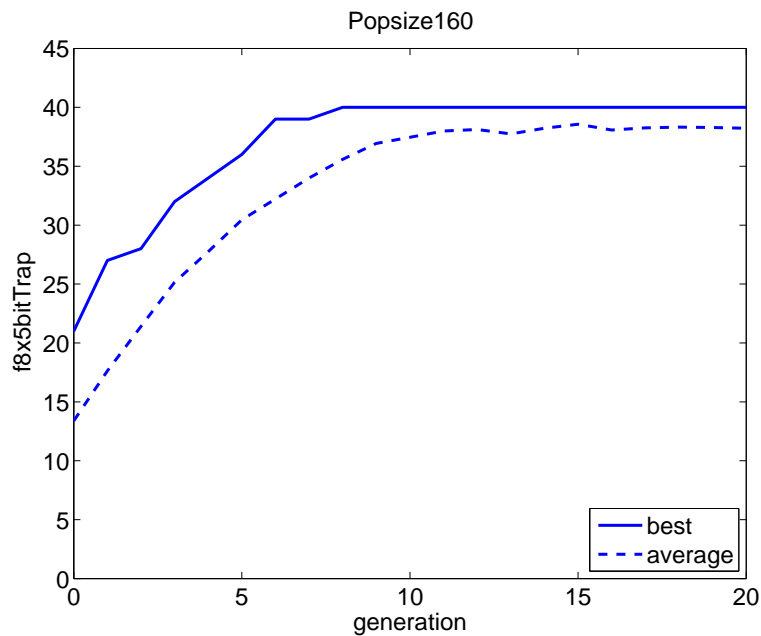
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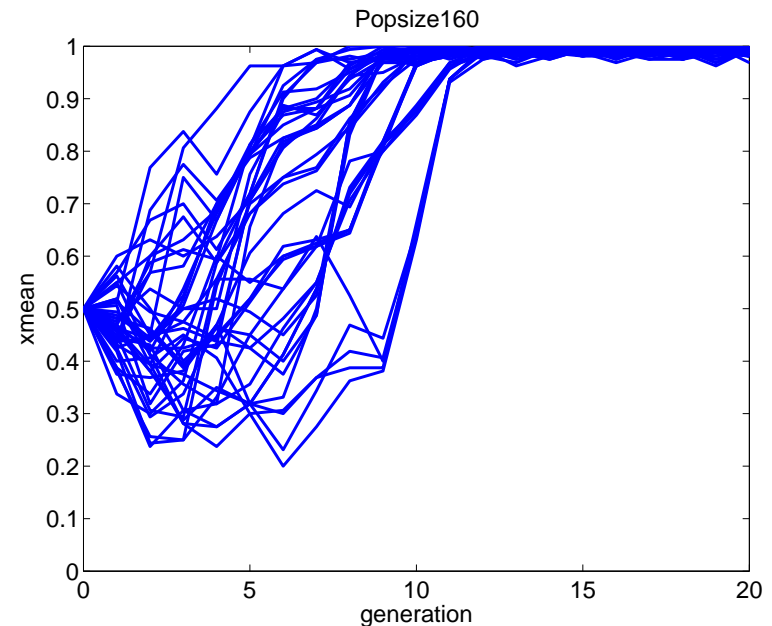
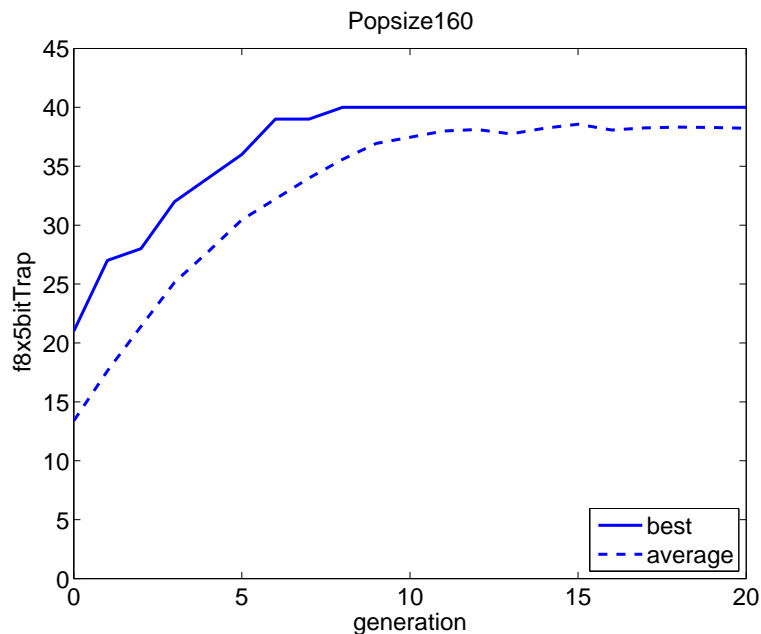
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Problem f_2 contains some interactions among variables and *GA knows about them*.



Epistasis

Epistasis:

- Effects of one gene are dependent on (influenced, conditioned by) other genes.
- Other names: dependencies, interdependencies, interactions.

Linkage:

- Tendency of certain loci or alleles to be inherited together.

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When optimizing the following functions, which of the variables are linked together?

$$f = x_1 + x_2 + x_3 \quad (1)$$

$$f = 0.1x_1 + 0.7x_2 + 3x_3 \quad (2)$$

$$f = x_1x_2x_3 \quad (3)$$

$$f = x_1 + x_2^2 + \sqrt{x_3} \quad (4)$$

$$f = \sin(x_1) + \cos(x_2) + e^{x_3} \quad (5)$$

$$f = \sin(x_1 + x_2) + e^{x_3} \quad (6)$$

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Notes:

- Almost all real-world problems contain interactions among design variables.
- The “amount” and “type” of interactions depend on the representation and the objective function.
- Sometimes, by a clever choice of the representation and the objective function, we can get rid of the interactions.

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Linkage Identification Techniques

Problems:

- How to detect dependencies among variables?

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Linkage Identification Techniques

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Linkage Identification Techniques

Problems:

- How to detect dependencies among variables?
- How to use them?

Techniques used for linkage identification:

1. Indirect detection along genetic search (messy GAs)
2. Direct detection of fitness changes by perturbation
3. Model-based approach: classification
4. Model-based approach: distribution estimation (EDAs)

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Introduction to EDAs



Genetic Algorithms

Algorithm 1: Genetic Algorithm

```
1 begin
2   Initialize the population.
3   while termination criteria are not met do
4     Select parents from the population.
5     Cross over the parents, create offspring.
6     Mutate offspring.
7     Incorporate offspring into the population.
```

“Select → cross over → mutate” approach

Conventional GA operators

- are not adaptive, and
- cannot (or usually do not) discover and use *the interactions among solution components*.

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- Genetic Algorithms
- GA vs EDA
- EDAs

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Conventional GA operators

- are not adaptive, and
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The goal of recombination operators:

- Intensify the search in areas which contained “good” individuals in previous iterations.
- Must be able to take the interactions into account.
- Why not directly describe the distribution of “good” individuals???

GA vs EDA

Algorithm 1: Genetic Algorithm

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Why not use directly...

Algorithm 2: Estimation-of-Distribution Alg.

```
1 begin
2   Initialize the population.
3   while termination criteria are not met do
4     Select parents from the population.
5     Learn a model of their distribution.
6     Sample new individuals.
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   population.
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“Select → update model → sample” approach

GA vs EDA

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      population.
```

“Select → update model → sample” approach

Or even...

Algorithm 3: Estimation-of-Distribution Alg. (Type 2)

```
1 begin
2   Initialize the model.
3   while termination criteria are not met do
4     Sample new individuals.
5     Select better ones.
6     Update the model based on selected
      ones.
```

“Sample → select → update model” approach



EDAs

Explicit probabilistic model:

- Sound and principled way of working with dependencies.
- Adaptation ability (different behavior in different stages of evolution).

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Names:

EDA Estimation-of-Distribution Algorithm

PMBGA Probabilistic Model-Building Genetic Algorithm

IDEA Iterated Density Estimation Algorithm

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Continuous EDAs (a very simplified view):

- Histograms and (Mixtures of) Gaussian distributions are used most often as the probabilistic model.
- Algorithms with Gaussians usually become very similar to CMA-ES.

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In the following, we shall talk only about discrete (binary) EDAs.

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Motivation Example



Example

5-bit OneMax (CountOnes) problem:

- $f_{D \times 1 \text{bitOneMax}}(\mathbf{x}) = \sum_{d=1}^D x_d$
- Optimum: 11111, fitness: 5

Algorithm: **Univariate Marginal Distribution Algorithm (UMDA)**

- Population size: 6
- Tournament selection: $t = 2$
- **Model:** vector of probabilities $p = (p_1, \dots, p_D)$
 - each p_d is the probability of observing 1 at d th element
- **Model learning:**
 - estimate p from selected individuals
- **Model sampling:**
 - generate 1 on d th position with probability p_d (independently of other positions)

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- Example
- UMDA Pipeline
- UMDA: OneMax
- Trap function
- UMDA: Traps
- Beating traps
- Good news!

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Selection, Modeling, Sampling

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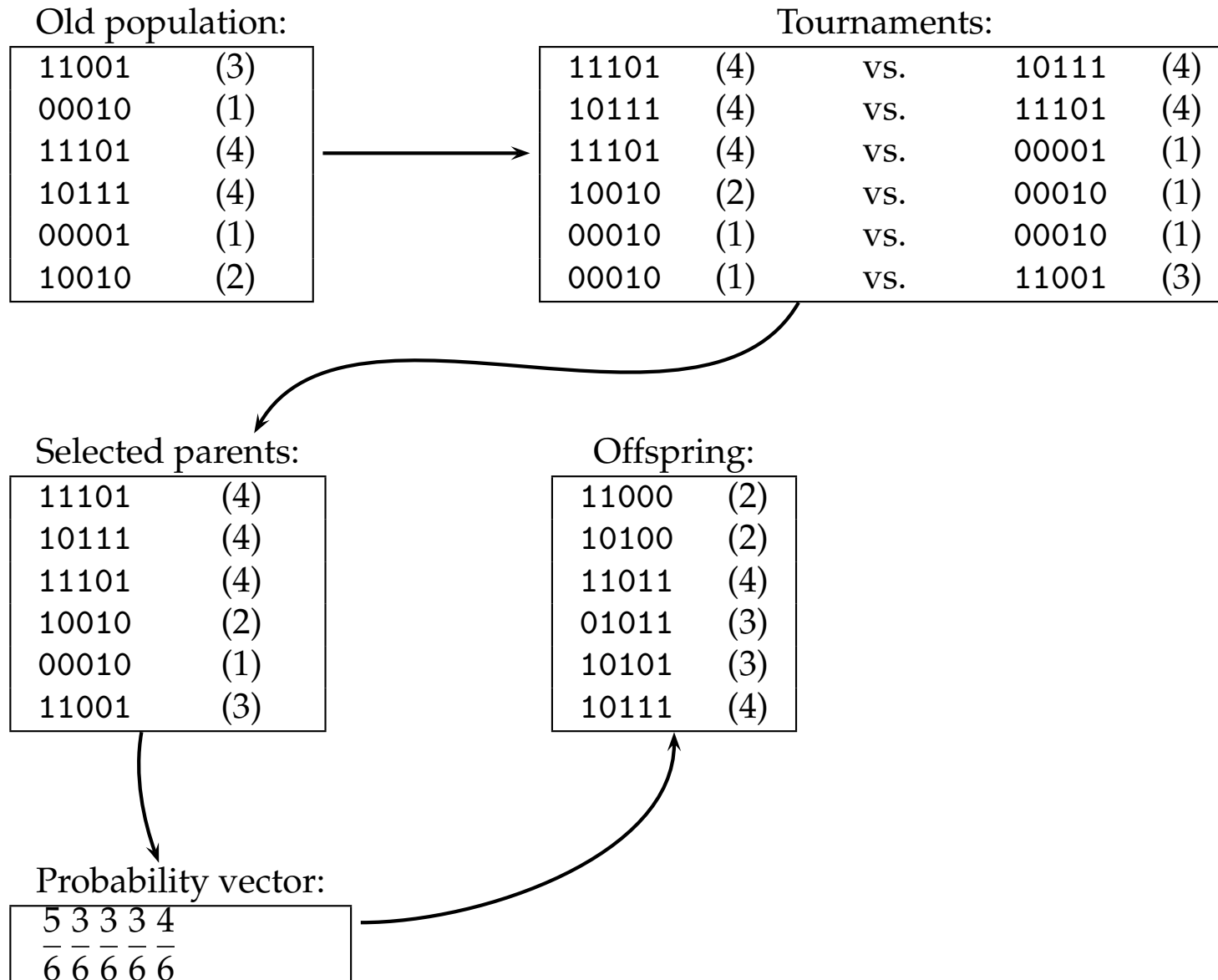
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UMDA Behaviour for OneMax problem

