DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

CONVEX HULL IN 3 DIMENSIONS

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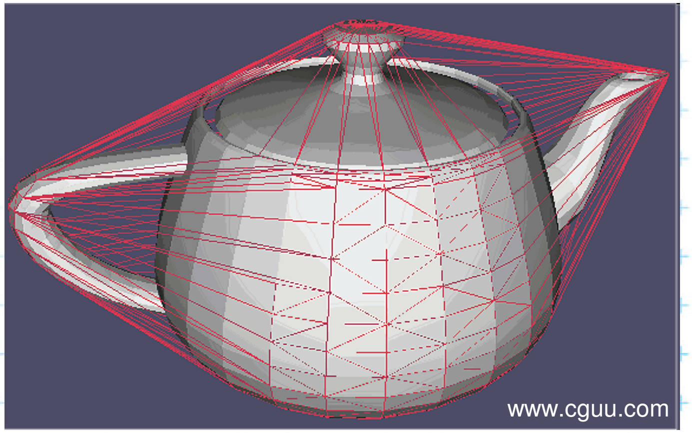
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Preparata], [Rourke] and [Boissonnat]

Version from 23.10.2014

Talk overview

- Lower bounds for convex hull in 2D and 3D
- Other criteria for CH algorithm classification
- Recapitulation of CH algorithms
- Terminology refresh
- Convex hull in 3D
  - Terminology
  - Algorithms
    - Gift wrapping
    - D&C Merge
    - Randomized Incremental



Lower bounds for Convex hull

- $O(n \log n)$ in E^2, E^3
  – output insensitive
- $O(n\ h)$, $O(n \log h)$, h is number of CH facets
  – output sensitive algs.
- $O(n)$ for sorted points and for polygon
- $O(\log n)$ for new point insertion in online algs.

Other criteria for CH algorithm classification

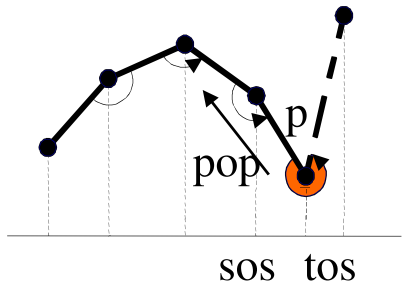
- Optimality – depends on data order (or distribution)

  In the worst case x In the expected case
- Output sensitivity – depends on the result ~ O(f(h))
- Extendable to higher dimensions?
- Off-line versus on-line
  - Off-line – all points available, preprocessing for search speedup
  - On-line – stream of points, new point p_i on demand, just one new point at a time, CH valid for $\{p_1, p_2, \ldots, p_i\}$
  - Real-time – points come as they “want”

    (not faster than optimal constant O(log n) inter-arrival delay)
- Parallelizable x serial
- Dynamic – points can be deleted
- Deterministic x approximate (lecture 13)

Graham scan

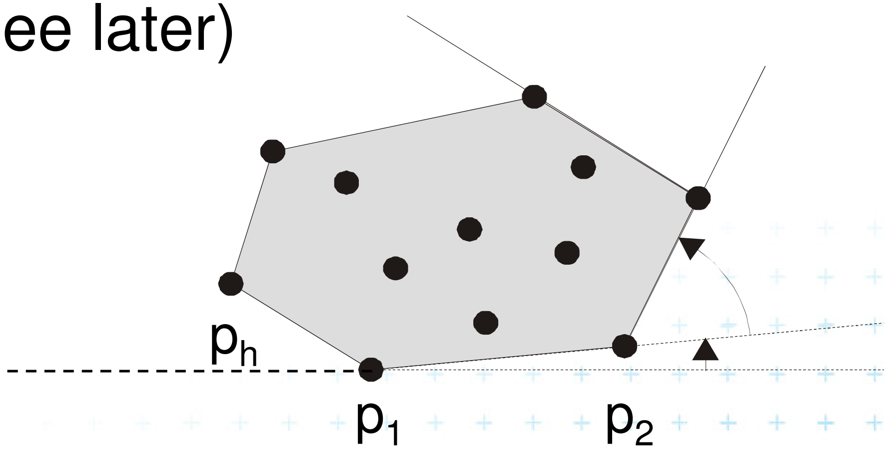
- O(n log n) time and O(n) space is
  - optimal in the worst case
  - not optimal in average case (not output sensitive)
  - only 2D
  - off-line
  - serial (not parallel)
  - not dynamic



