DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

ARRANGEMENTS (uspořádání)

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Based on [Berg], [Mount]

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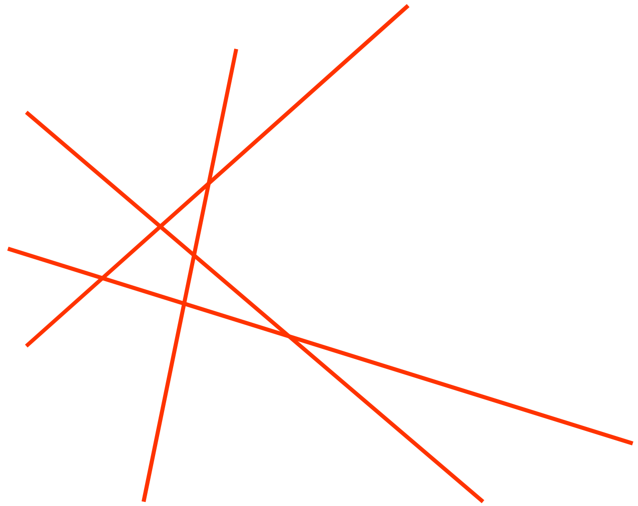
Talk overview

- Arrangements of lines
  - Incremental construction
  - Topological plane sweep



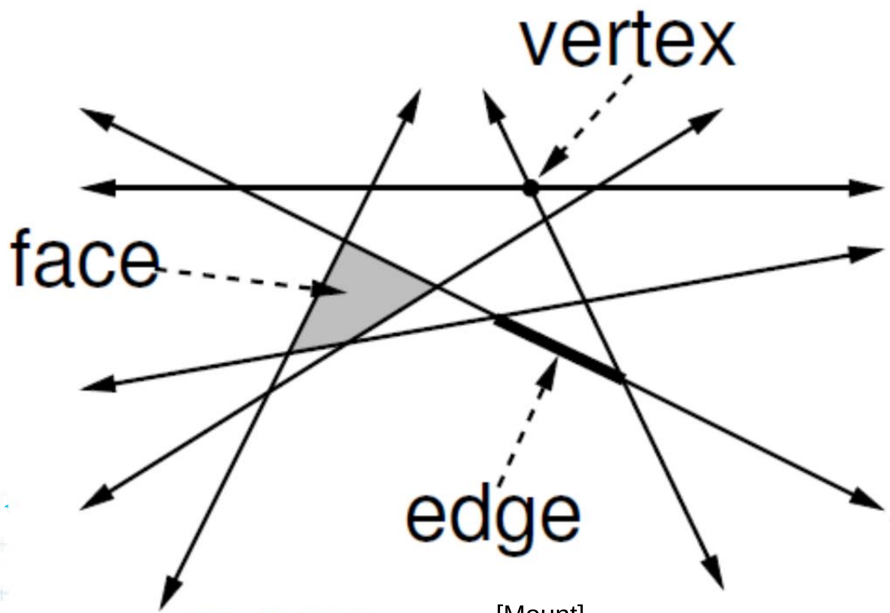
- Duality – next lesson

Arrangements

- The next most important structure in CG after CH, VD, and DT
- Possible in any dimension
  arrangement of (d-1)-dimensional hyperplanes
- We concentrate on arrangement of lines in plane
- Typical application: problems of point sets in dual plane (collinear points, point on circles, …)

Line arrangement

- A finite set L of lines subdivides the plane into a cell complex, called arrangement $A(L)$
- In plane, arrangement defines a planar graph
  - Vertices – intersections of (2 or more) lines
  - Edges – intersection free segments (or rays or lines)
  - Faces – convex regions containing no line (possibly unbounded)



[Mount]

Line arrangement

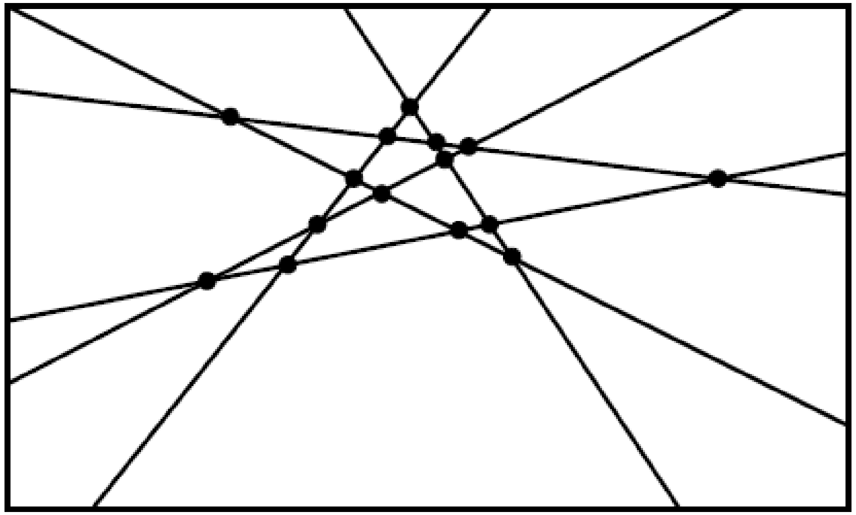
- Simple arrangement assumption

  = no three lines intersect in a single point

  - Can be solved by careful implementation or symbolic perturbation

Line arrangement

- Formal problem: graph must have bounded edges
  - Topological fix: add vertex in infinity
  - Geometrical fix: BBOX, often enough as abstract with corners $\{-\infty,-\infty\},\{\infty,\infty\}$

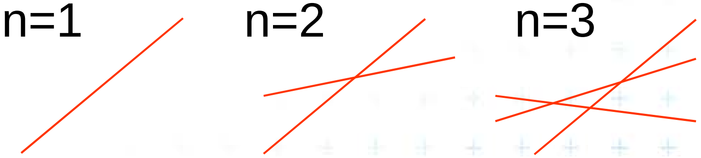


bounding box [Mount]

Combinatorial complexity of line arrangement

- $O(n^2)$
- Given *n* lines in general position, max numbers are
  - Vertices $\binom{n}{2} = \frac{n(n-1)}{2}$ - each line intersect n – 1 others
  - Edges n^2 - *n*–1 intersections create *n* edges on each of *n* lines
  - Faces $\frac{n(n+1)}{2} + 1 = \binom{n}{2} + n + 1$ $\quad$ $f_0 = 1$ (celá rovina), $f_n = f_{n-1} + n$

n=1 n=2 n=3



$f_1 = 2$ $\quad$ $f_2 = 4$ $\quad$ $f_3 = 7$

$$f_n = f_0 + \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + 1$$

DCGI

Construction of line arrangement

(0. Plane sweep method)

– $O(n^2 \log n)$ time and $O(n)$ storage plus $O(n^2)$ storage for the arrangement (log n - heap & BVS access, n^2 vertices, edges, faces)

A. Incremental method

– $O(n^2)$ time and $O(n^2)$ storage

– Optimal method

B. Topological plane sweep

– $O(n^2)$ time and $O(n)$ storage only

– Does not store the result arrangement

– Useful for applications that may throw out the arrangement after processing

DCGI

A. Incremental construction of arrangement

- $O(n^2)$ time, $O(n^2)$ space
  ~size of arrangement => it is an optimal algorithm
- Not randomized – depends on n only, not on order
- Add line l_i one by one (i = 1 .. n)
  – Find the leftmost intersection with BBOX
  among $2(i-1)+4$ edges on the BBOX …O(i)
  – Trace the line through the arrangement $A(L_{i-1})$ and split the intersected faces …O(i) – why? See later
  – Update the subdivision (cell split) …O(1)
- Altogether $O(ni) = O(n^2)$

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A. Incremental construction of arrangement

Arrangement(L)

Input: Set of lines L in general position (no 3 intersect in 1 common point)

Output: Line arrangement $A(L)$ (resp. part of the arrangement stored in BBOX $B(L)$ containing all the vertices of $A(L)$)

1. Compute the BBOX $B(L)$ containing all the vertices of $A(L)$ …O(n^2)
2. Construct DCEL for the subdivision induced by BBOX $B(L)$ …O(1)
3. **for** $i = 1$ **to** n ***do*** ***// insert line*** l_i
4. find edge e, where line l_i intersects the BBOX of $2(i-1)+4$ edges …O(i)
5. f = bounded face incident to the edge e
6. **while** f is in $B(L)$ (bounded face f = f is in the BBOX) … O(???)
7. split f and set f to be the next intersected face across the intersected edge
8. update the DCEL (split the cell) …O(1)

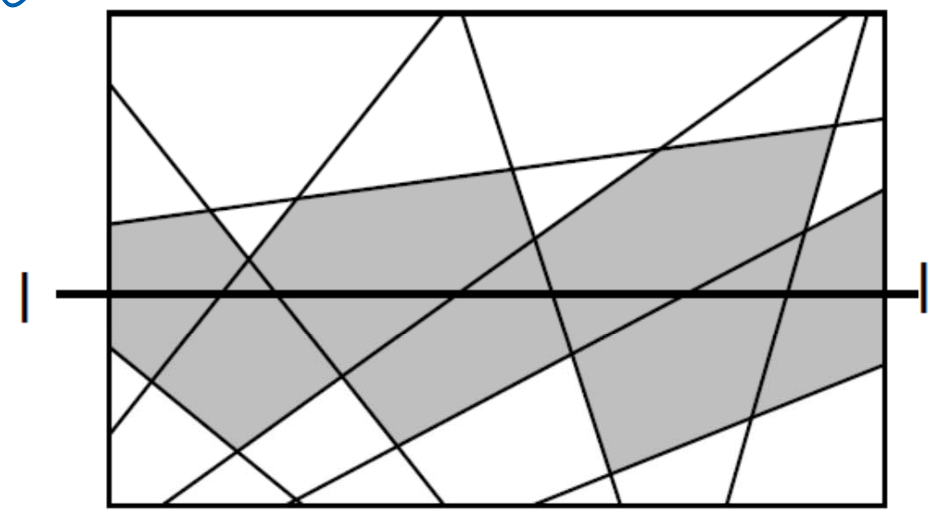
See later

DCGI

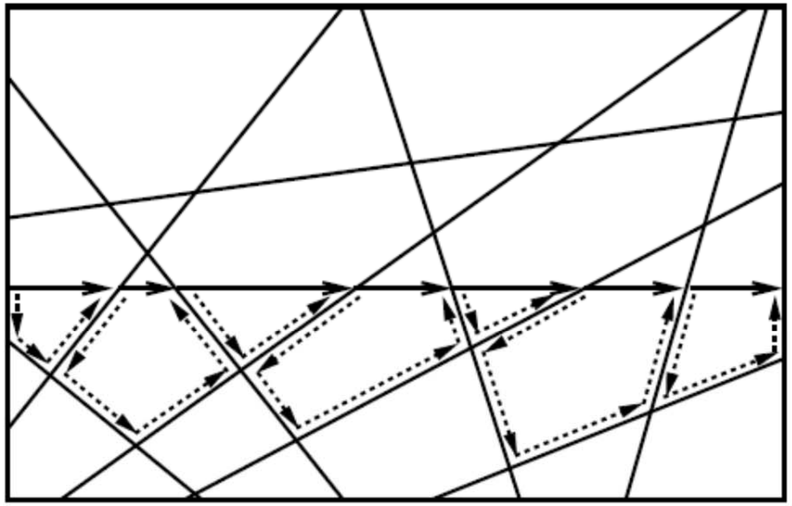
Tracing the line through the arrangement

- Walk around edges of current face (face walking)
- Determine if the line l_i intersects current edge e
- When intersection found, jump to the face on the other side of this edge e

n=8 lines, 7 faces in the zone, 22 edges tested of max 48



The zone of l



Walking the lower part of the zone

[Berg]

Tracing the line through the arrangement

- Number of traversed edges determines the insertion complexity
- Naïve estimation would be $O(i^2)$ traversed edges (i faces, i lines per face, i^2 edges)
- According to the Zone theorem, it is $O(i)$ edges only!