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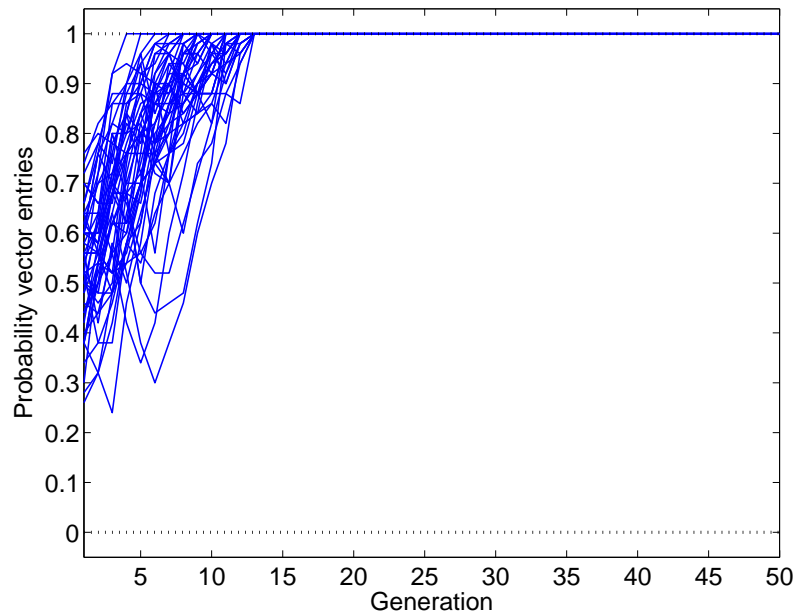
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- 1s are better than 0s on average, selection increases the proportion of 1s
- Recombination preserves and combines 1s, the ratio of 1s increases over time
- If we have many 1s in population, we cannot miss the optimum

The number of evaluations needed for reliable convergence:

Algorithm	Nr. of evaluations
UMDA	$\mathcal{O}(D \ln D)$
Hill-Climber	$\mathcal{O}(D \ln D)$
GA with uniform crossover	approx. $\mathcal{O}(D \ln D)$
GA with 1-point crossover	a bit slower

UMDA behaves similarly to GA with uniform crossover!



What about a different fitness?

For OneMax function:

- UMDA works well, all the bits probably eventually converge to the right value.

Will UMDA be similarly successful for other fitness functions?

- Well,no. :-)

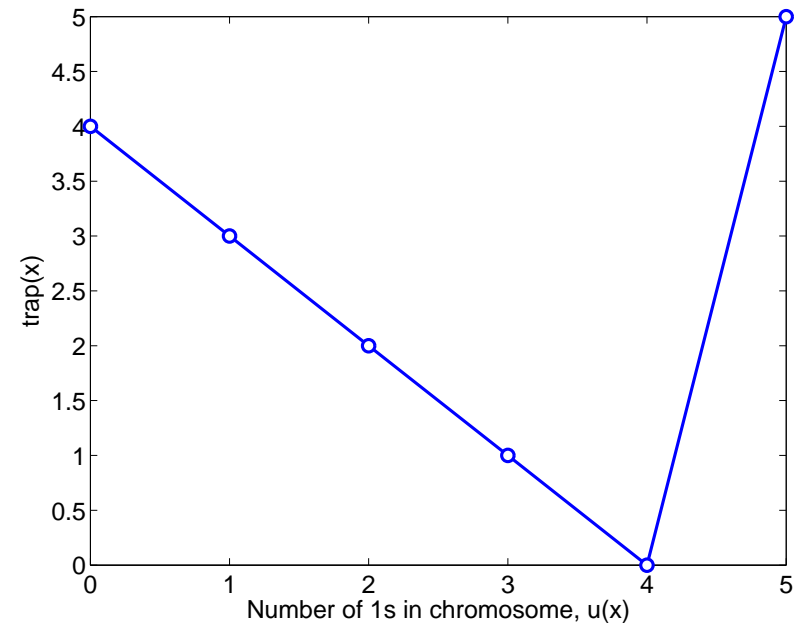
Problem: Concatanated 5-bit traps

$$f = f_{\text{trap}}(x_1, x_2, x_3, x_4, x_5) + \\ + f_{\text{trap}}(x_6, x_7, x_8, x_9, x_{10}) + \\ + \dots$$

The *trap* function is defined as

$$f_{\text{trap}}(\mathbf{x}) = \begin{cases} 5 & \text{if } u(\mathbf{x}) = 5 \\ 4 - u(\mathbf{x}) & \text{otherwise} \end{cases}$$

where $u(\mathbf{x})$ is the so called *unity* function and returns the number of 1s in \mathbf{x} (it is actually the One Max function).



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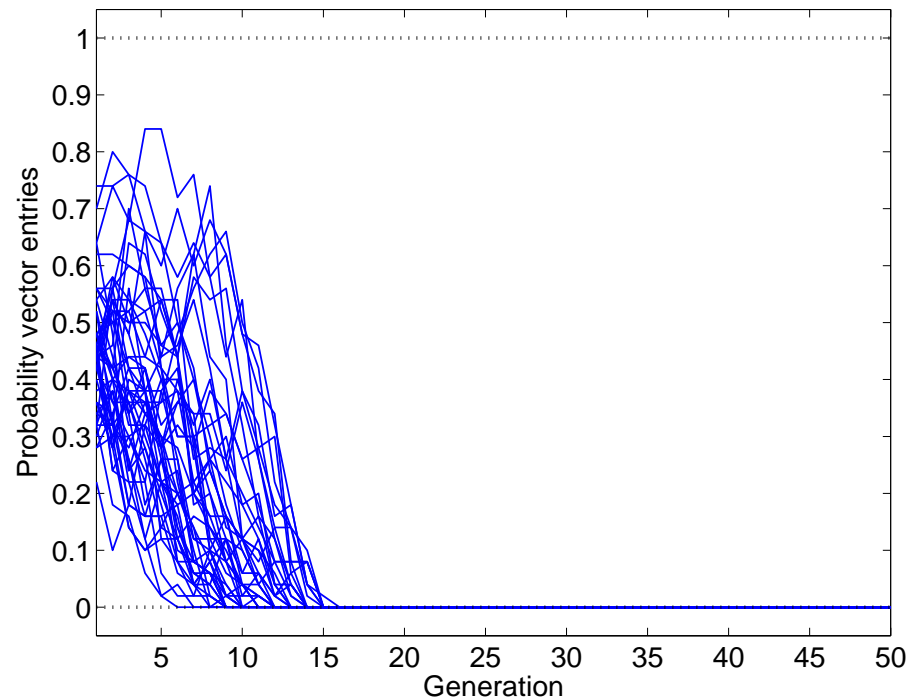
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UMDA behaviour on concatenated traps

Traps:

- Optimum in 111111...1
- But $f_{\text{trap}}(0****) = 2$ while $f_{\text{trap}}(1****) = 1.375$
- 1-dimensional probabilities lead the GA to the wrong way!
- Exponentially increasing population size is needed, otherwise GA will not find optimum reliably.



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What can be done about traps?

The f_{trap} function is *deceptive*:

- Statistics over 1**** and 0**** do not lead us to the right solution
- The same holds for statistics over 11*** and 00***, 111** and 000**, 1111* and 0000*

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- Harder than the *needle-in-the-haystack* problem:
 - regular haystack simply does not provide any information, where to search for the needle
 - f_{trap} -haystack actively lies to you—it points you to the wrong part of the haystack

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- But: $f_{\text{trap}}(00000) < f_{\text{trap}}(11111)$, 11111 will be better than 00000 on average
- 5bit statistics should work for 5bit traps in the same way as 1bit statistics work for OneMax problem!

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Model learning:

- build model for each 5-tuple of bits
- compute $p(00000), p(00001), \dots, p(11111)$,

Epistasis

EDAs

Motivation Example

- Example
- UMDA Pipeline
- UMDA: OneMax
- Trap function
- UMDA: Traps
- **Beating traps**
- Good news!

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

Scalability Analysis

Summary



What can be done about traps?

The f_{trap} function is *deceptive*:

- Statistics over 1**** and 0**** do not lead us to the right solution
- The same holds for statistics over 11*** and 00***, 111** and 000**, 1111* and 0000*
- Harder than the *needle-in-the-haystack* problem:
 - regular haystack simply does not provide any information, where to search for the needle
 - f_{trap} -haystack actively lies to you—it points you to the wrong part of the haystack
- But: $f_{\text{trap}}(00000) < f_{\text{trap}}(11111)$, 11111 will be better than 00000 on average
- 5bit statistics should work for 5bit traps in the same way as 1bit statistics work for OneMax problem!

Model learning:

- build model for each 5-tuple of bits
- compute $p(00000), p(00001), \dots, p(11111)$,

Model sampling:

- Each 5-tuple of bits is generated independently
- Generate 00000 with probability $p(00000)$, 00001 with probability $p(00001)$, ...

Epistasis

EDAs

Motivation Example

- Example
- UMDA Pipeline
- UMDA: OneMax
- Trap function
- UMDA: Traps
- **Beating traps**
- Good news!

Discrete EDAs

Pairwise Interactions

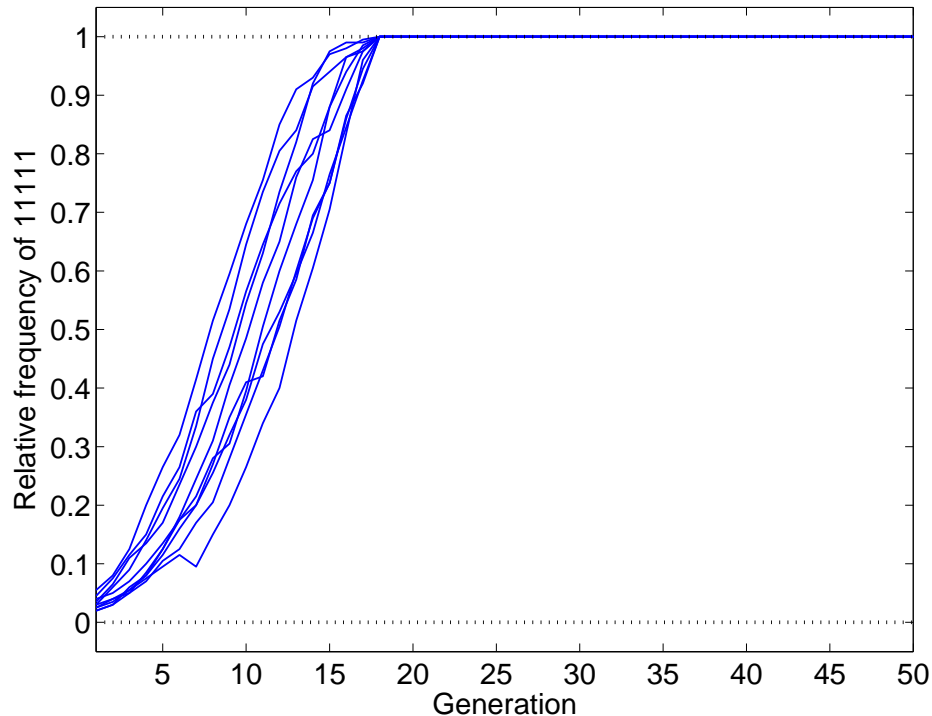
Multivar. Interactions

Scalability Analysis

Summary

Good news!