O(*n*) for polygon (will be discussed in seminar [9])

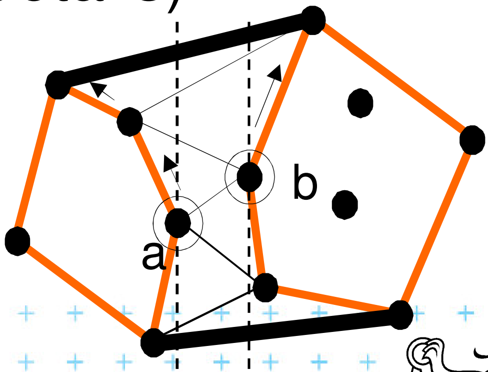
Jarvis March – Gift wrapping

- *O(hn)* time and O(*n*) space is
  - not optimal in worst case $O(n^2)$
  - may be optimal if *h* << *n* (output sensitive)
  - 3D or higher dimensions (see later)
  - off-line
  - serial (not parallel)
  - not dynamic



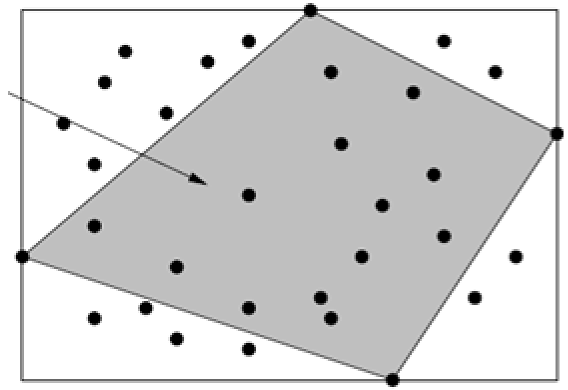
Divide & Conquer

- O(n log n) time and O(n) space is
  - optimal in worst case (in 2D or 3D)
  - not optimal in average case (not output sensitive)
  - 2D or 3D (circular ordering), in higher dims not optimal
  - off-line
  - Version with sorting (the presented one) – serial
  - Parallel for overlapping merged hulls
    (see Chapter 3.3.5 in Preparata for details)
  - not dynamic



Quick hull

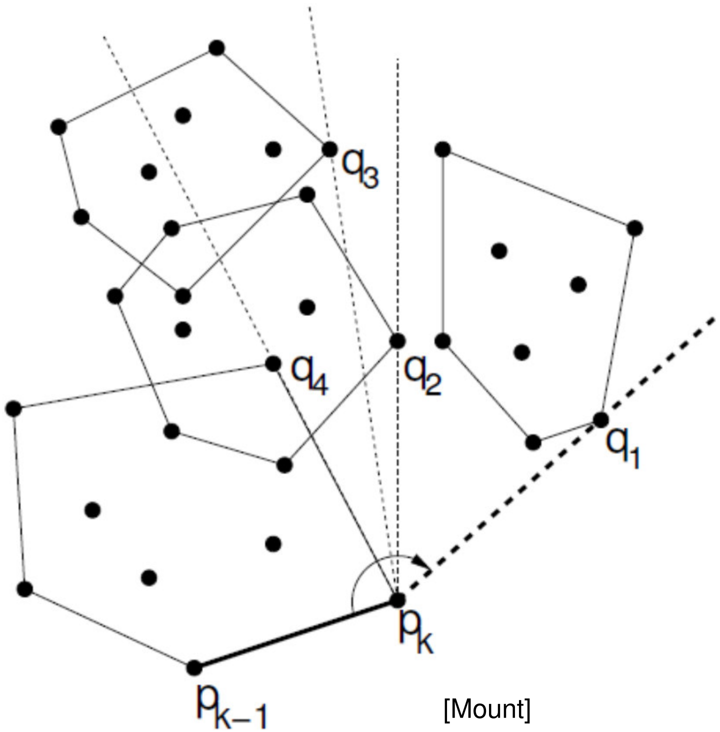
- O(n log n) expected time, O(n^2) the worst case and O(n) space *in 2D* is
  - not optimal in worst case O(n^2)
  - optimal if uniform distribution then $h << n$ (output sensitive)
  - 2D, or higher dimensions [see http://www.qhull.org/]
  - off-line
  - serial (not parallel)
  - not dynamic