Zone theorem

= given an arrangement $A(L)$ of n lines in the plane and given any line l in the plane, the total number of edges in all the cells of the zone $Z_A(L)$ is at most $6n$.

For proof see [Mount, page 69]

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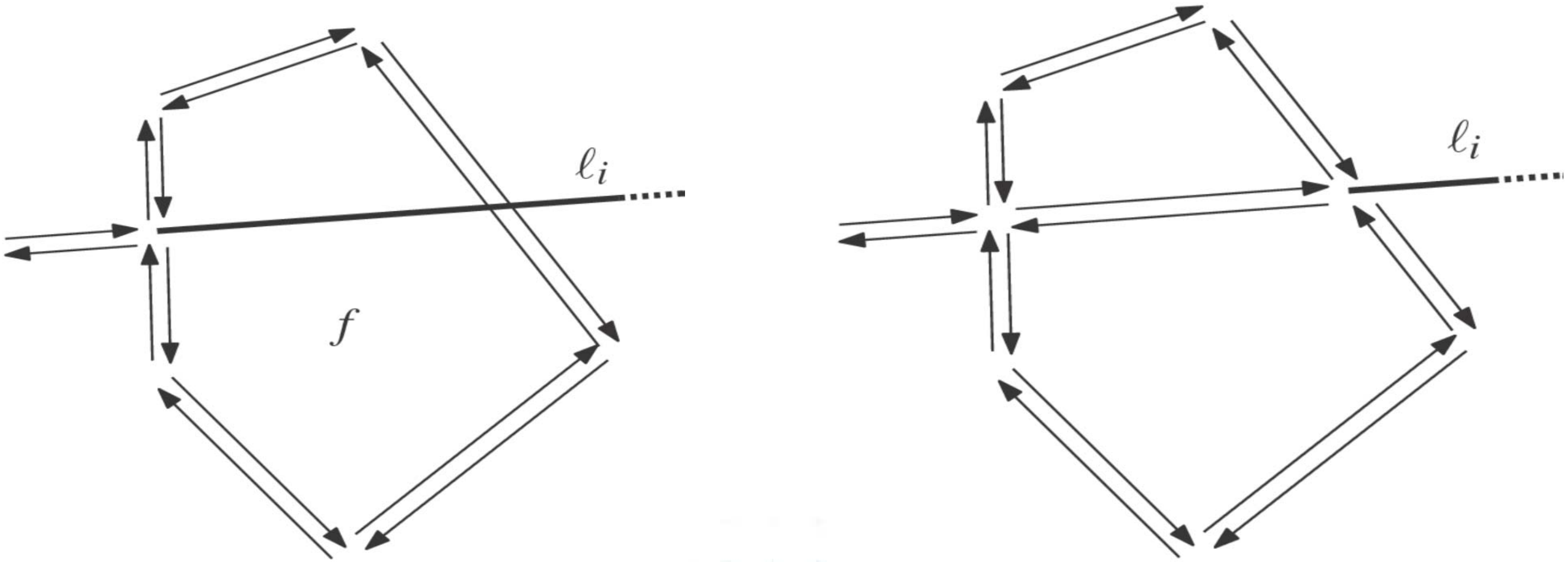
Cell split

- 2 new face records, 1 new vertex, 2+2 new half-edges + update pointers … O(1)



[Berg]

Complexity of incremental algorithm

- n insertions
- $O(i) = O(n)$ time for one line insertion
  (Zone theorem)

=> Complexity: $O(n^2) + n.O(i) = O(n^2)$

bbox edges walked

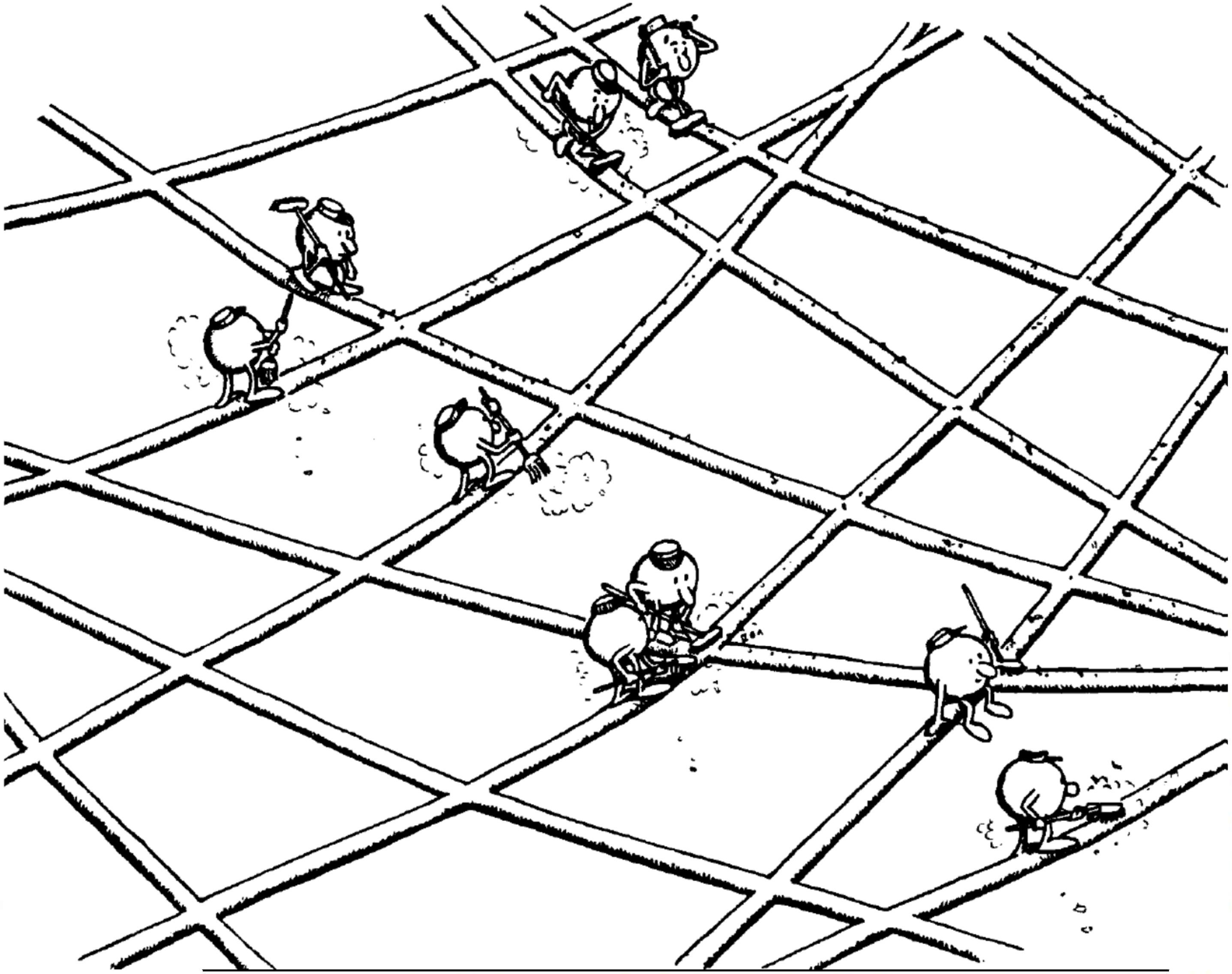
B. Topological plane sweep algorithm

- Complete arrangement needs $O(n^2)$ storage
- Often we need just to process each arrangement element just once – and we can throw it out then
- Classical Sweep line algorithm
  - needs $O(n)$ storage
  - needs log n for heap manipulation in $O(n^2)$ event points

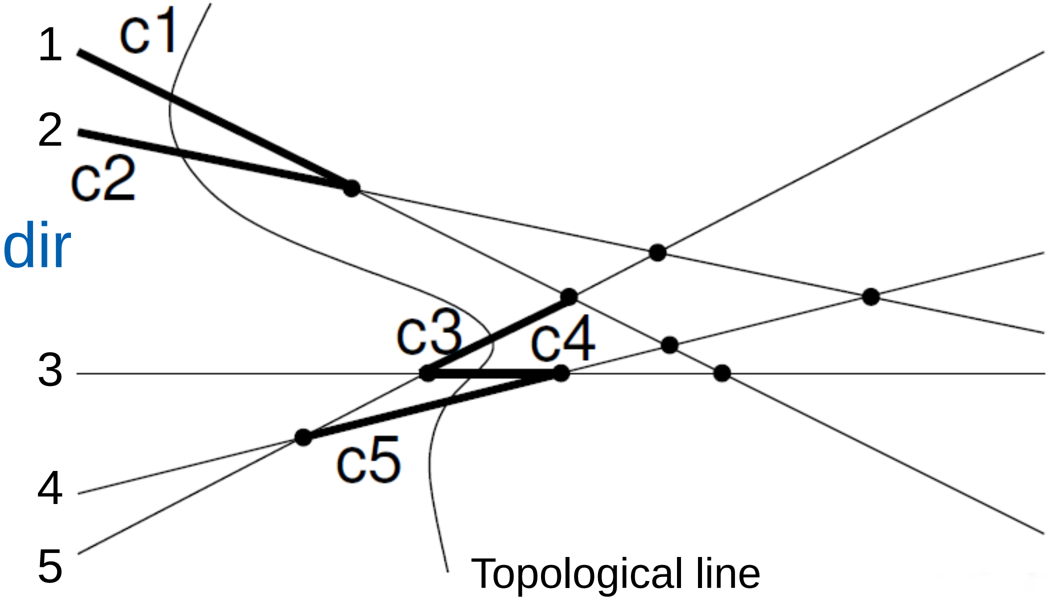
  => $O(n^2 \log n)$ algorithm
- Topological sweep line - TSL
  - disperses $O(\log n)$ factor in time
  - array of neighbors and a stack of ready vertices

  => $O(n^2)$ algorithm

Illustration from Edelsbrunner & Guibas



Topological line and cut



Topological line (curve)

(an intuitive notion)

- Monotonic curve in y-dir
- intersects each line exactly once (as a sweep line)

Cut in an arrangement A

- is an ordered sequence of edges $c_1, c_2, \ldots, c_n$ in A (one taken from each line), such that for $1 \leq i \leq n-1$, c_i and c_{i+1} are incident to the same face of A and c_i is above and c_{i+1} below the face
- Edges not necessarily connected (as c_2 and c_3)