The right statistics work great!



Algorithm	Nr. of evaluations
UMDA with 5bit BB	$\mathcal{O}(D \ln D)$ (WOW!)
Hill-Climber	$\mathcal{O}(D^k \ln D)$, $k = 5$
GA with uniform xover	approx. $\mathcal{O}(2^D)$
GA with 1-point xover	similar to unif. xover

What shall we do next?

If we were able to

- find the right statistics with a small overhead, and
- use them in the UMDA framework,

we would be able to solve order- k separable problems using $\mathcal{O}(D^2)$ evaluations.

- ...and there are many problems of this type.

The problem solution is closely related to the so-called *linkage learning*, i.e. discovering and using statistical dependencies among variables.



Discrete EDAs



EDAs without interactions

Epistasis

EDAs

Motivation Example

Discrete EDAs

- EDAs without interactions

Pairwise Interactions

Multivar. Interactions

Scalability Analysis

Summary

1. **Population-based incremental learning (PBIL)**
Baluja, 1994
2. **Univariate marginal distribution algorithm (UMDA)**
Mühlenbein and Paaß, 1996
3. **Compact genetic algorithm (cGA)**
Harik, Lobo, Goldberg, 1998

Similarities:

- all of them use a vector of probabilities

Differences:

- PBIL and cGA do not use population (only the vector p); UMDA does
- PBIL and cGA use different rules for the adaptation of p

Advantages:

- Simplicity
- Speed
- Simple simulation of large populations

Limitations:

- Reliable only for order-1 decomposable problems (i.e., problems without interactions).



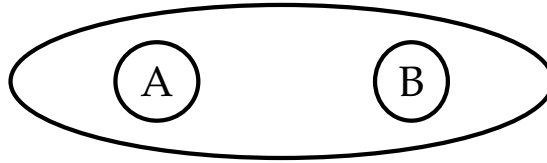
EDAs with Pairwise Interactions



From single bits to pairwise models

How to describe two positions together?

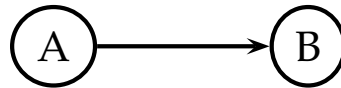
- Using the joint probability distribution:



Number of free parameters:

		$p(A, B)$	
		B 0	1
A	0	$p(0, 0)$	$p(0, 1)$
	1	$p(1, 0)$	$p(1, 1)$

- Using conditional probabilities:



Number of free parameters:

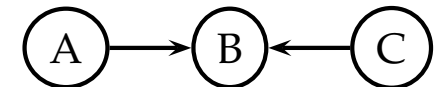
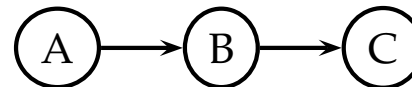
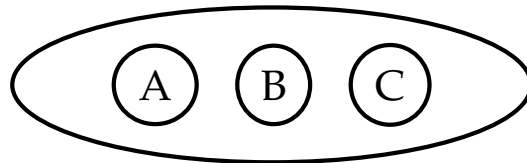
$$p(A, B) = p(B|A) \cdot p(A):$$

$$p(B = 1|A = 0)$$

$$p(B = 1|A = 1)$$

$$p(A = 1)$$

Question: what is the number of parameters in case of the following models?



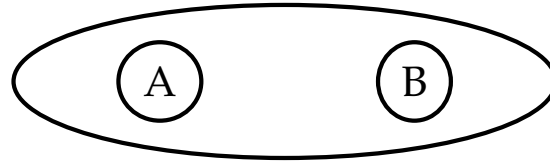
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From single bits to pairwise models

How to describe two positions together?

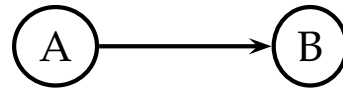
- Using the joint probability distribution:



Number of free parameters: 3

		$p(A, B)$	
		B = 0	B = 1
A	0	$p(0, 0)$	$p(0, 1)$
	1	$p(1, 0)$	$p(1, 1)$

- Using conditional probabilities:



Number of free parameters:

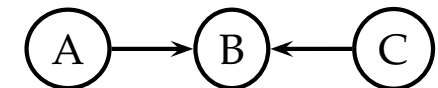
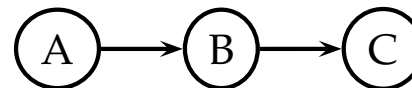
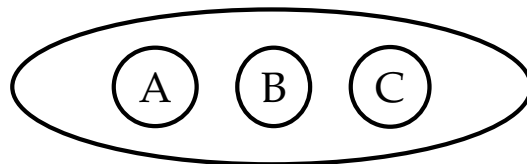
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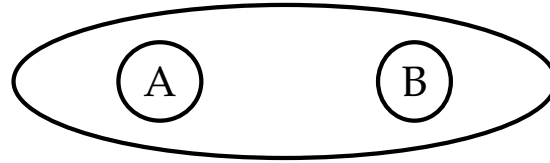
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From single bits to pairwise models

How to describe two positions together?

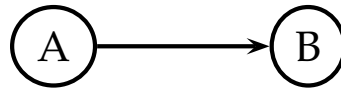
- Using the joint probability distribution:



Number of free parameters: 3

		$p(A, B)$	
		0	1
A	0	$p(0, 0)$	$p(0, 1)$
	1	$p(1, 0)$	$p(1, 1)$

- Using conditional probabilities:



Number of free parameters: 3

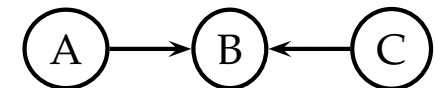
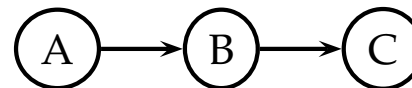
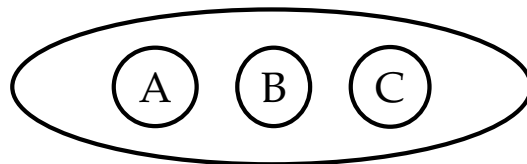
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Question: what is the number of parameters in case of the following models?



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Example with pairwise dependencies: dependency tree

- Nodes: binary variables (loci of chromosome)
- Edges: dependencies among variables
- Features:
 - Each node depends at most on 1 other node
 - Graph does not contain cycles
 - Graph is connected

Learning the structure of dependency tree:

1. Score the edges using mutual information:

$$I(X, Y) = \sum_{x,y} p(x, y) \cdot \log \frac{p(x, y)}{p(x)p(y)}$$

2. Use any algorithm to determine the maximum spanning tree of the graph, e.g. Prim (1957)
 - (a) Start building the tree from any node
 - (b) Add such a node that is connected to the tree by the edge with maximum score

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

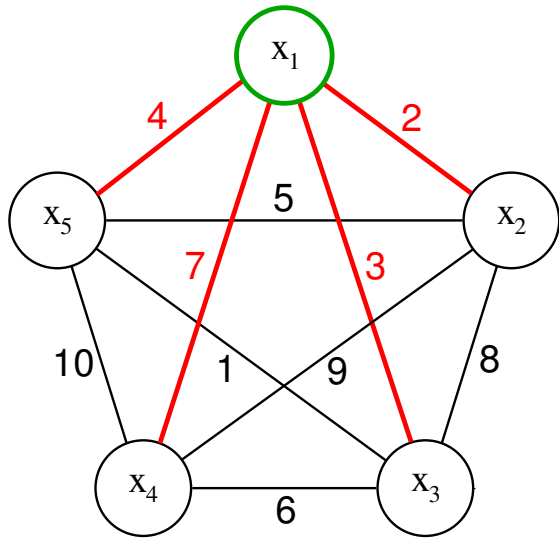
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Multivar. Interactions