[Mount]

Chan

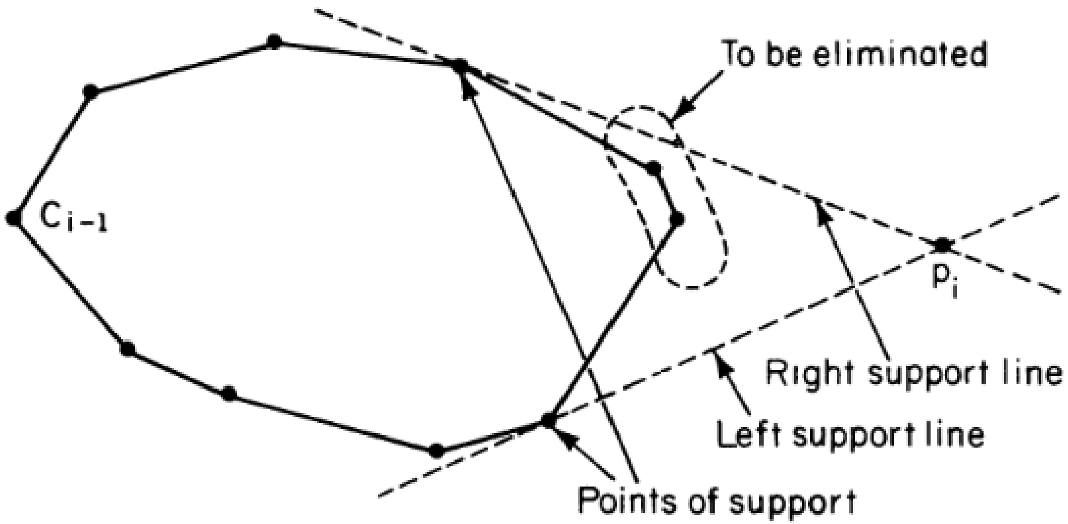
- *O(n* log *h*) time and O(*n*) space is
  - optimal for *h* points on convex hull (output sensitive)
  - 2D and 3D --- gift wrapping
  - off-line
  - Serial (not parallel)
  - not dynamic



[Mount]