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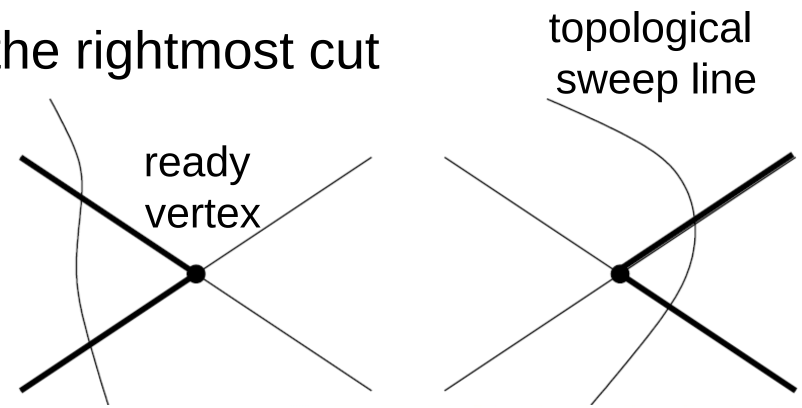
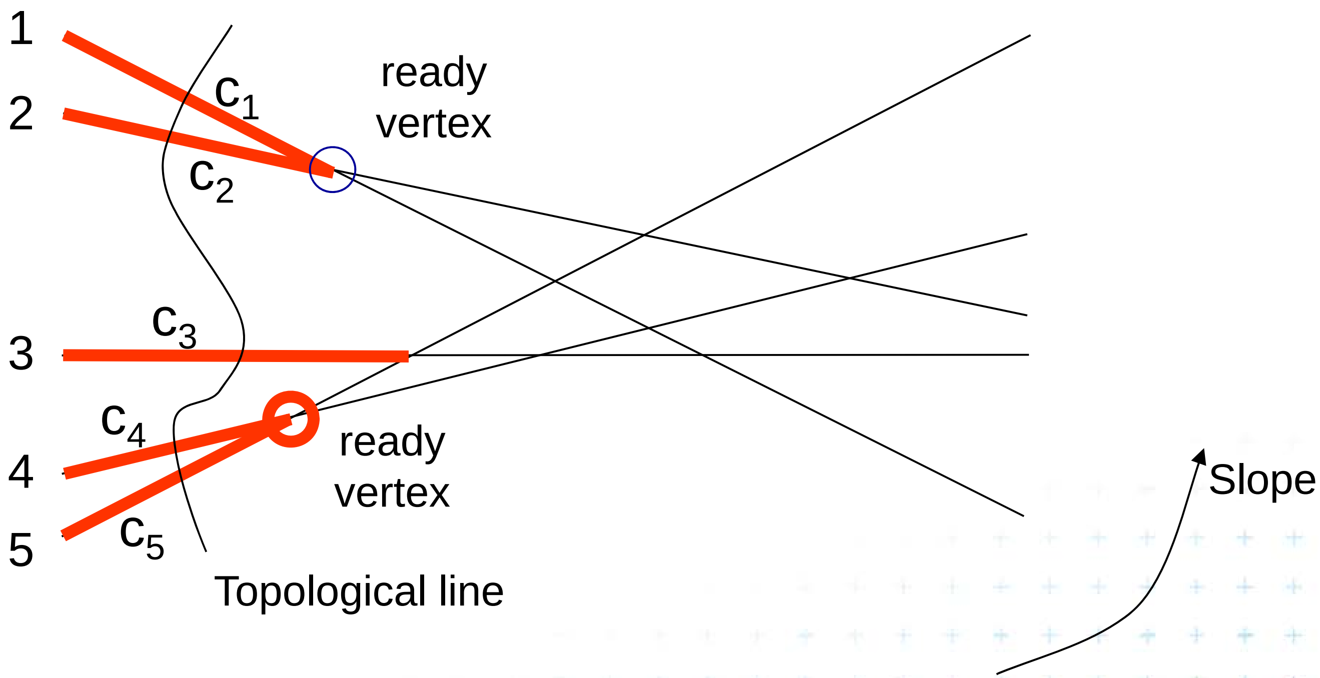
Topological plane sweep algorithm

- Starts at the leftmost cut
  - Consist of left-unbounded edges of *A* (ending at ——)
  - Computed in O(*n* log *n*) time – order of slopes
- The sweep line is
  - pushed from the leftmost cut to the rightmost cut
  - Advances in elementary steps



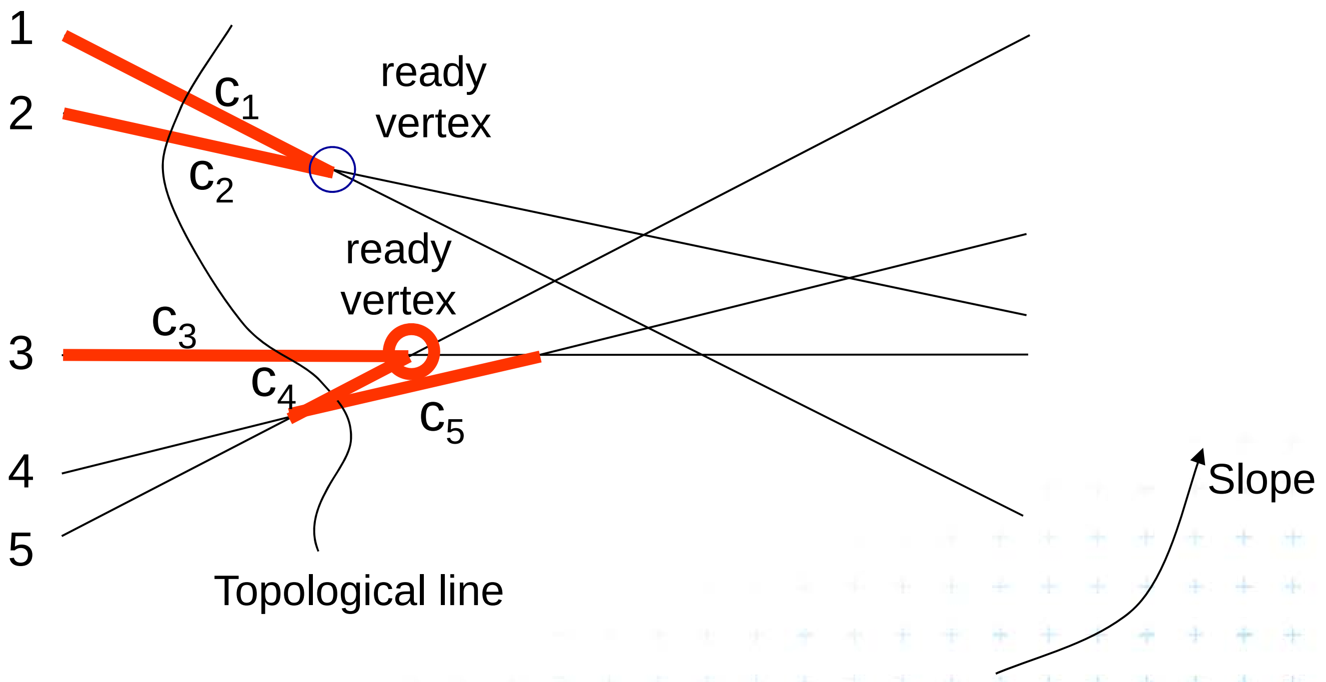
- Elementary step
  = Processing of any *ready vertex*
  (intersection of consecutive edges at their right-point)
  - Swaps the order of lines along the sweep line
  - Is always possible (e.g., the point with smallest *x*)
  - Searching of smallest x would need O(log *n*) time …

Step 0 – the leftmost cut

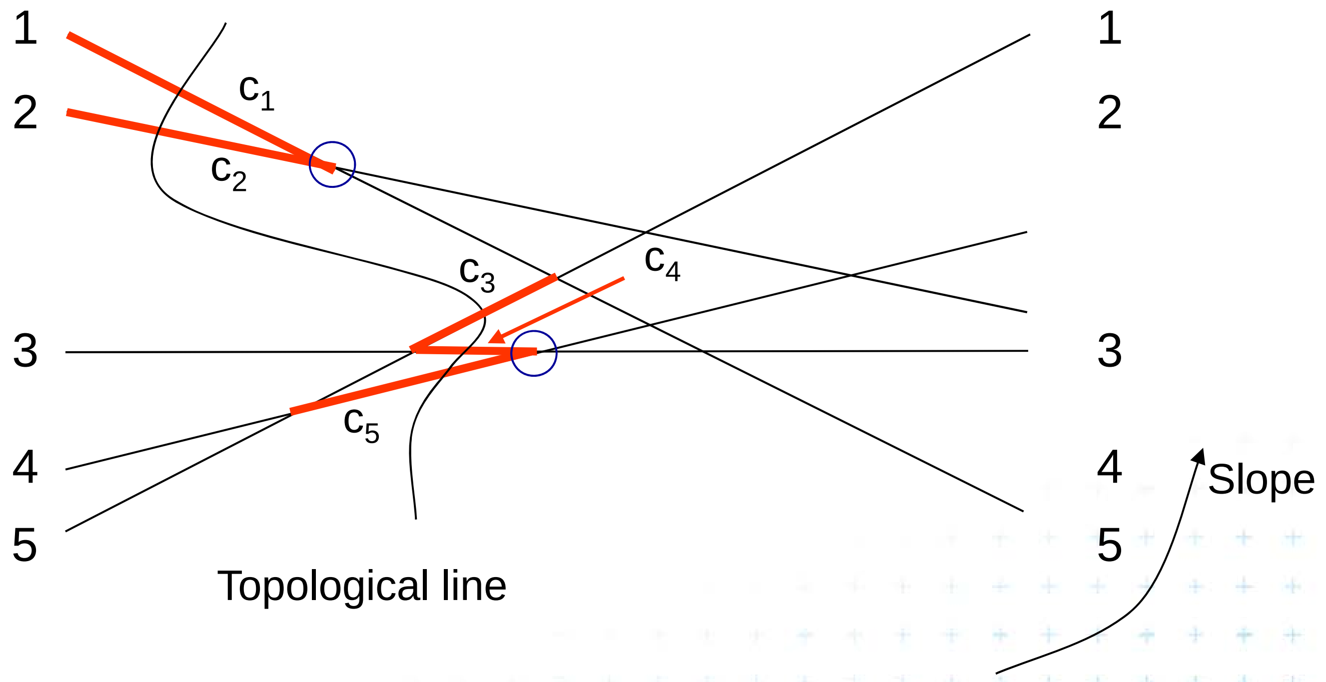


c_i = ordered sequence of edges along the topological sweep line

Step 1 – after processing of c4 x c5

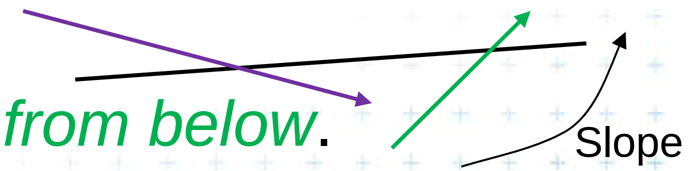


Step 2 – after processing of c3 x c4



How to determine the next right point?

- Elementary step (intersection at edges right-point)
  - Is always possible (e.g., the point with smallest *x*)
  - But searching the smallest x would need O(log *n*) time
  - We need O(1) time
- Right endpoint of the edge in the cut results from
  - UHT a line of *smaller slope* intersecting it *from above* (traced from L to R) or
  - LHT line of *larger slope* intersecting it *from below*.