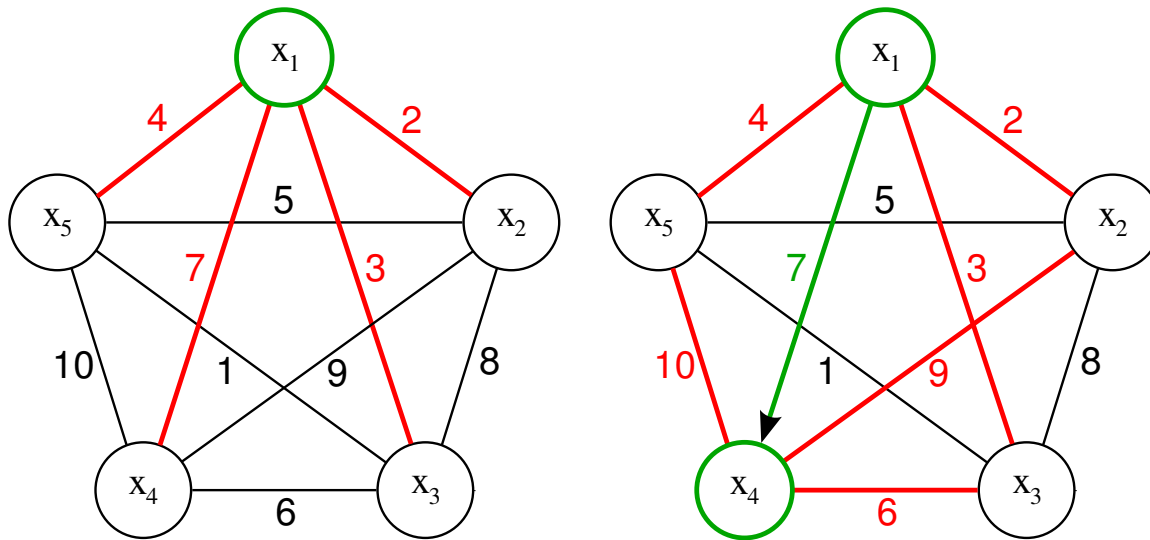
Scalability Analysis

Summary

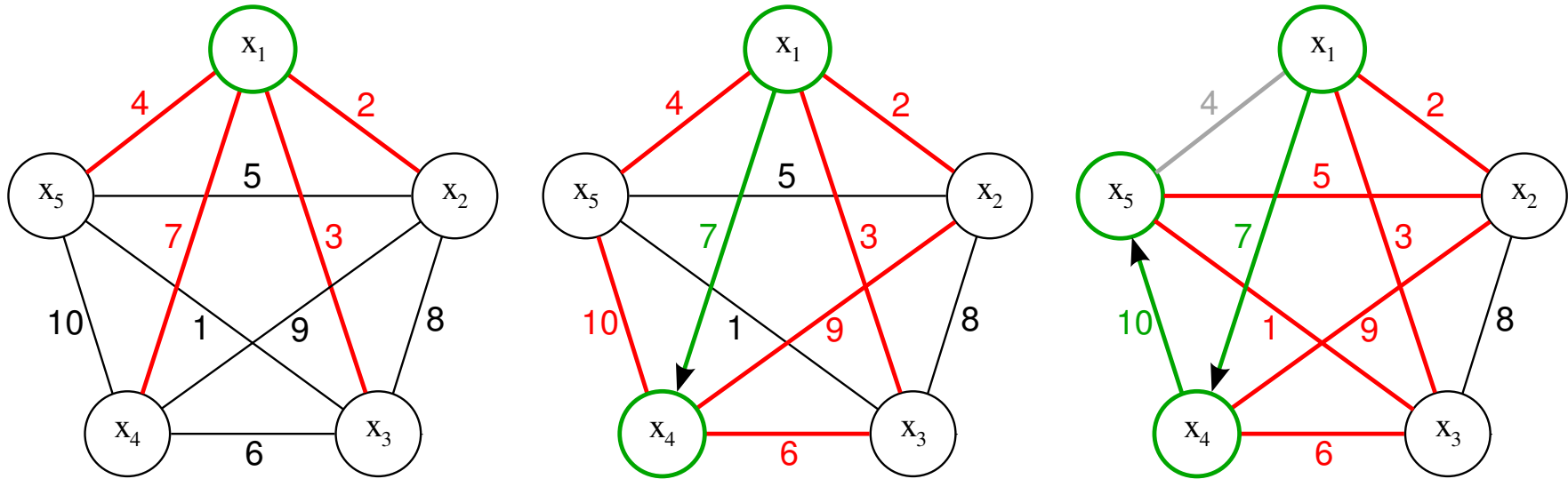
Example of dependency tree learning



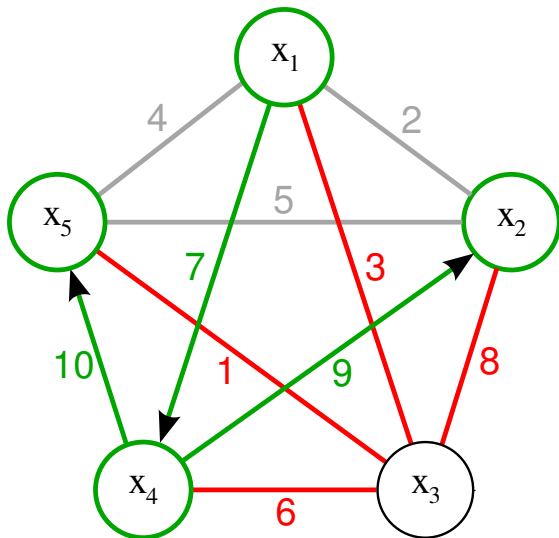
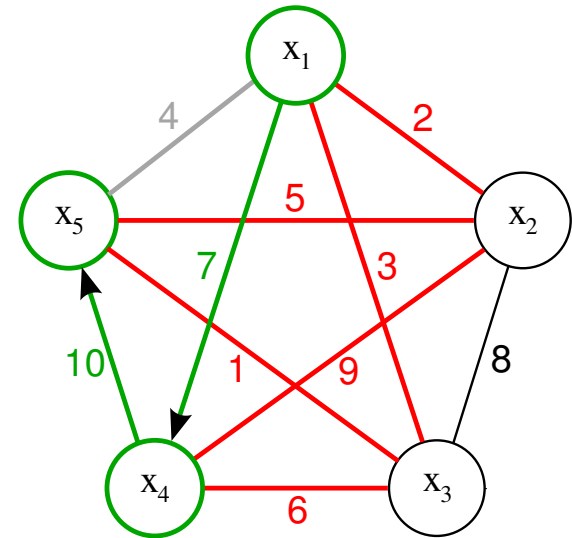
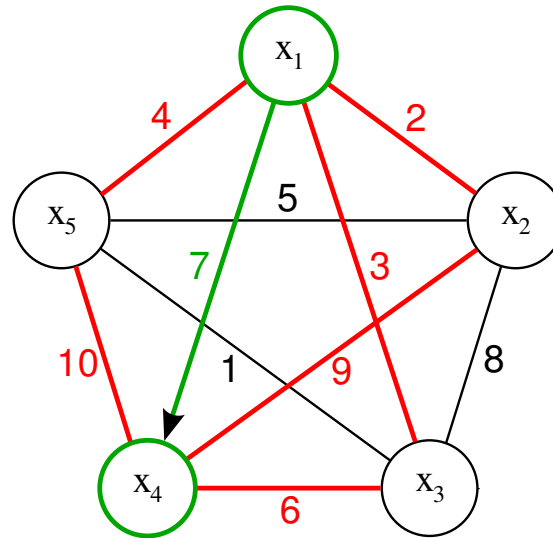
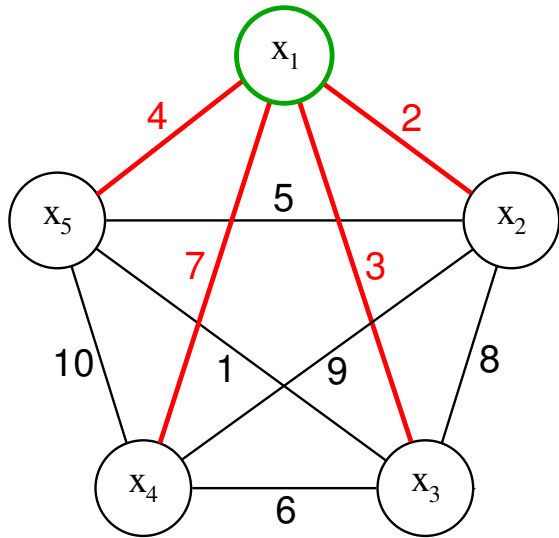
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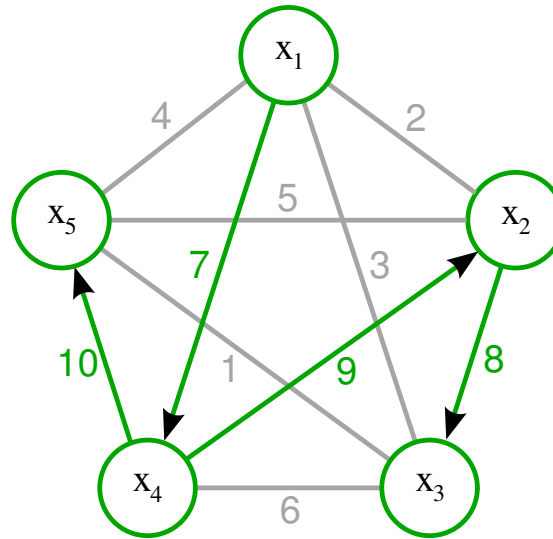
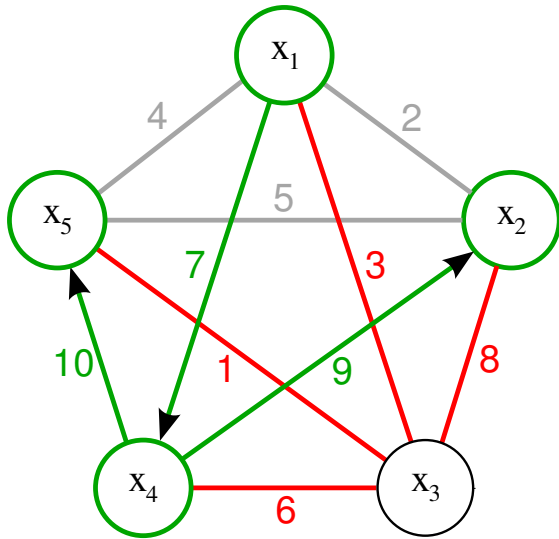
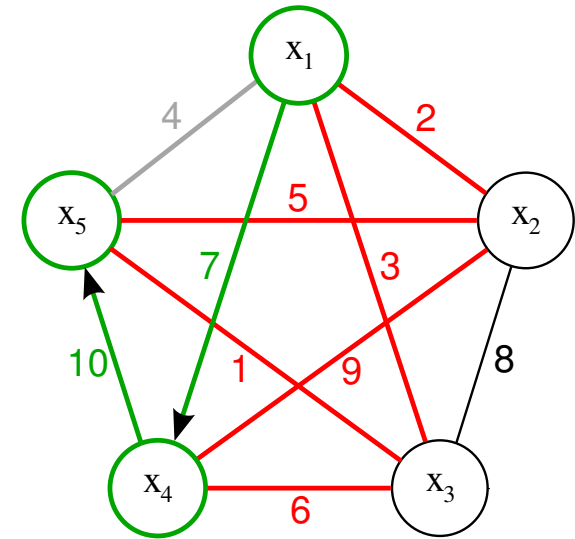
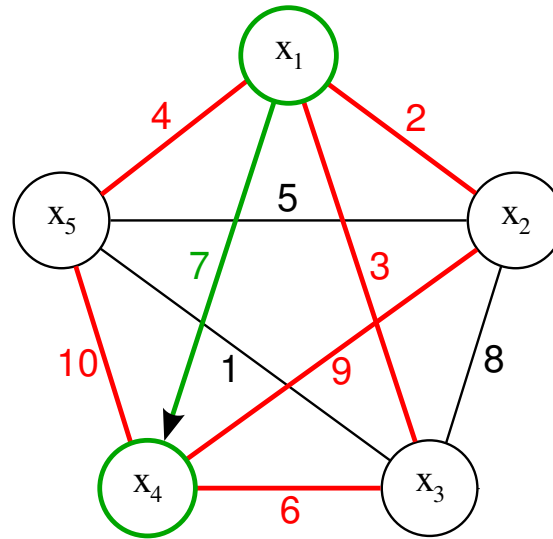
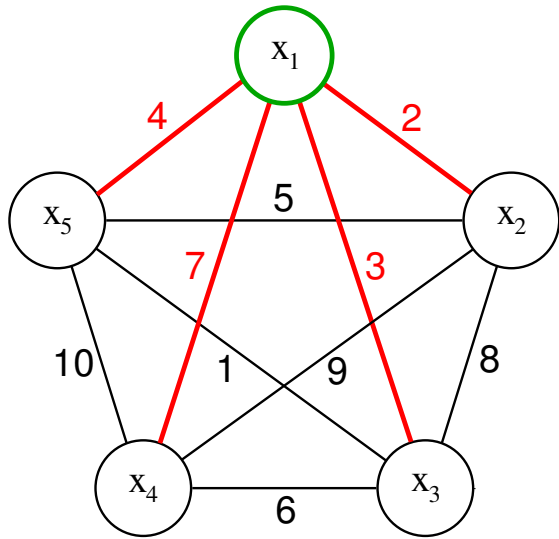
Example of dependency tree learning



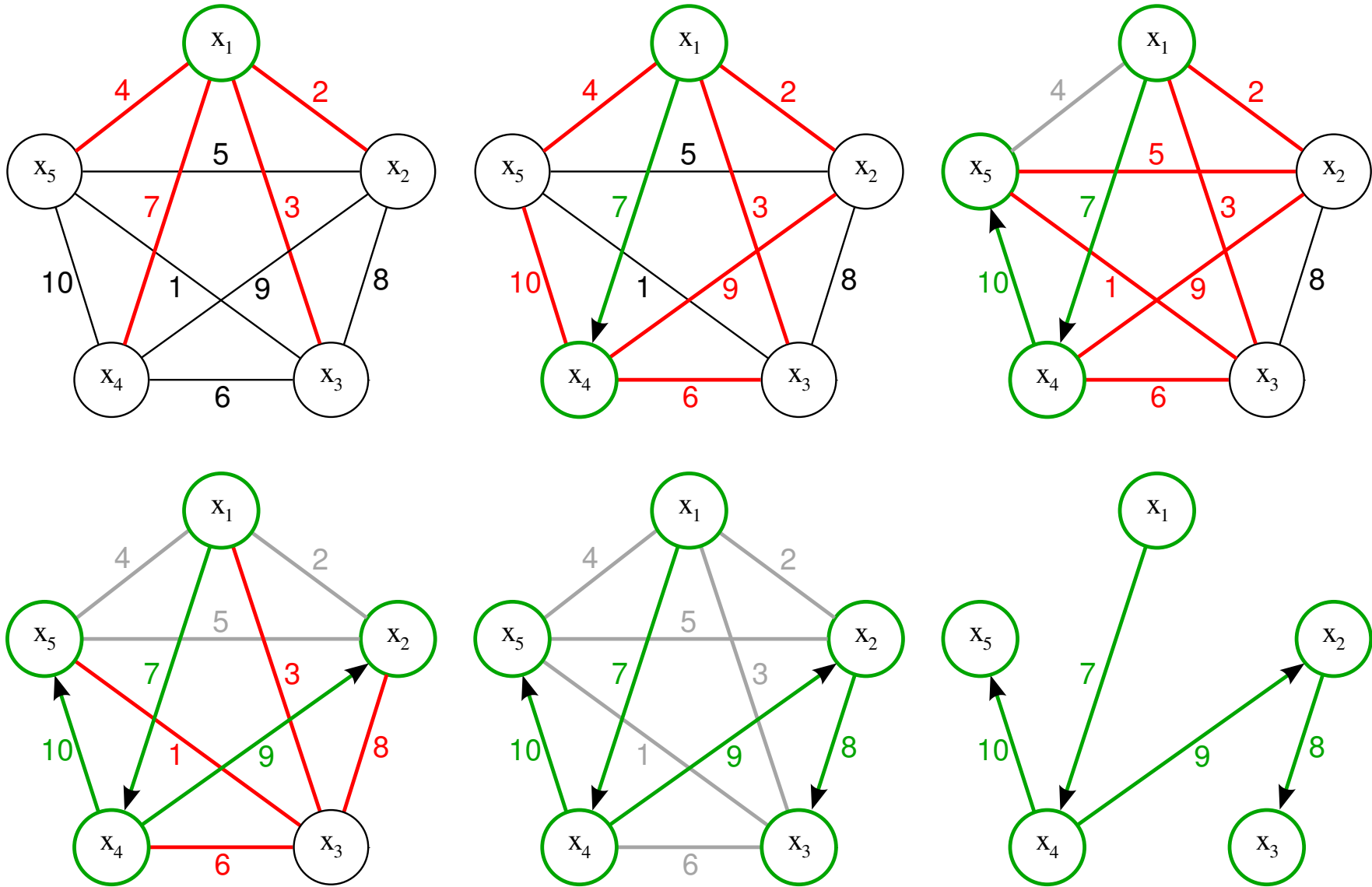
Example of dependency tree learning



Example of dependency tree learning



Example of dependency tree learning





Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

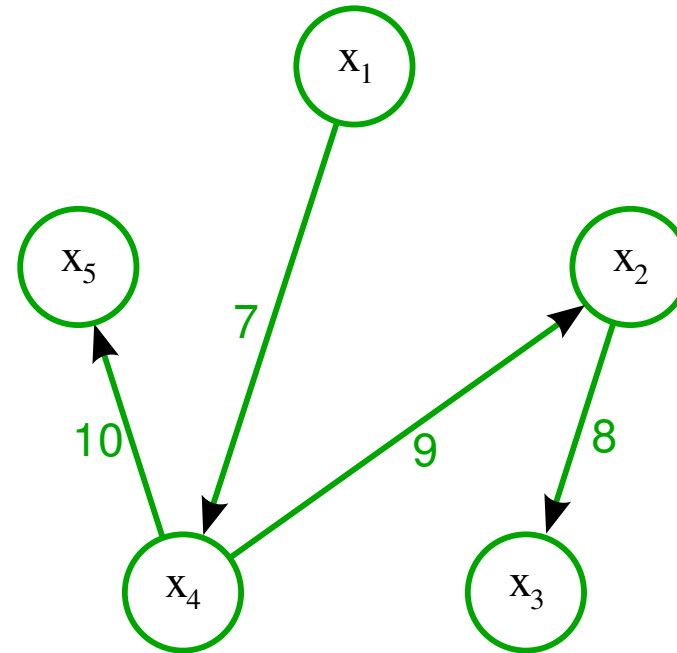
Pairwise Interactions

- Graph. models
- Dependency tree
- DT learning
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- Summary

Multivar. Interactions

Scalability Analysis

Summary



Probability	Number of params
-------------	------------------

$$p(X_1 = 1)$$

$$p(X_4 = 1 | X_1)$$

$$p(X_5 = 1 | X_4)$$

$$p(X_2 = 1 | X_4)$$

$$p(X_3 = 1 | X_2)$$

Whole model	
-------------	--



Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

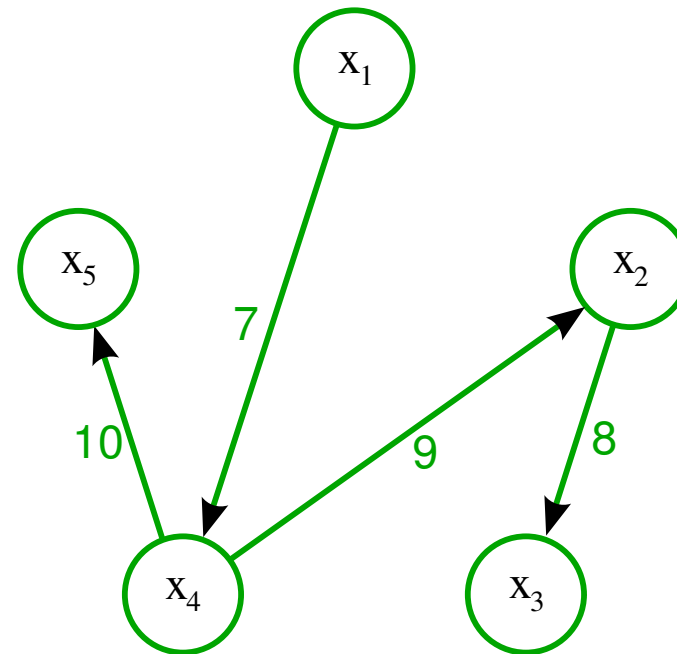
Pairwise Interactions

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Multivar. Interactions

Scalability Analysis

Summary



Probability	Number of params
$p(X_1 = 1)$	1
$p(X_4 = 1 X_1)$	
$p(X_5 = 1 X_4)$	
$p(X_2 = 1 X_4)$	
$p(X_3 = 1 X_2)$	
Whole model	



Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

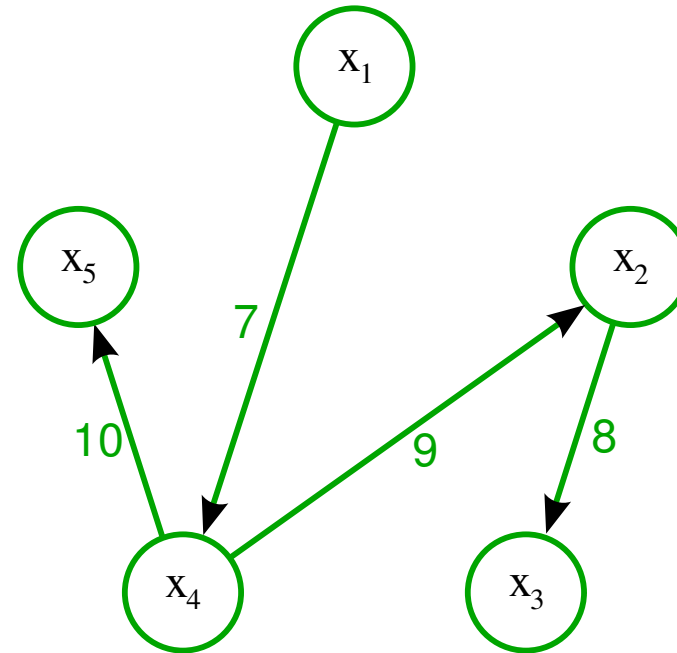
Pairwise Interactions

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Multivar. Interactions

Scalability Analysis

Summary



Probability	Number of params
$p(X_1 = 1)$	1
$p(X_4 = 1 X_1)$	2
$p(X_5 = 1 X_4)$	
$p(X_2 = 1 X_4)$	
$p(X_3 = 1 X_2)$	
Whole model	



Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

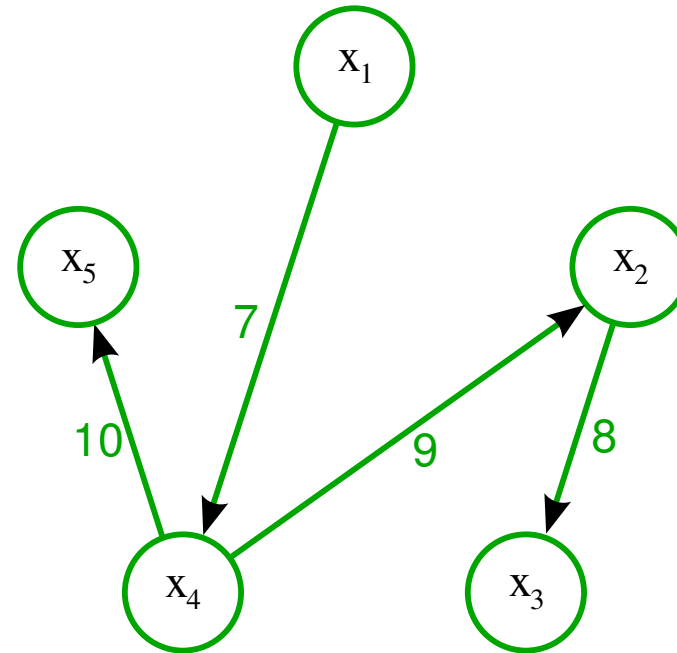
Pairwise Interactions

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Multivar. Interactions

Scalability Analysis

Summary



Probability	Number of params
$p(X_1 = 1)$	1
$p(X_4 = 1 X_1)$	2
$p(X_5 = 1 X_4)$	2
$p(X_2 = 1 X_4)$	
$p(X_3 = 1 X_2)$	
Whole model	



Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

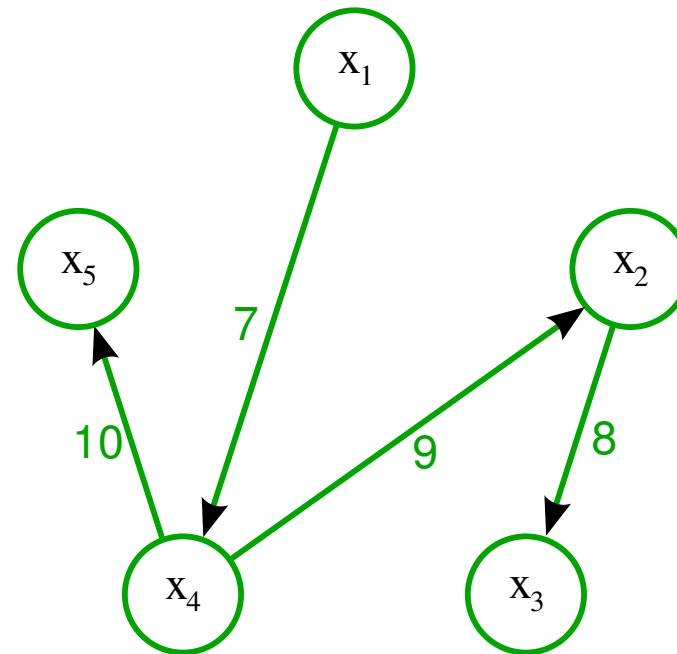
Pairwise Interactions

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Multivar. Interactions

Scalability Analysis

Summary



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Whole model	



Dependency tree: probabilities

Epistasis

EDAs

Motivation Example

Discrete EDAs

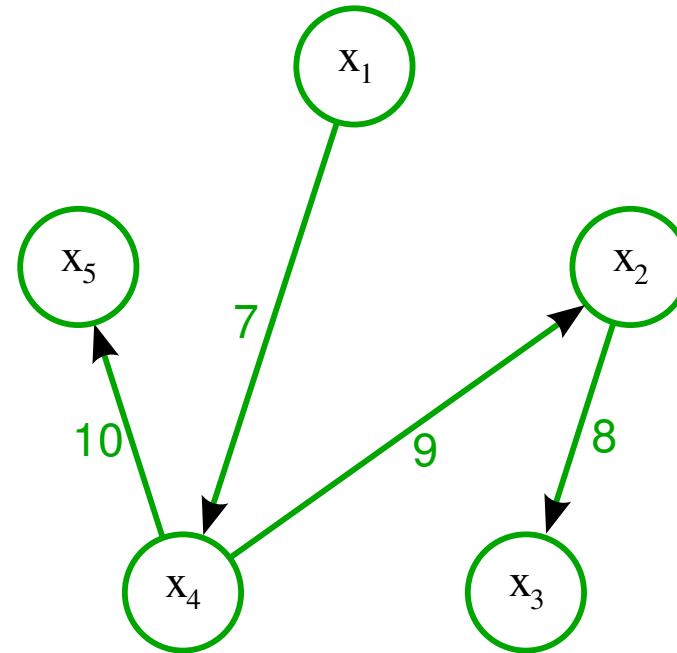
Pairwise Interactions

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Multivar. Interactions

Scalability Analysis

Summary



Probability	Number of params
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$p(X_5 = 1 X_4)$	2
$p(X_2 = 1 X_4)$	2
$p(X_3 = 1 X_2)$	2
Whole model	9