- Use Upper and Lower Horizon Trees (UHT, LHT)
  - Common segments of UHT and LHT belong to the cut
  - Intersect the trees, find pairs of consecutive edges
  - use the right points as legal steps (push to stack)

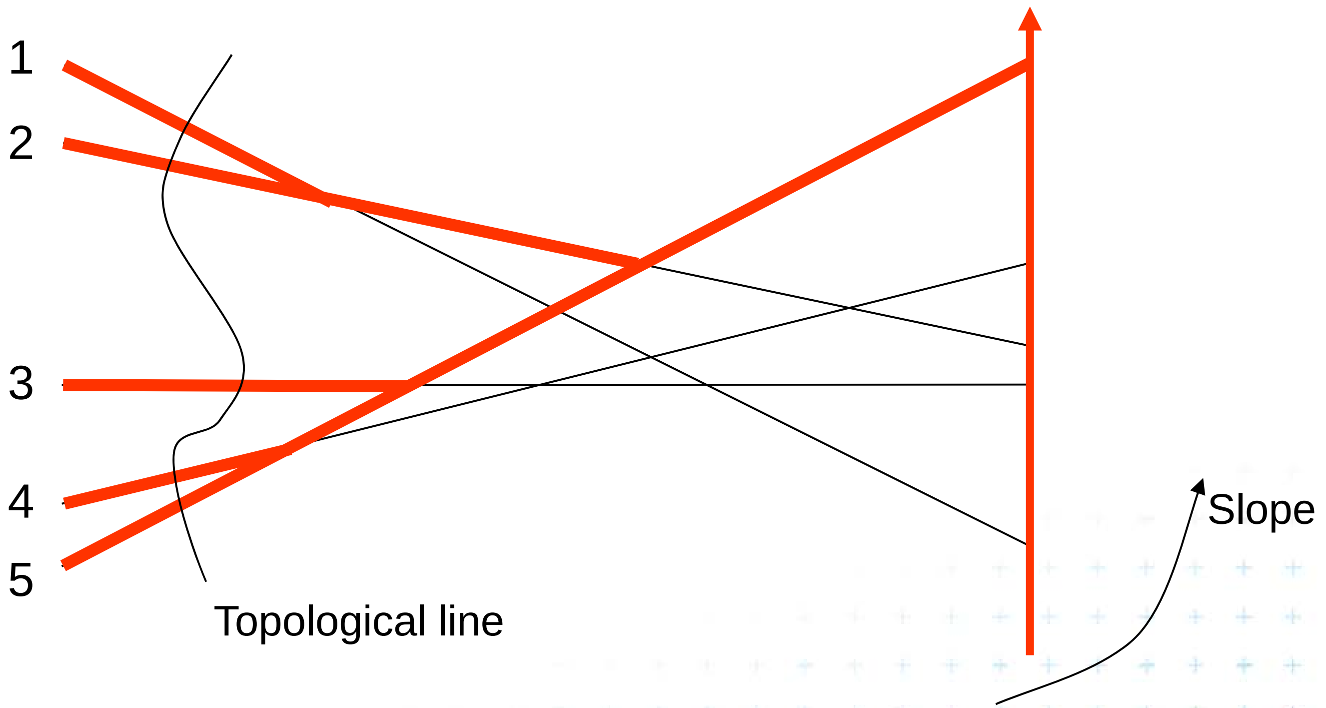
Upper and lower horizon tree

- Upper horizon tree (UHT)
  - Insert lines in order of decreasing slope (cw)
  - When two edges meet, keep the edge with higher slope and trim the inserted edge (with lower slope)
  - To get one tree and not the forest of trees (if not connected) add a vertical line in +∞ (slope +90°)
  - Left endpoints of the edges in the cut do not belong to the tree

- Lower horizon tree (LHT) construction is symmetrical
- UHT and LHT serve for right endpts determination

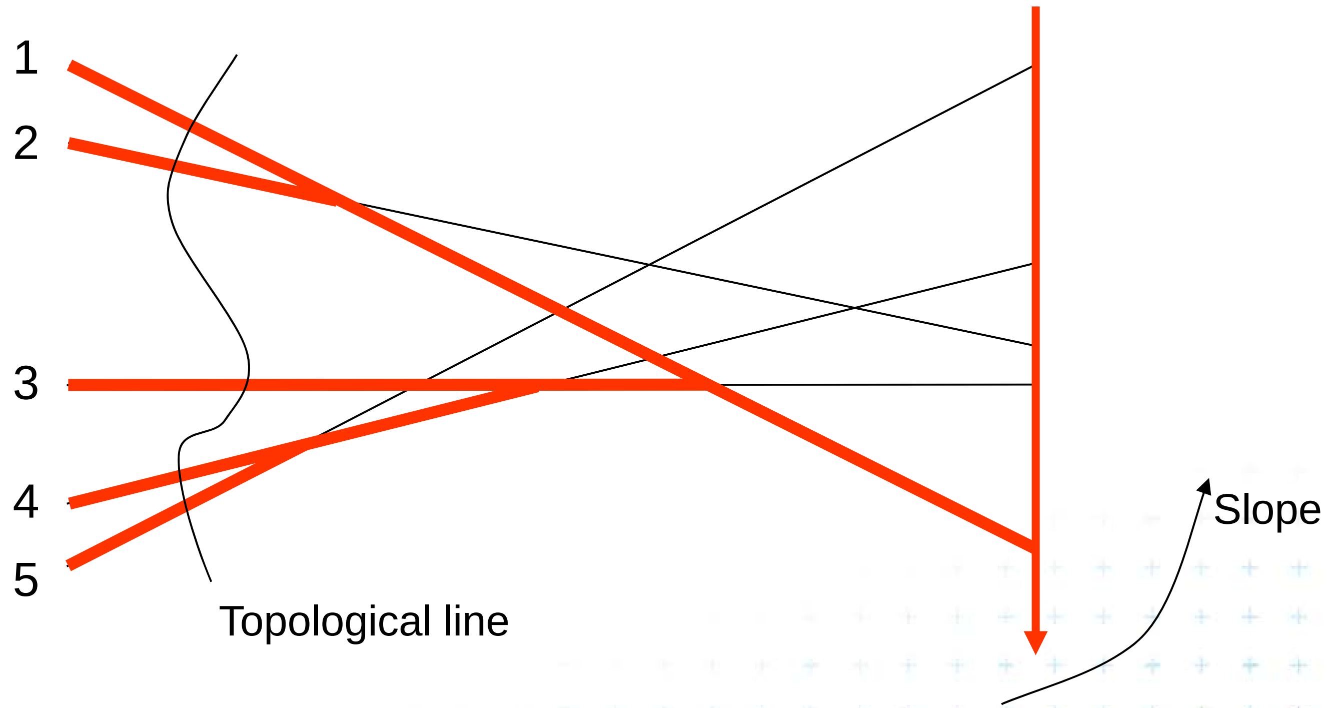
Upper horizon tree (UHT) – initial tree

- Insert lines in order of decreasing slope (“cw”)

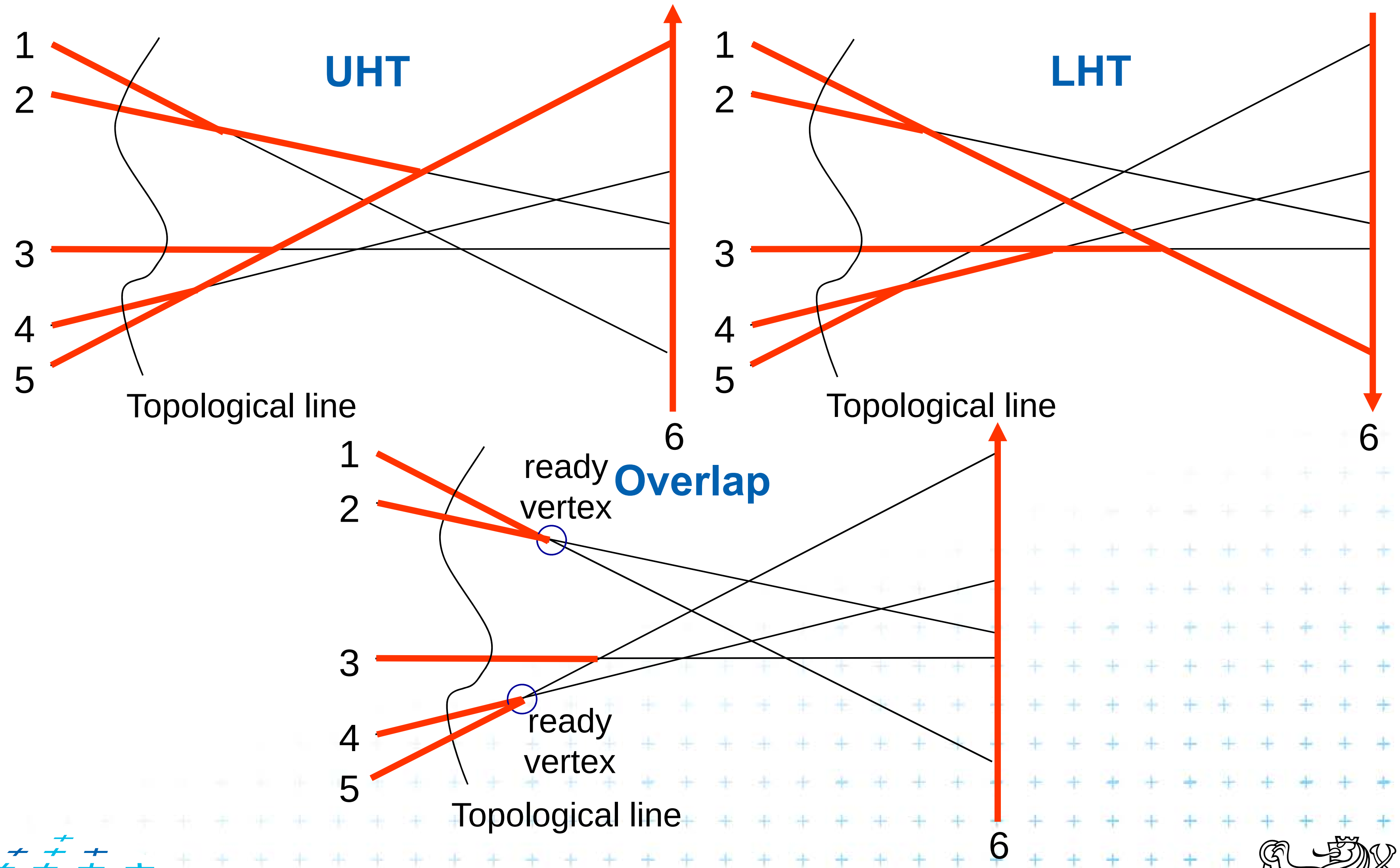


Lower horizon tree (LHT) – initial tree

- Insert lines in order of increasing slope ("ccw")