EDAs with pairwise interactions

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

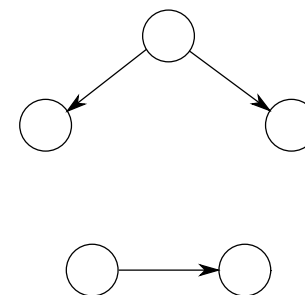
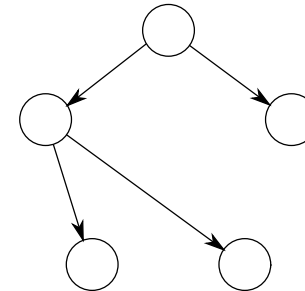
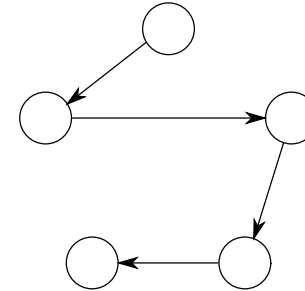
- Graph. models
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Multivar. Interactions

Scalability Analysis

Summary

1. **MIMIC** (sequences)
 - Mutual Information Maximization for Input Clustering
 - de Bonet et al., 1996
2. **COMIT** (trees)
 - Combining Optimizers with Mutual Information Trees
 - Baluja and Davies, 1997
3. **BMDA** (forrest)
 - Bivariate Marginal Distribution Algorithm
 - Pelikan and Mühlenbein, 1998





Summary

- Advantages:
 - Still simple
 - Still fast
 - Can learn *something* about the structure
- Limitations:
 - Reliable only for order-1 or order-2 decomposable problems

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

- Graph. models
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- **Summary**

Multivar. Interactions

Scalability Analysis

Summary



EDAs with Multivariate Interactions



ECGA

Extended Compact GA, Harik, 1999

Marginal Product Model (MPM)

- Variables are treated in groups
- Variables in different groups are considered statistically independent
- Each group is modeled by its joint probability distribution
- The algorithm adaptively searches for the groups during evolution

Problem	Ideal group configuration
OneMax	[1] [2] [3] [4] [5] [6] [7] [8] [9] [10]
5bitTraps	[1 2 3 4 5] [6 7 8 9 10]

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

- ECGA
- MDL Metric
- BOA
- BOA: Learning

Scalability Analysis

Summary



ECGA

Extended Compact GA, Harik, 1999

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Problem	Ideal group configuration
OneMax	[1] [2] [3] [4] [5] [6] [7] [8] [9] [10]
5bitTraps	[1 2 3 4 5] [6 7 8 9 10]

Learning the structure

1. Evaluation metric: Minimum Description Length (MDL)
2. Search procedure: greedy
 - (a) Start with each variable belonging to its own group
 - (b) Perform such a join of two groups which improves the score best
 - (c) Finish if no join improves the score

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

- ECGA
- MDL Metric
- BOA
- BOA: Learning

Scalability Analysis

Summary



ECGA: Evaluation metric

Minimum description length:

Minimize the number of bits required to store the model and the data encoded using the model

$$DL(Model, Data) = DL_{Model} + DL_{Data}$$

Model description length:

Each group g has $|g|$ dimensions, i.e. $2^{|g|} - 1$ frequencies, each of them can take on values up to N

$$DL_{Model} = \log N \sum_{g \in G} (2^{|g|} - 1)$$

Data description length using the model:

Defined using the entropy of marginal distributions (X_g is $|g|$ -dimensional random vector, x_g is its realization):

$$DL_{Data} = N \sum_{g \in G} h(X_g) = -N \sum_{g \in G} \sum_{x_g} p(X_g = x_g) \log p(X_g = x_g)$$

Epistasis

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Motivation Example

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Pairwise Interactions

Multivar. Interactions

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- BOA: Learning

Scalability Analysis

Summary

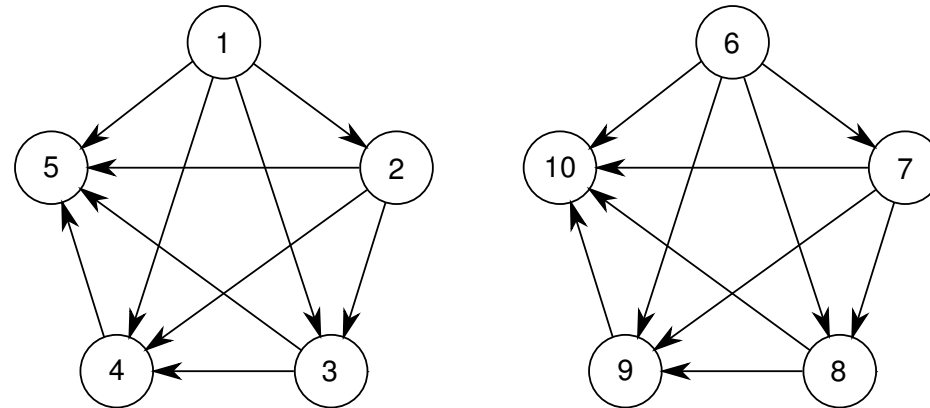


BOA

Bayesian Optimization Algorithm: Pelikán, Goldberg, Cantù-Paz, 1999

Bayesian network (BN)

- Conditional dependencies (instead groups)
- Sequence, tree, forrest — special cases of BN
- For trap function:



- The same model used independently in
 - Estimation of Bayesian Network Alg. (EBNA), Etxeberria et al., 1999
 - Learning Factorized Density Alg. (LFDA), Mühlenbein et al., 1999

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

- ECGA
- MDL Metric
- **BOA**
- BOA: Learning

Scalability Analysis

Summary



BOA: Learning the structure

1. Evaluation metric:
 - Bayesian-Dirichlet metric, or
 - Bayesian information criterion (BIC)
2. Search procedure: greedy
 - (a) Start with graph with no edges (univariate marginal product model)
 - (b) Perform one of the following operations, choose the one which improves the score best
 - Add an edge
 - Delete an edge
 - Reverse an edge
 - (c) Finish if no operation improves the score

BOA solves order- k decomposable problems in less than $\mathcal{O}(D^2)$ evaluations!

$$n_{evals} = \mathcal{O}(D^{1.55}) \text{ to } \mathcal{O}(D^2)$$

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

- ECGA
- MDL Metric
- BOA
- **BOA: Learning**

Scalability Analysis

Summary



Scalability Analysis

Test functions

One Max:

$$f_{D \times 1 \text{bitOneMax}}(\mathbf{x}) = \sum_{d=1}^D x_d$$

Trap:

$$f_{D \text{bitTrap}}(\mathbf{x}) = \begin{cases} D & \text{if } u(\mathbf{x}) = D \\ D - 1 - u(\mathbf{x}) & \text{otherwise} \end{cases}$$

Equal Pairs:

$$f_{D \text{bitEqualPairs}}(\mathbf{x}) = 1 + \sum_{d=2}^D f_{\text{EqualPair}}(x_{d-1}, x_d)$$

$$f_{\text{EqualPair}}(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{if } x_1 \neq x_2 \end{cases}$$

Sliding XOR:

$$f_{D \text{bitSlidingXOR}}(\mathbf{x}) = 1 + f_{\text{AllEqual}}(\mathbf{x}) + \sum_{d=3}^D f_{\text{XOR}}(x_{d-2}, x_{d-1}, x_d)$$

$$f_{\text{AllEqual}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = (000 \dots 0) \\ 1 & \text{if } \mathbf{x} = (111 \dots 1) \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\text{XOR}}(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 \oplus x_2 = x_3 \\ 0 & \text{otherwise} \end{cases}$$

Concatenated short basis functions:

$$f_{N \times K \text{bitBasisFunction}} = \sum_{k=1}^K f_{\text{BasisFunction}}(x_{K(k-1)+1}, \dots, x_{Kk})$$

Test function (cont.)

1. $f_{40 \times 1 \text{bitOneMax}}$
 - order-1 decomposable function, no interactions
2. $f_{1 \times 40 \text{bitEqualPairs}}$
 - non-decomposable function
 - weak interactions: optimal setting of each bit depends on the value of the preceding bit
3. $f_{8 \times 5 \text{bitEqualPairs}}$
 - order-5 decomposable function
4. $f_{1 \times 40 \text{bitSlidingXOR}}$
 - non-decomposable function
 - stronger interactions: optimal setting of each bit depends on the value of the 2 preceding bits
5. $f_{8 \times 5 \text{bitSlidingXOR}}$
 - order-5 decomposable function
6. $f_{8 \times 5 \text{bitTrap}}$
 - order-5 decomposable function
 - interactions in each 5-bit block are very strong, the basis function is deceptive



Scalability analysis

Facts:

- using small population size, population-based optimizers can solve only easy problems
- increasing the population size, the optimizers can solve increasingly harder problems
- ... but using a too big population is wasting of resources.

Epistasis

EDAs

Motivation Example

Discrete EDAs

Pairwise Interactions

Multivar. Interactions

Scalability Analysis

- Test functions
- Test function (cont.)
- **Scalability analysis**
- OneMax
- Non-dec. Eq. Pairs
- Decomp. Eq. Pairs
- Non-dec. Sl. XOR
- Decomp. Sl. XOR
- Decomp. Trap
- Model evolution

Summary



Scalability analysis

Facts:

- using small population size, population-based optimizers can solve only easy problems
- increasing the population size, the optimizers can solve increasingly harder problems
- ... but using a too big population is wasting of resources.

Scalability analysis:

- determines the optimal (smallest) population size, with which the algorithm solves the given problem reliably
 - reliably: algorithm finds the optimum in 24 out of 25 runs)
 - for each problem complexity, the optimal population size is determined e.g. using the bisection method
- studies the influence of the problem complexity (dimensionality) on the optimal population size and on the number of needed evaluations

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Summary



Scalability on the One Max function

Epistasis

EDAs

Motivation Example

Discrete EDAs

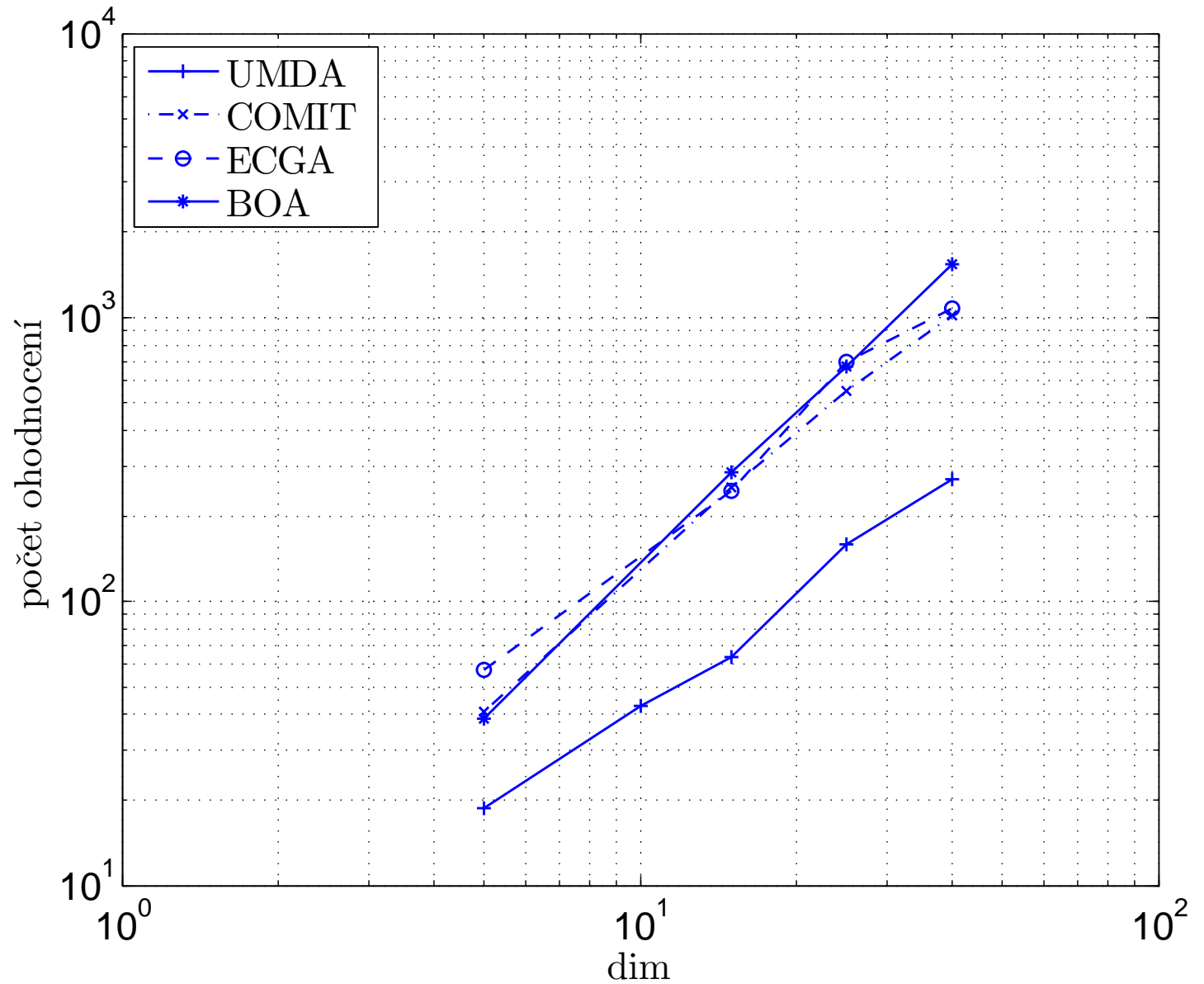
Pairwise Interactions

Multivar. Interactions

Scalability Analysis

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Summary





Scalability on the non-decomposable Equal Pairs function

Epistasis

EDAs

Motivation Example

Discrete EDAs

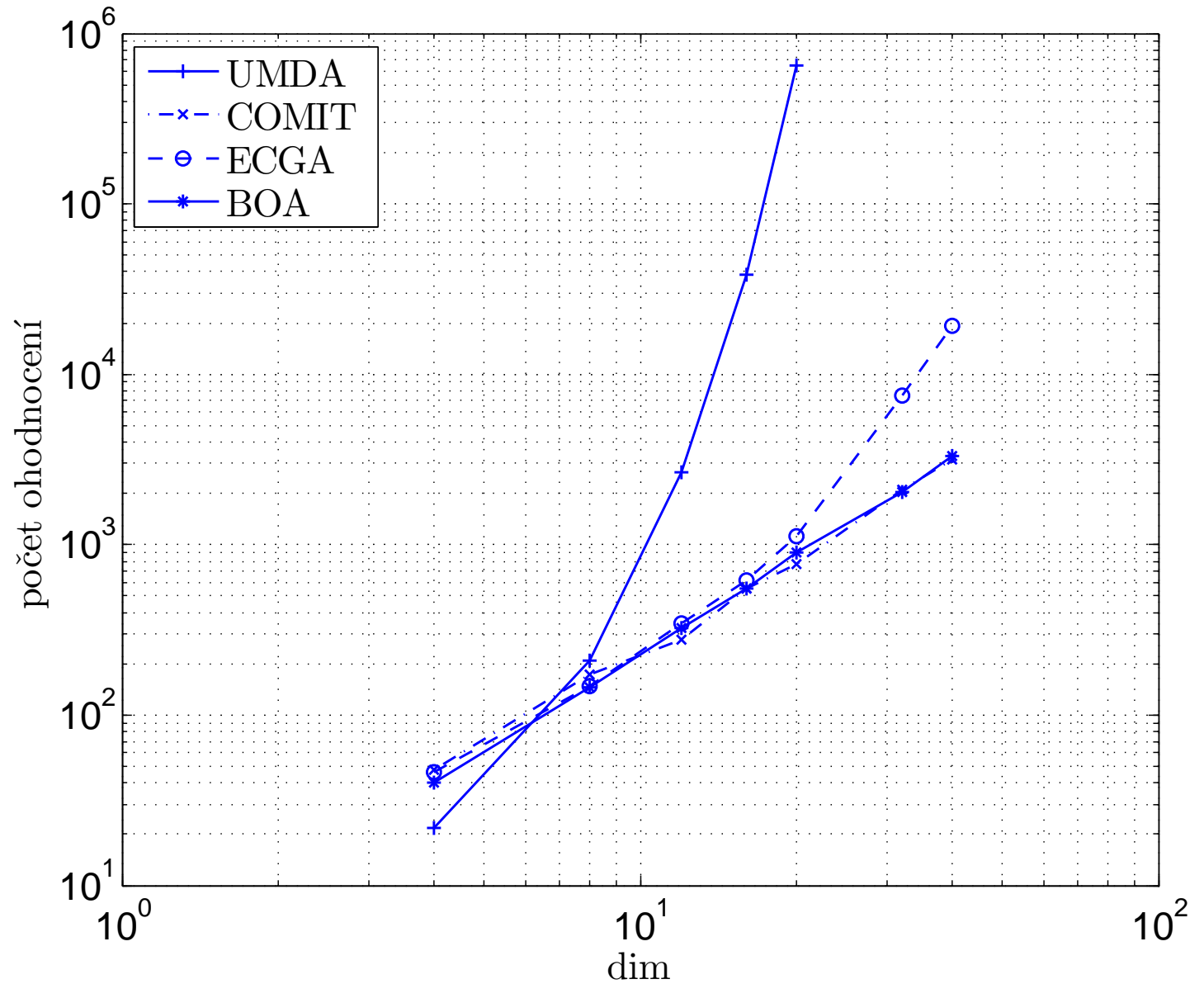
Pairwise Interactions

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Summary





Scalability on the decomposable Equal Pairs function

Epistasis

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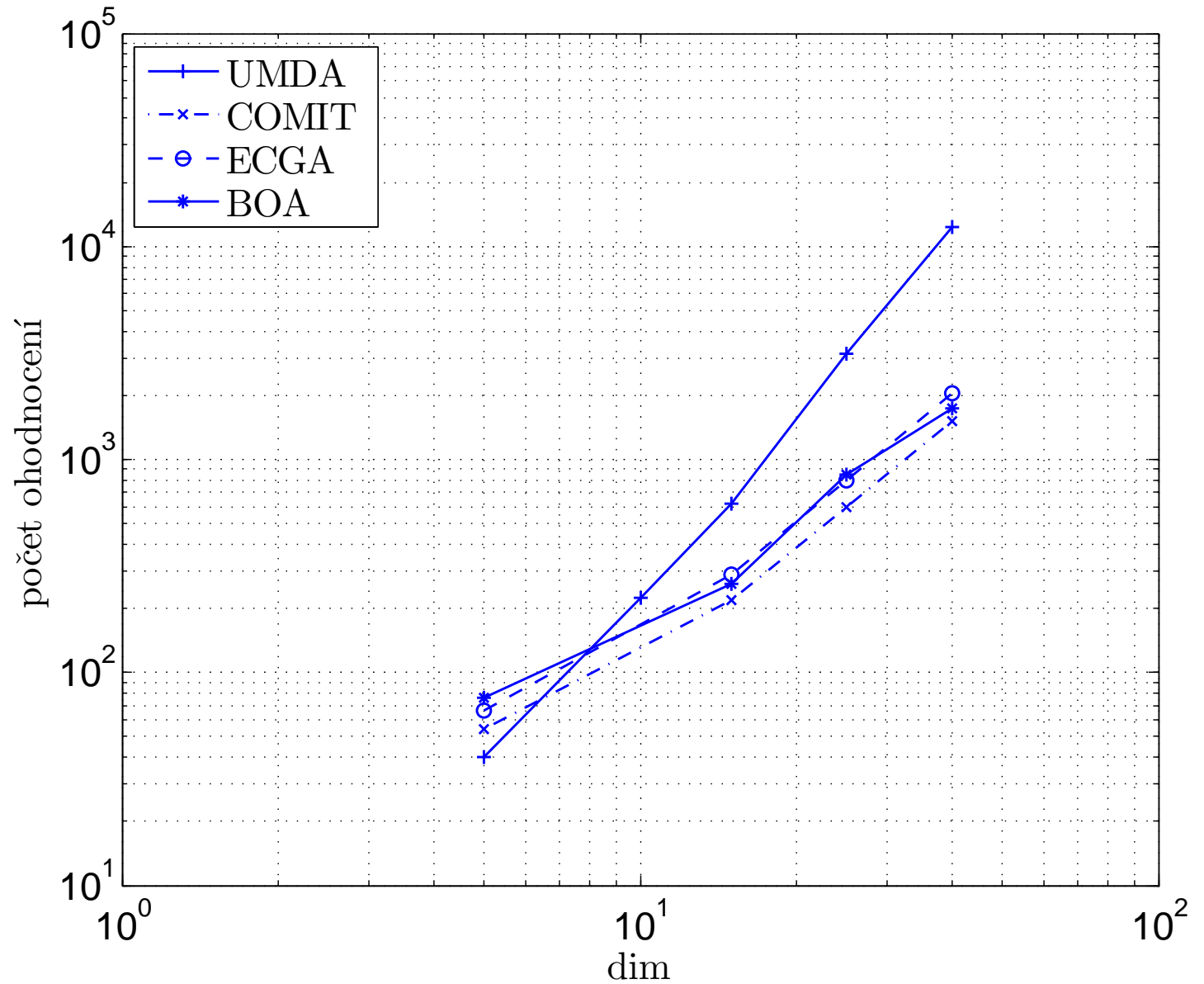
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Summary