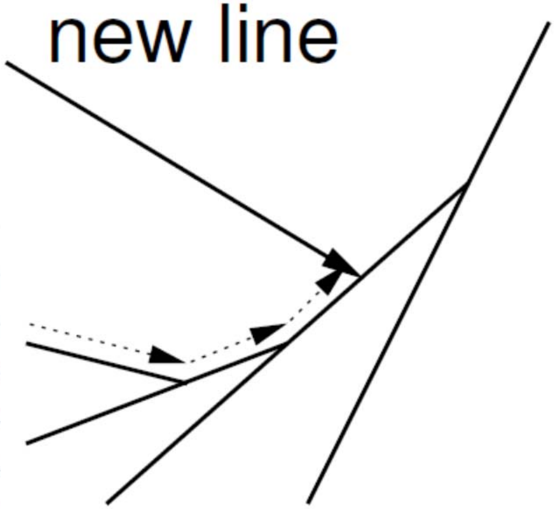
Overlap UHT and LHT – detect ready vertices



Upper horizon tree (UHT) – init. construction

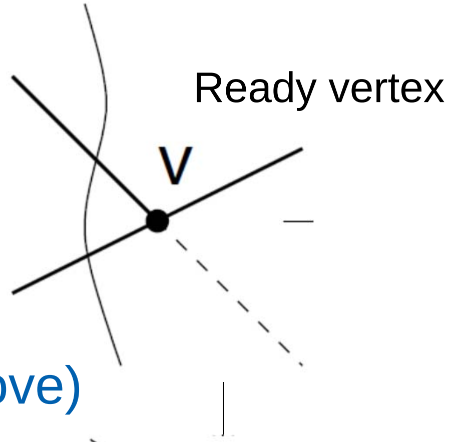
- Insert lines in order of decreasing slope (cw)
- Each new line starts above all the current lines
- The uppermost face = convex polygonal chain
- Walk left to right along the chain
  to determine the intersection
- Never walk twice over a segment
  - Such segment is no longer part of the upper chain
  - O(n) segments in UHT

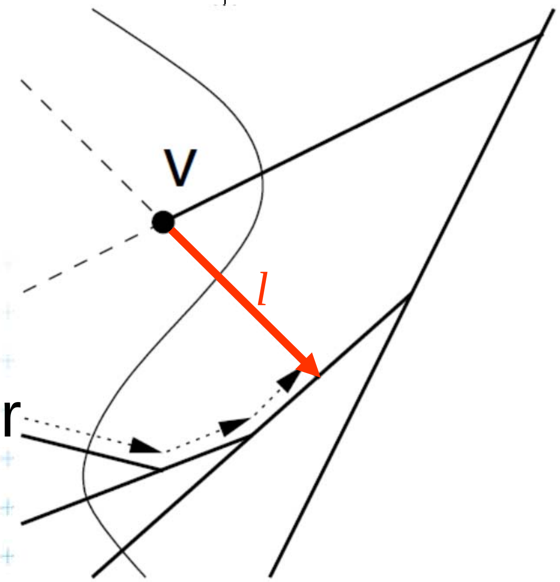
  => O(n) initial construction
  (after n log n sorting of the lines ~slope)



Upper horizon tree (UHT) – update

- After the elementary step
- Two edges swap position along the sweep line
- Lower edge *l* (lower slope, comes from above)
  - Reenter to UHT
  - Terminate at nearest edge of UHT
  - Start in edge below in the current cut
  - Traverse the face in CCW order
  - Intersection must exist, as *l* has lower slope than the other edge from *v* and both belong to the same face





Data structures for topological sweep alg.

Topological sweep line algorithm uses 5 arrays:

1) Line equation coefficients – E [1:n]
2) Upper horizon tree – UHT [1:n]
3) Lower horizon tree – LHT [1:n]
4) Order of lines cut by the sweep line – C [1:n]
5) Edges along the sweep line – N [1:n]
6) Stack for ready vertices (events) – S

(n number of lines)

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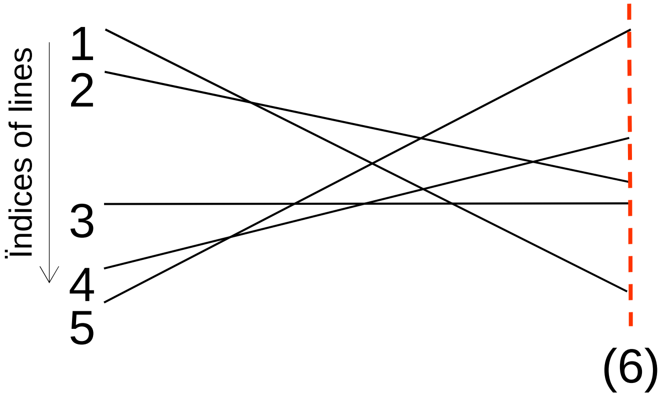
1) Line equation coefficients *E* [1:*n*]



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

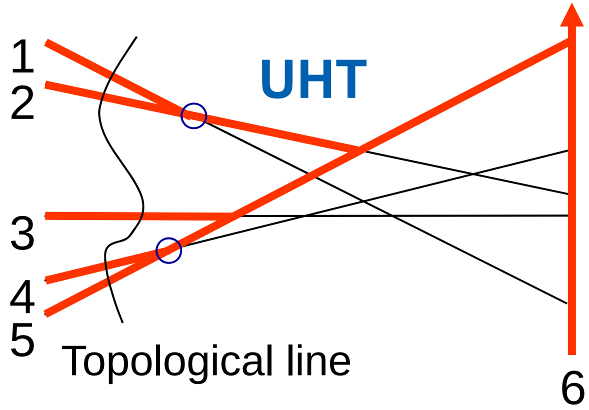
- Array of line equation coefs. *E*
  - Contains coefficients a_i and b_i of line equations $y = a_i x + b_i$
  - *E* is indexed by the line index
  - Lines are ordered according to their slope (angle from -90° to 90°)

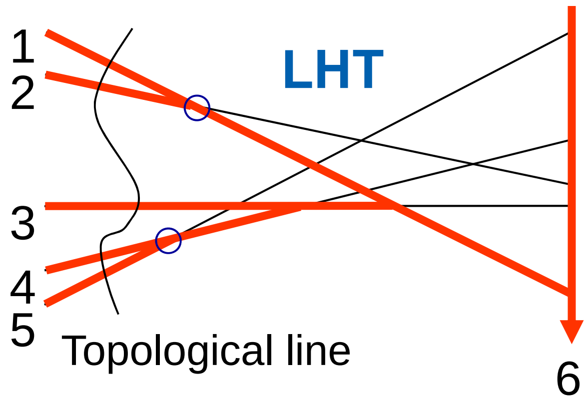
Slope

2) and 3) – Horizon trees UHT and LHT

Their intersection is used for searching of legal steps (right points)

- the shorter edge wins





UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

- Store pairs of line indices in E that delimit segment l_i to the left and to the right
- Segments are half open
- Unlimited line has "indices"
  (—, +—] (+—, —]
- One additional vertical line
  - prevents the tree from splitting into forest of trees
  - is inserted first and never trimmed
  - is (—, +—] for UHT
  - is (+—, —] for LHT

4) Order of lines cut by sweep line – C [1:*n*]

- The topological sweep line cuts each line once
- Order of the cuts (along the topological sweep line) is stored in array C as a sequence of line indices
- Array C "points" to the array E of line equations
- For the initial leftmost cut, the order is the same as in E
- Index *ci* addresses *i-th* line from top along the sweep line

CUT Lines C

Indexes of sup-porting lines

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

5) Edges along the sweep line – N [1:*n*]

- Edges intersected by the topological sweep line are stored here (edges along the sweep line)
- Instead of endpoints themselves, we store the indices of lines whose intersections delimit the edge
- Order of these edges is the same as in C (both use the index *ci*)
- Index *ci* stores the index of *i-th* edge from top along the sweep line

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

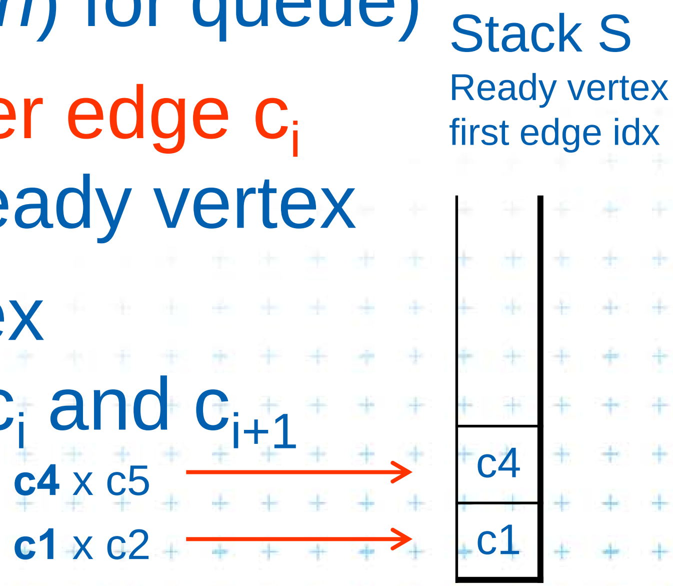
The first edge along the sweep line:

- lies on line C[c1]
- Comes from infinity
- is delimited by edge E[2]

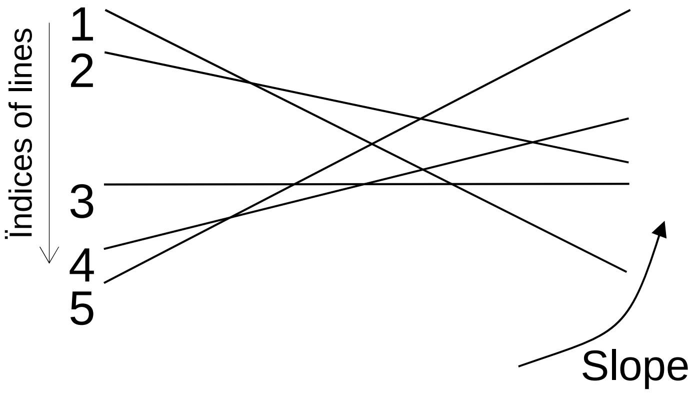
DCGI

6) Stack S

- The exact order of events is not important
  (event = intersection in ready vertex)
- Alg. can process any of the “ready vertex”
- Event queue is therefore replaced by a stack
  (faster: O(1) instead of O(log *n*) for queue)
- The stack stores just the upper edge c_i
  from the pair intersecting in ready vertex
- Intersection in the ready vertex
  is computed between stored c_i and c_{i+1}



Topological sweep line demo



Array of line equations E

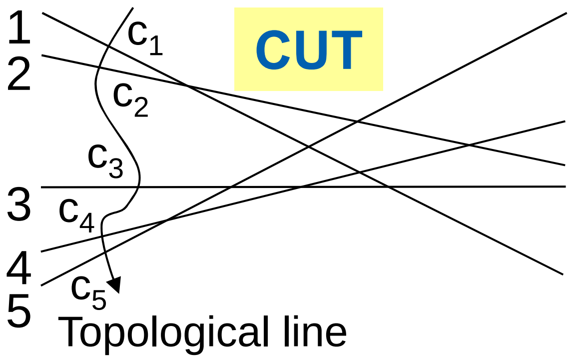
$y = a_i x + b$

Ïndices of lines		
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

Input

- set of lines *L* in the plane
- ordered in increasing slope ($\angle$ -90° to 90°), simple, not vertical
- line parameters in array *E*

1) Initial leftmost cut - C



- Store the indices of lines in E into the Cut lines array C in increasing slope order

Array of line equations E

$y = a_i x + b$

Indices of lines

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

CUT Lines C

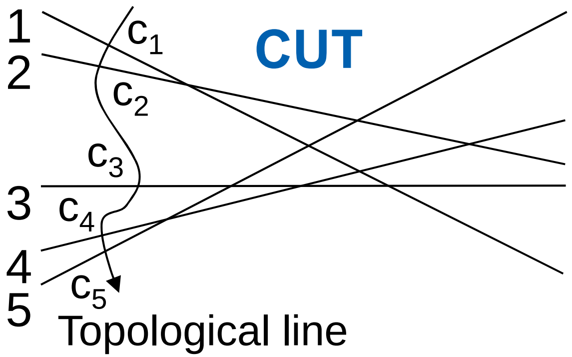
Indexes of supporting lines

Line indices along the cut

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

1) Initial leftmost cut - N



- Prepare array *N* for endpoints of the cutted edges (resp. for line indices delimiting these edges)
- Init it by line "ends" $-\infty, +\infty$

Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

indices of lines

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	–
c2	$-\infty$	–
c3	$-\infty$	–
c4	$-\infty$	–
c5	$-\infty$	–

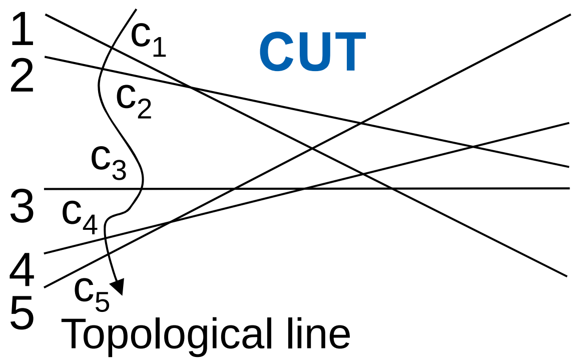
CUT Lines C

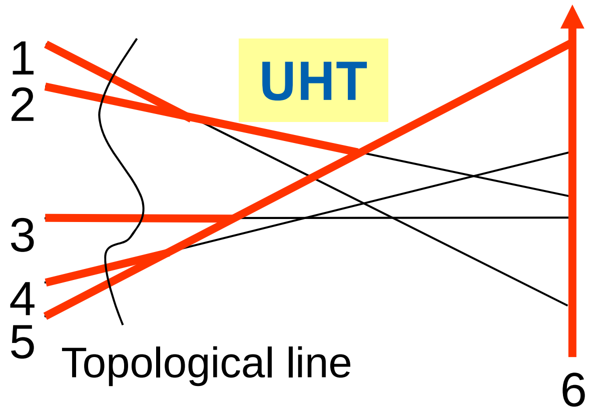
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

2a) Compute Upper Horizon Tree - UHT





Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+–

Order of insertion into UHT

Inserted first, never changed

CUT edges N

Pairs of line indices delimiting the edge

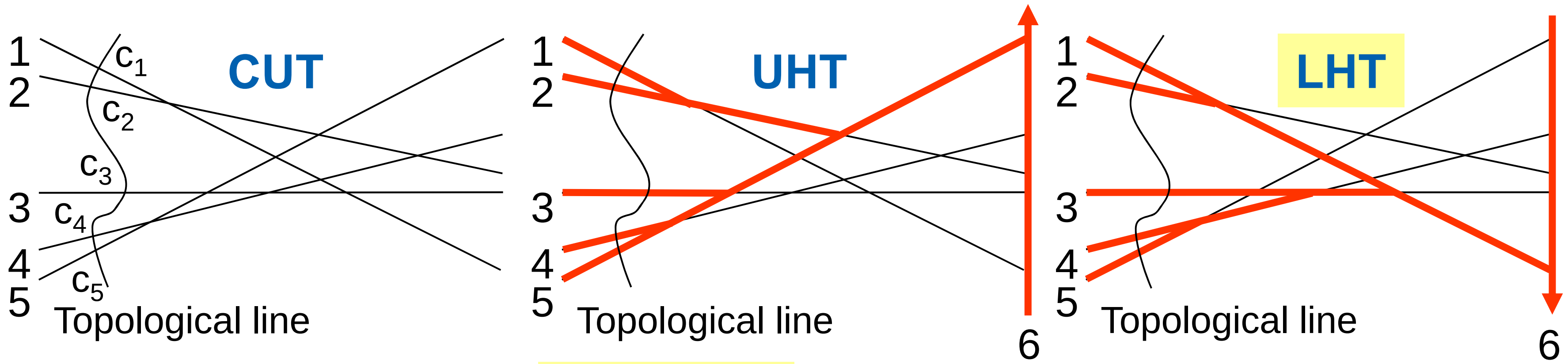
c1	$-\infty$	–
c2	$-\infty$	–
c3	$-\infty$	–
c4	$-\infty$	–
c5	$-\infty$	–

CUT Lines C

Indexes of sup-porting lines

c1	1
c2	2
c3	3
c4	4
c5	5

2b) Compute Lower Horizon Tree - LHT



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+–

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	—

Order of insertion into LHT

Inserted first, never changed

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	–
c2	$-\infty$	–
c3	$-\infty$	–
c4	$-\infty$	–
c5	$-\infty$	–

CUT Lines C

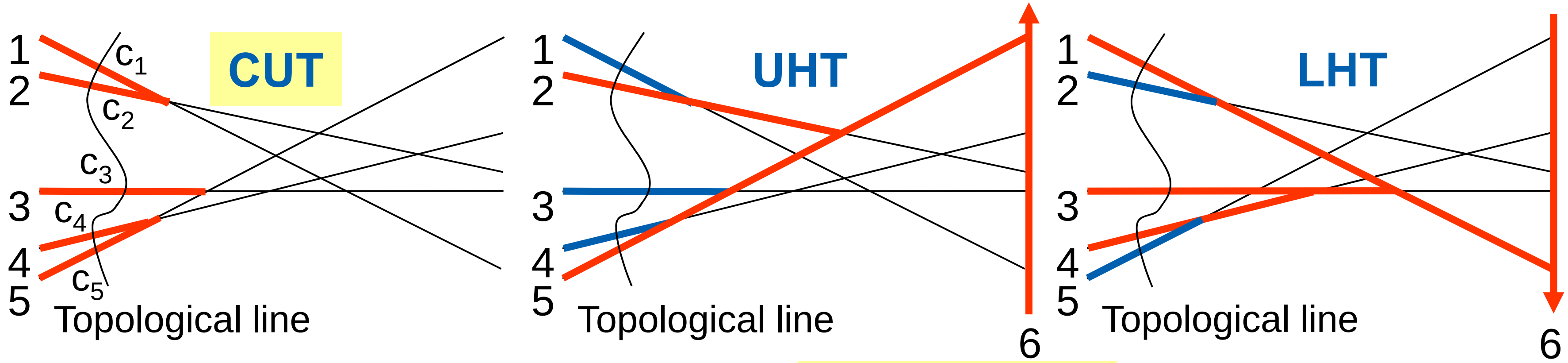
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

Stack S

Ready vertex first edge idx

3a) Determine right delimiters of edges - N



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+–

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	—

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

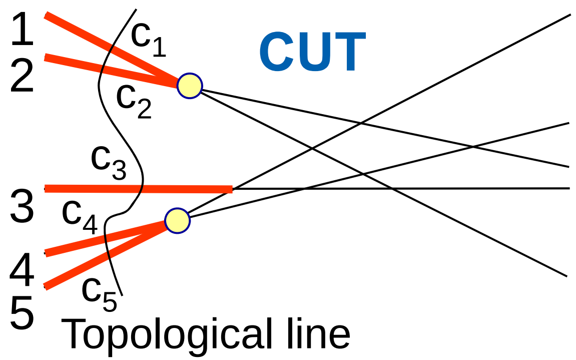
Stack S

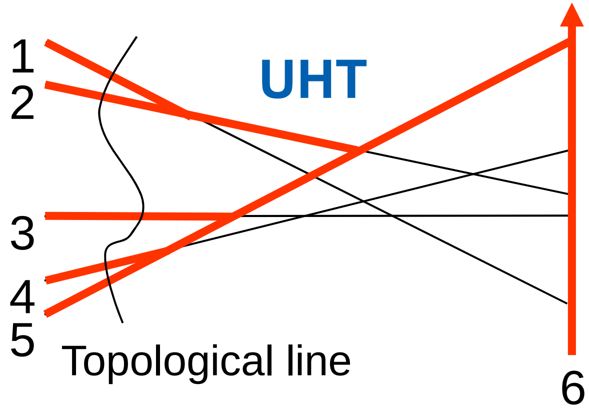
Ready vertex first edge idx

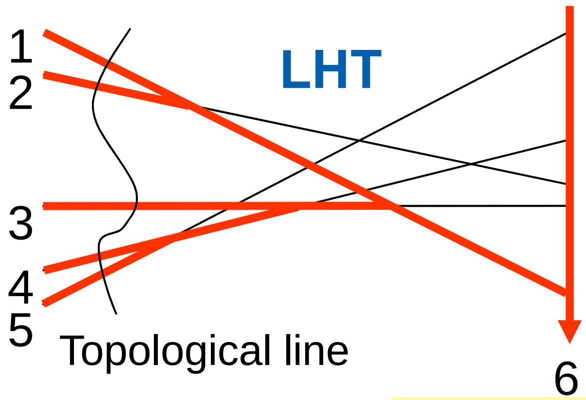
Intersect the trees – take the shorter edge