Scalability on the non-decomposable Sliding XOR function

Epistasis

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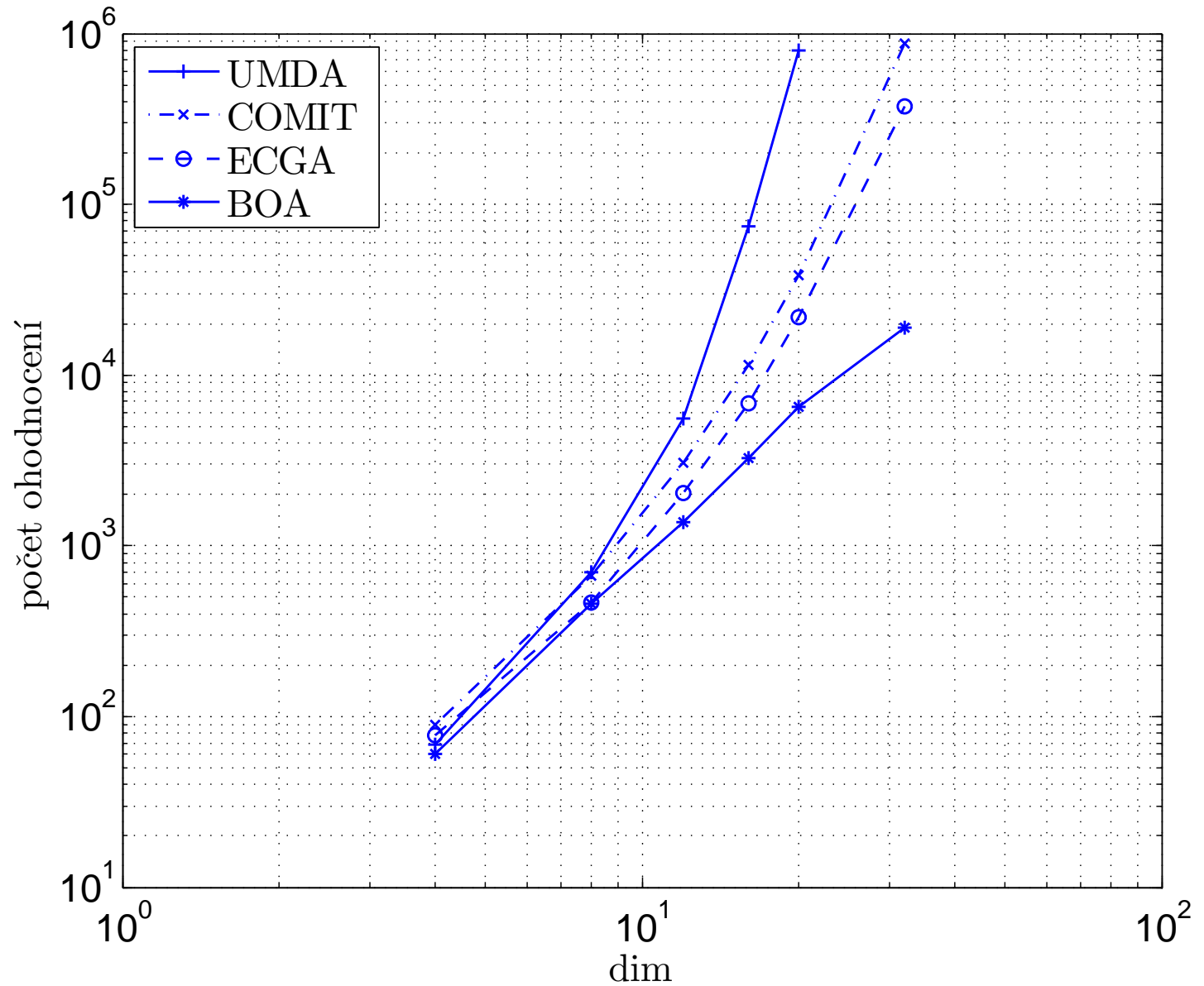
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Scalability on the decomposable Sliding XOR function

Epistasis

EDAs

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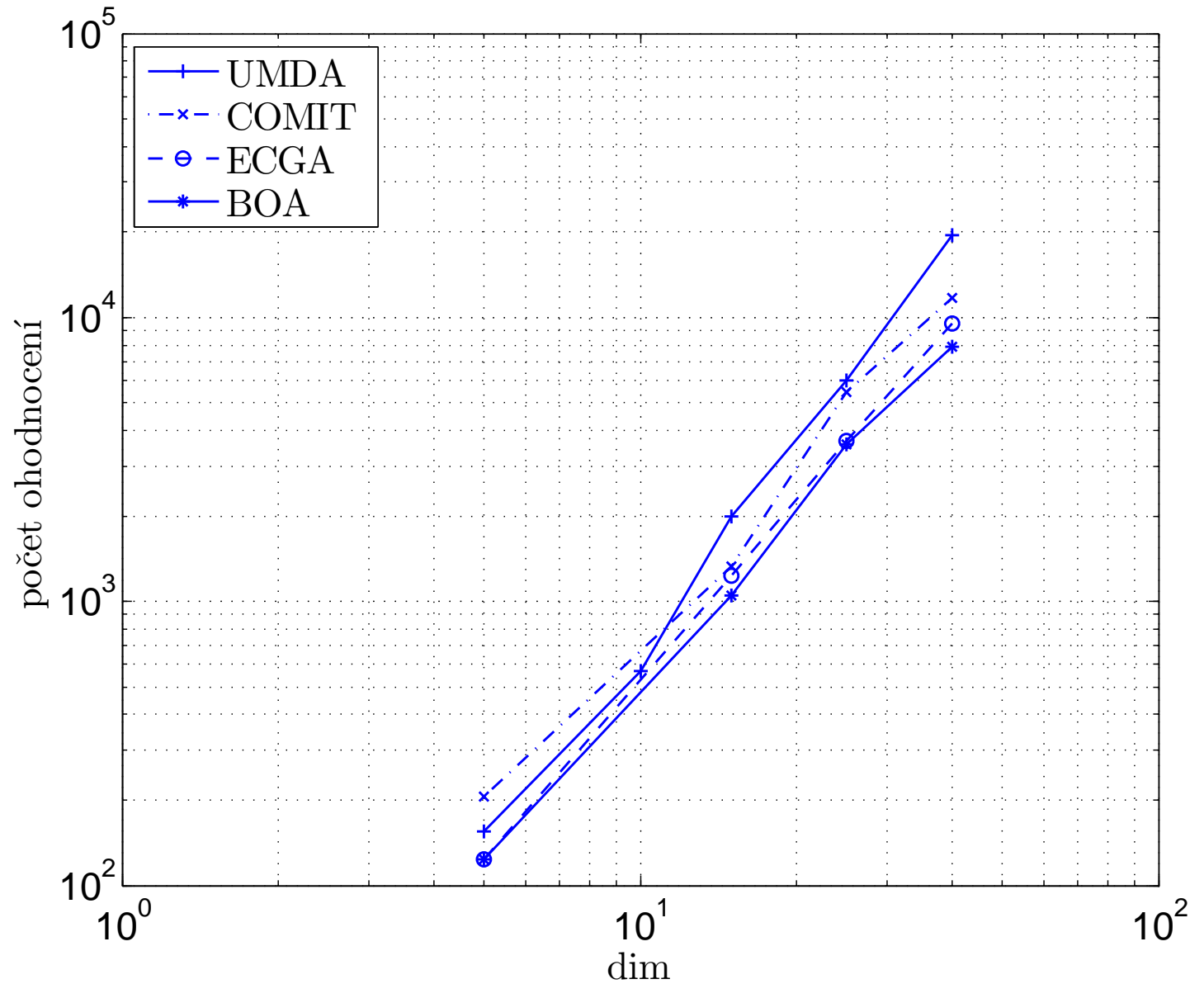
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Summary





Scalability on the decomposable Trap function

Epistasis

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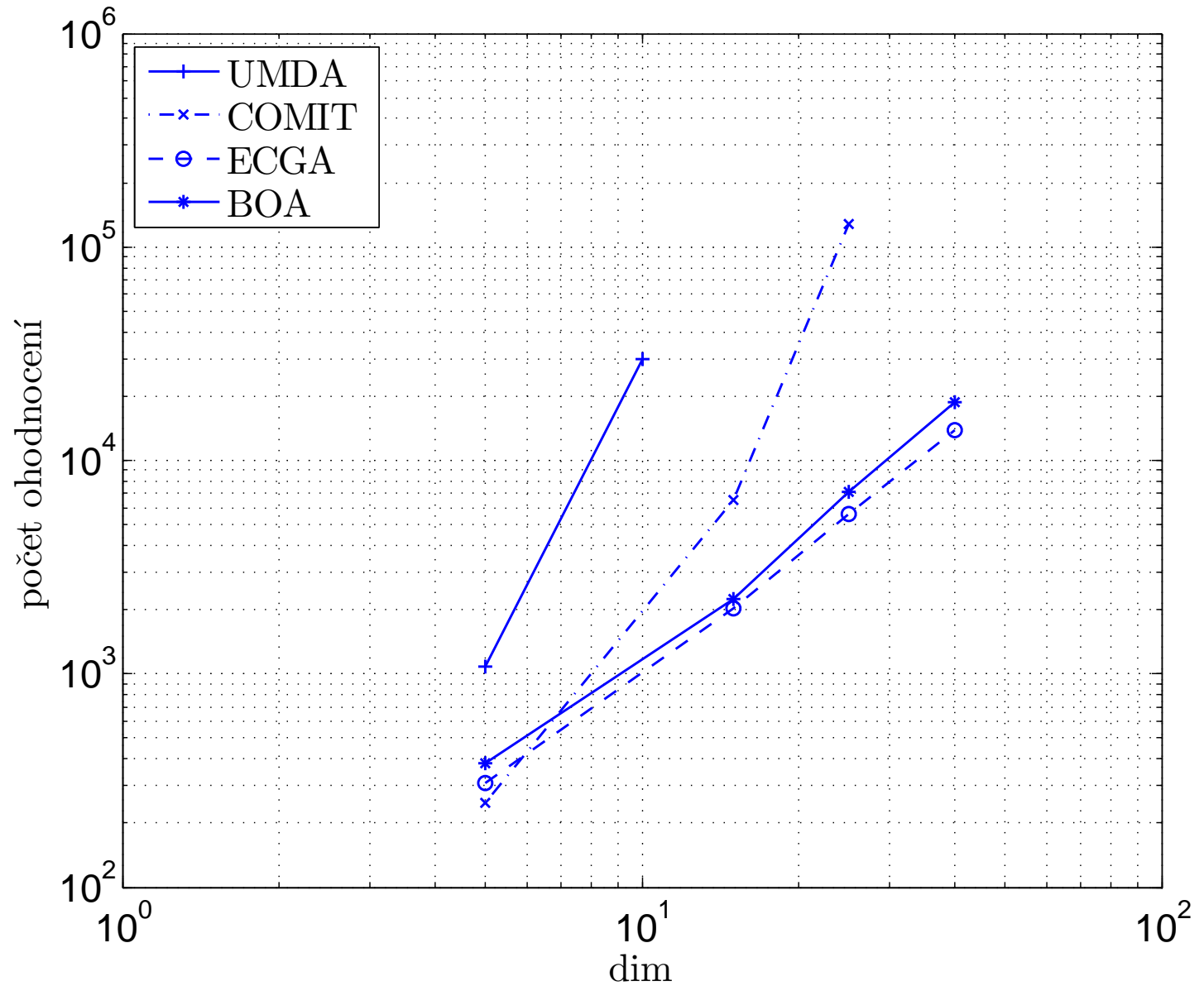
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Summary





Model structure during evolution

During the evolution, the model structure is increasingly precise and at the end of the evolution, the model structure describes the problem structure exactly.

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Summary

NO! That's not true!

Why?

- In the beginning, the distribution patterns are not very discernible, models similar to uniform distributions are used.
- In the end, the population converges and contains many copies of the same individual (or a few individuals). No interactions among variables can be learned. Model structure is wrong (all bits independent), but the model describes the position of optimum very precisely.
- The model with the best matching structure is found somewhere in the middle of the evolution.
- Even though the right structure is never found during the evolution, the problem can be solved successfully.



Summary



Learning outcomes

After this lecture, a student shall be able to

- explain what an epistasis is and show an example of functions with and without epistatic relations;
- demonstrate how epistatic relationships can destroy the efficiency of the search performed by an optimization algorithm, and explain it using schemata;
- describe an Estimation-of-Distribution algorithm and explain its differences from ordinary EA;
- describe in detail and implement a simple UMDA algorithm for binary representations;
- understand, fit to data, and use simple Bayesian networks;
- explain the commonalities and differences among EDAs not able to work with any interactions (PBIL, cGA, UMDA);
- explain the commonalities and differences among EDAs able to work with only pairwise interactions (MIMIC, COMIT, BMDA);
- explain the commonalities and differences among EDAs able to work with multivariate interactions (ECGA, BOA);
- explain the model learning procedures used in ECGA and BOA;
- understand what effect the use of a more complex model has on the efficiency of the algorithm when used on problems with increasingly hard interactions.

Epistasis

EDAs

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Scalability Analysis

Summary

- Learning outcomes