DCGI

3b) Ready vertices = inters. of neighbors – S







Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+—

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	—

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

CUT Lines C
Indexes of supporting lines

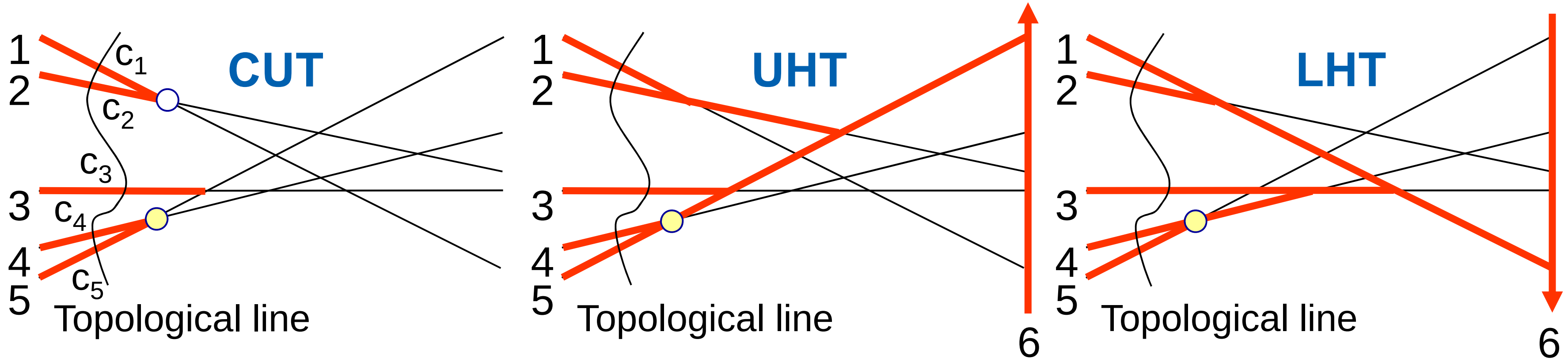
c1	1
c2	2
c3	3
c4	4
c5	5

Stack S
Ready vertex first edge idx

c4
c1

Compute intersections of neighbors – push into stack

4a) Pop ready vertex from S – process c4



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+–

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	—

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	5
c5	$-\infty$	4

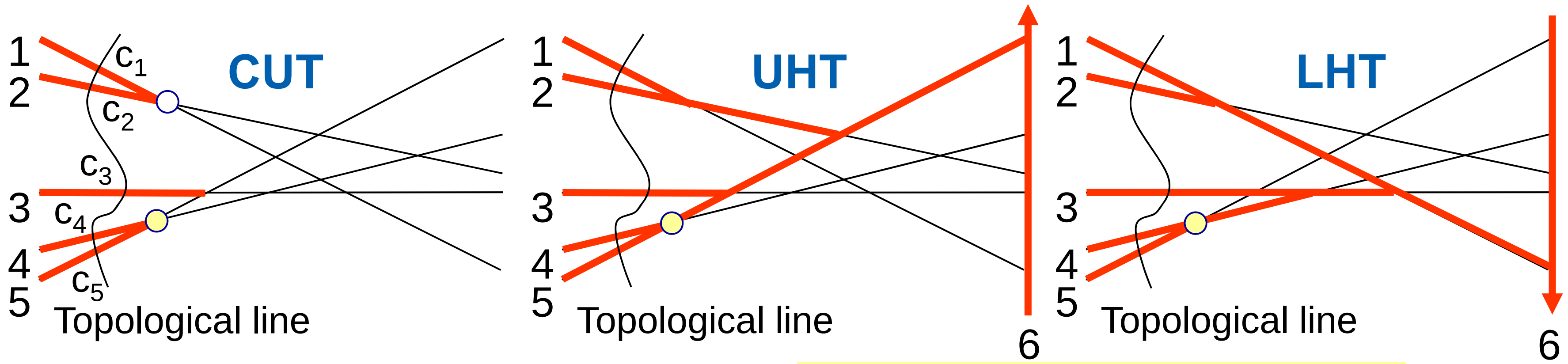
CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

Stack S
Ready vertex first edge idx

c4
c1

4b) Swap lines c4 and c5 – swap 4 and 5



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	+–

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	—

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	4
c5	$-\infty$	5

CUT Lines C
Indexes of sup-porting lines

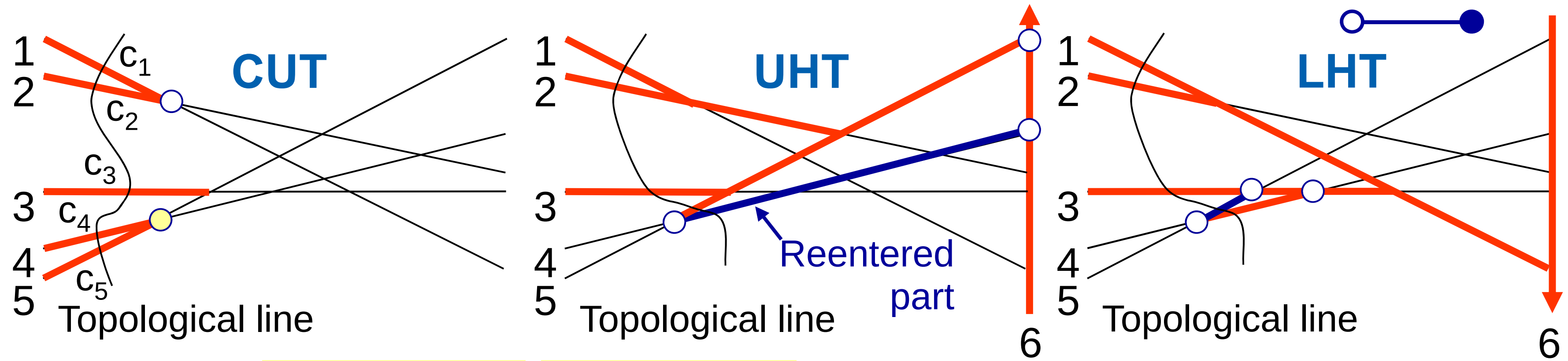
c1	1
c2	2
c3	3
c4	**5**
c5	**4**

Stack S
Ready vertex first edge idx

c1

DCGI

4c) Update the horizon trees – UHT and LHT



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	—	2
2	—	5
3	—	5
4	**5**	**6**
5	**4**	**6**
6	—	+—

LHT array

Delimiting lines indices

1	—	6
2	—	1
3	—	1
4	**5**	3
5	**4**	**3**
6	+—	—

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	4
c5	$-\infty$	5

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

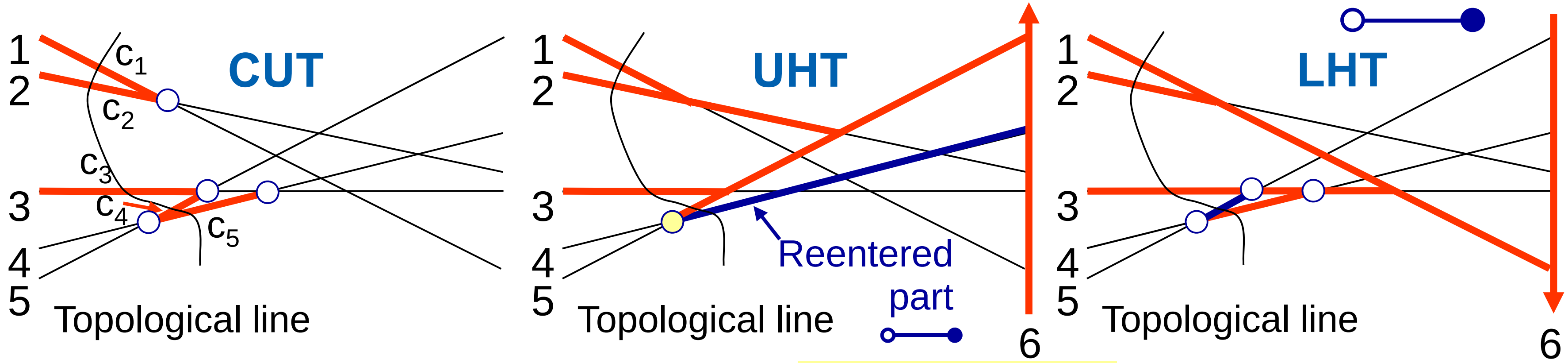
Stack S

Ready vertex upper edge id›

c1

Note: Edges are half open to prevent the tree after reinsertion

4d) Determine new cut edges endpoints – N



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	—	2
2	—	5
3	—	5
4	**5**	**6**
5	**4**	**6**
6	—	+—

LHT array

Delimiting lines indices

1	—	6
2	—	1
3	—	1
4	**5**	**3**
5	**4**	**3**
6	+—	—

CUT edges N

Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	—	5
c4	**4**	**3**
c5	**5**	**3**

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

Stack S

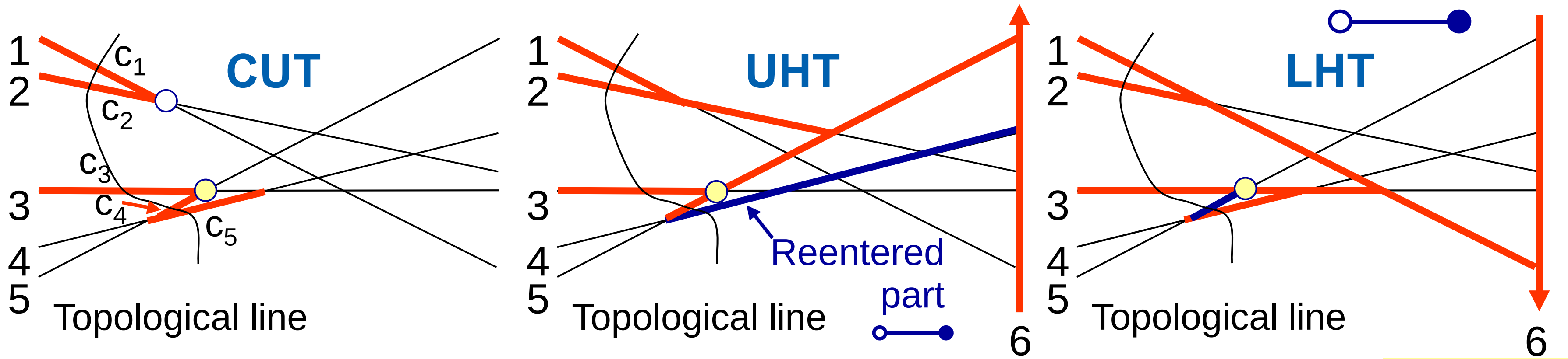
Ready vertex upper edge idx

c1

Intersect the trees – take the shorter edge

DCGI

4e) Intersect with neighbors – push into S



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	—	2
2	—	5
3	—	5
4	**5**	**6**
5	**4**	**6**
6	—	+—

LHT array
Delimiting lines indices

1	—	6
2	—	1
3	—	1
4	**5**	3
5	**4**	**3**
6	+—	—

CUT edges N
Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	—	5
c4	**4**	**3**
c5	**5**	**3**

CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

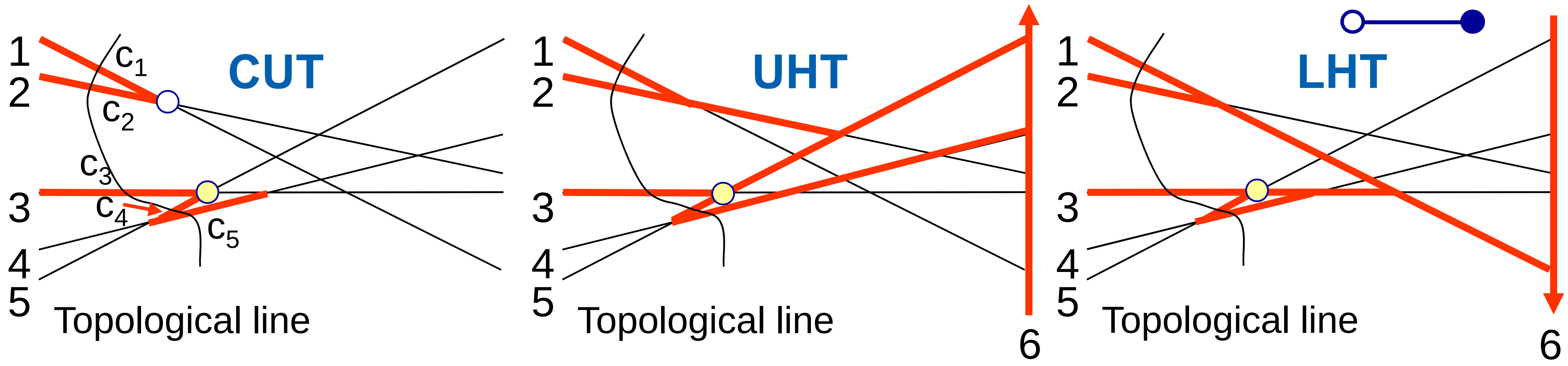
Stack S
Ready vertex upper edge idx

c3
c1

Intersections of neighbors - into stack

DCGI

4a) Pop ready vertex from S – process c3



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	—	2
2	—	5
3	—	5
4	5	6
5	4	6
6	—	+—

LHT array

Delimiting lines indices

1	—	6
2	—	1
3	—	1
4	5	3
5	4	3
6	+—	—

CUT edges N

Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	—	5
c4	4	3
c5	5	3

CUT Lines C

Indexes of supporting lines

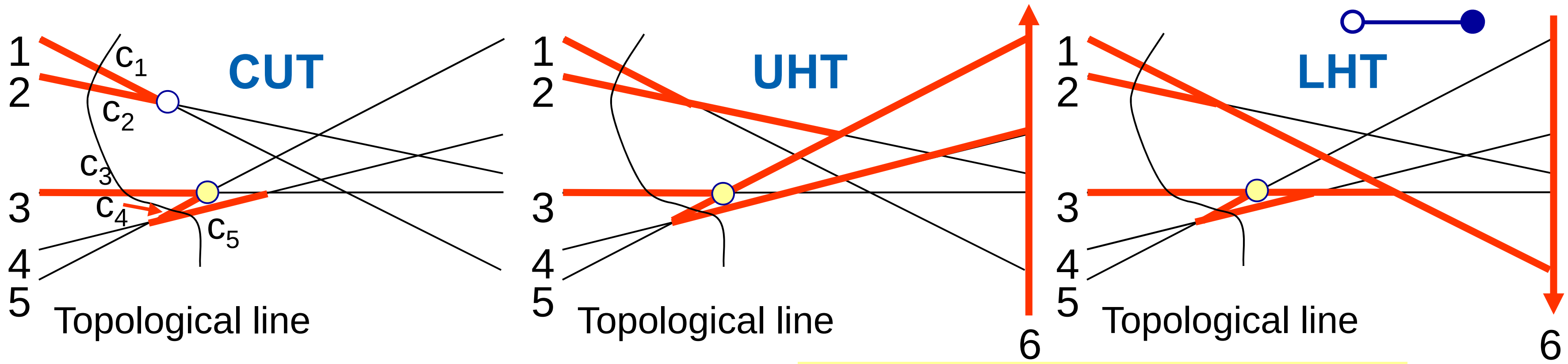
c1	1
c2	2
c3	**3**
c4	**5**
c5	4

Stack S

Ready vertex first edge idx

c3
c1

4b) Swap lines c4 and c5 – swap 4 and 5



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	—	2
2	—	5
3	—	5
4	5	6
5	4	6
6	—	+—

LHT array
Delimiting lines indices

1	—	6
2	—	1
3	—	1
4	5	3
5	4	3
6	+—	—

CUT edges N
Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	4	3
c4	—	5
c5	5	3

CUT Lines C
Indexes of supporting lines

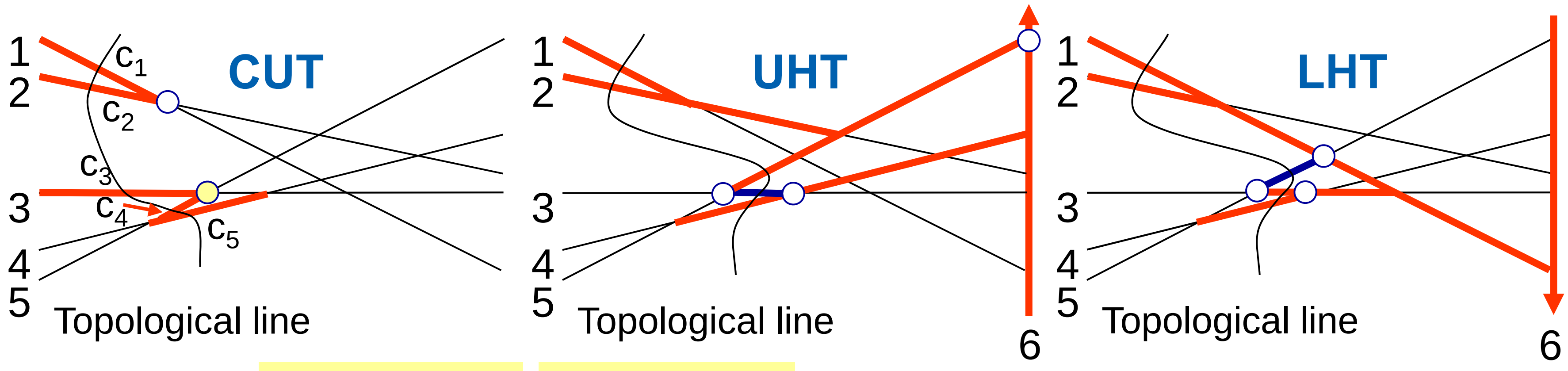
c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c3
c1

DCGI

4c) Update the horizon trees – UHT and LHT



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	—	2
2	—	5
3	**5**	**4**
4	5	6
5	**3**	6
6	—	+—

LHT array
Delimiting lines indices

1	—	6
2	—	1
3	**5**	1
4	5	3
5	**3**	**1**
6	+—	—

CUT edges N
Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	4	3
c4	—	5
c5	5	3

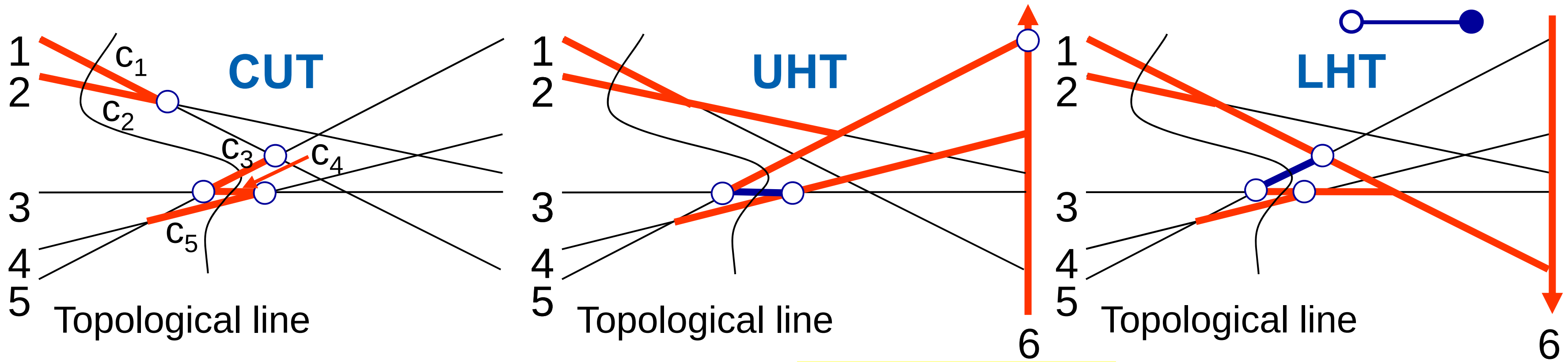
CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c3
c1

4d) Determine new cut edges endpoints



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	—	2
2	—	5
3	**5**	**4**
4	5	6
5	**3**	6
6	—	+—

LHT array
Delimiting lines indices

1	—	6
2	—	1
3	**5**	1
4	5	3
5	**3**	**1**
6	+—	—

CUT edges N
Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	3	1
c4	5	4
c5	5	3

CUT Lines C
Indexes of supporting lines

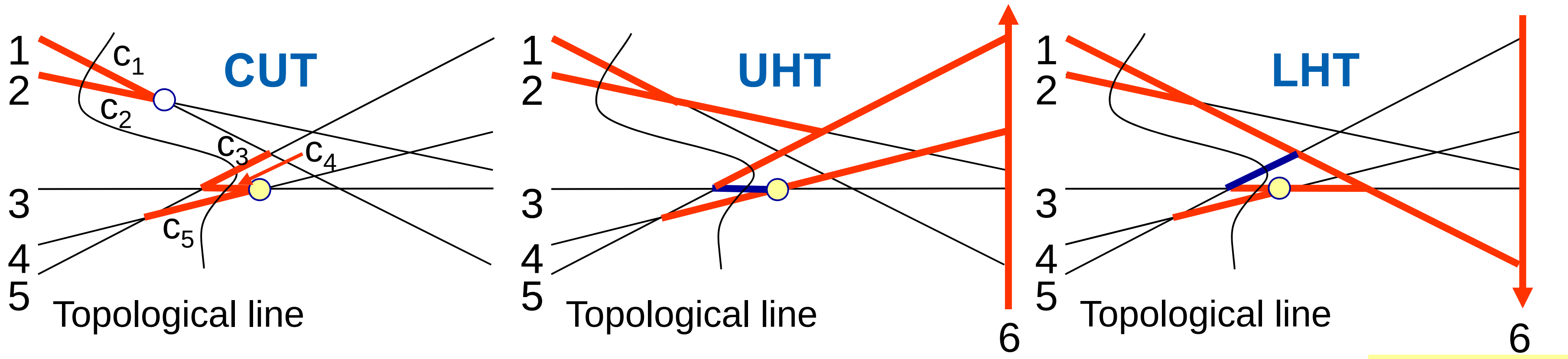
c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c3
c1

Intersect the trees – take the shorter edge

4e) Intersect with neighbors – push into S



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	—	2
2	—	5
3	**5**	**4**
4	5	6
5	**3**	6
6	—	+—

LHT array
Delimiting lines indices

1	—	6
2	—	1
3	**5**	1
4	5	3
5	**3**	**1**
6	+—	—

CUT edges N
Pairs of line indices delimiting the edge

c1	—	2
c2	—	1
c3	3	1
c4	5	4
c5	5	3

CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c4
c1

Topological sweep algorithm

Slope

TopoSweep(*L*)

Input: Set of **lines *L* sorted by slope (-90° to 90°)**, simple, not vertical

Output: All parts of an **Arrangement *A*(*L*)** detected and then destroyed

1. Let *C* be the initial (leftmost) cut – lines in increasing order of slope
2. Create the initial UHT and LHT incrementally:
   a) UHT by inserting lines in decreasing order of slope
   b) LHT by inserting lines in increasing order of slope
3. By consulting UHT and LHT
   a) Determine the right endpoints N of all edges of the initial cut C
   b) Store neighboring lines with common endpoints into stack *S* *(ready vertices)*
4. Repeat until stack not empty
   a) Pop next ready vertex from stack *S* (its upper edge c_i)
   b) Swap these lines within the cut *C* (c_i <-> c_{i+1})
   c) Update the horizon trees UHT and LHT (reenter edge parts)
   d) Consulting UHT and LHT determine new cut edges endpoints N
   e) If new neighboring edges share an endpoint -> push them on *S*

DCGI

Determining cut edges from UHT and LHT

- for lines *i* = 1 to n
  - Compare UHT and LHT edges on line *i*
  - Set the cut lying on edge *i* to the shorter edge of these
- Order of the cuts along the sweep line
  - Order changes at the intersection *v* only (neighbors)
  - Order of remaining cuts not incident with intersection *v* does not change
- After changes of the order, test the new neighbors for intersections
  - Store intersections right from sweep line into the stack

Complexity

- $O(n^2)$ intersections
  => $O(n^2)$ events (elementary steps)
- $O(1)$ amortized time for one step – 4c)
  => $O(n^2)$ time for the algorithm

Amortized time

= even though a single elementary step can take more than $O(1)$ time, the total time needed to perform $O(n^2)$ elementary steps is $O(n^2)$, hence the average time for each step is $O(1)$.

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DCGI