Preparata's on-line algorithm

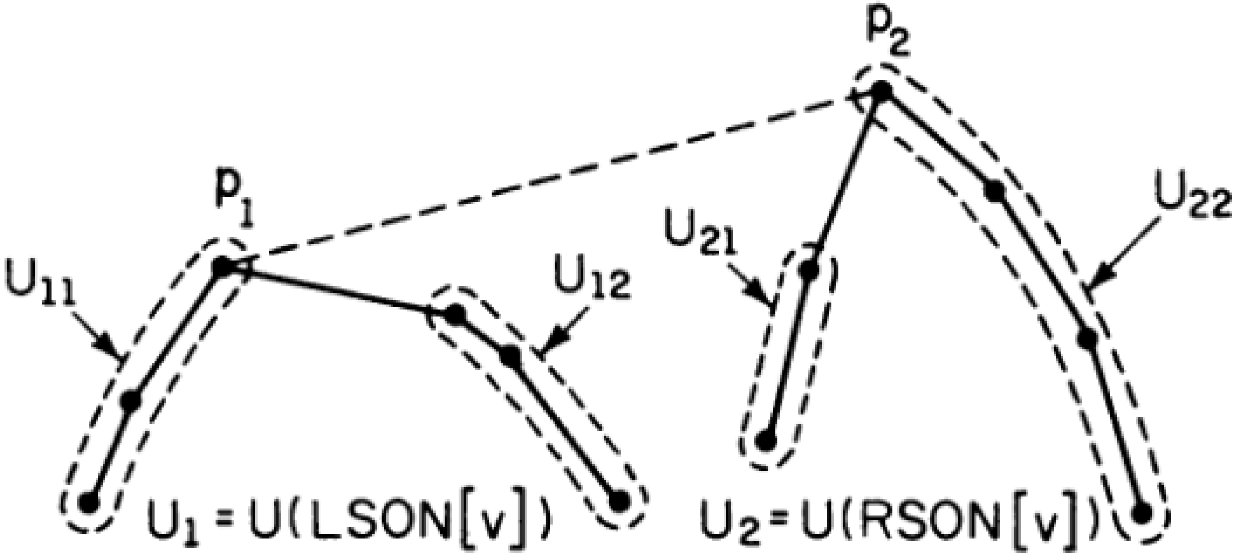
- New point *p* is tested
  - Inside –> ignored
  - Outside –> added to hull
    - Find left and right supporting lines (touch at supporting points)
    - Remove points between supporting points
    - Add *p* to CH between supporting lines



[Preparata]

Overmars and van Leeuven

- Allow dynamic CH
  (on-line insert & delete)
- Manage special tree with all intermediate CHs
- Will be discussed on seminar [7]



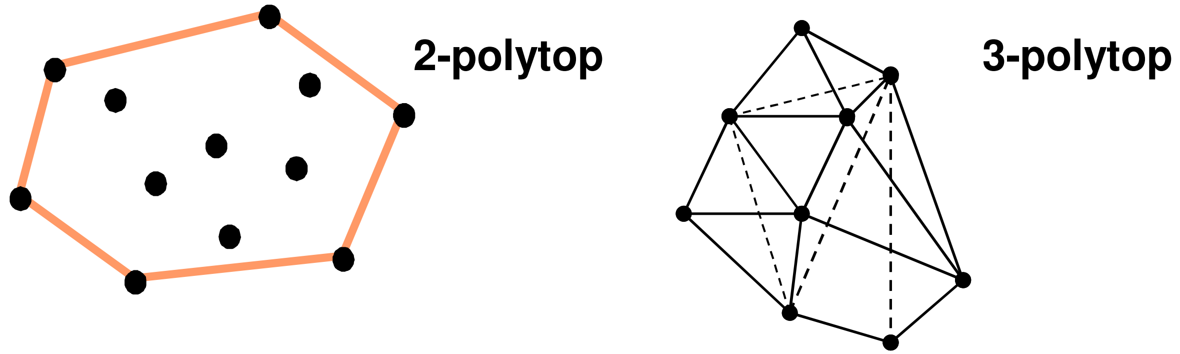
[Preparata]

Convex hull in 3D

- Terminology
- Algorithms
  1. Gift wrapping
  2. D&C Merge
  3. Randomized Incremental

Terminology

- Polytope (d-polytope)
  = convex hull of finite set of points in E^d



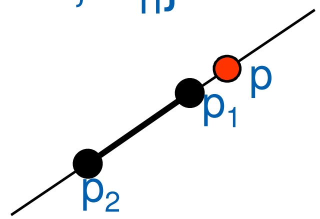
- Simplex (k-simplex, d-simplex)
  = CH of $k + 1$ affine independent points

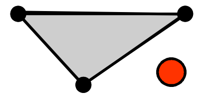
  = “Special” Polytope with all the points are on the CH

Terminology (2)

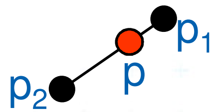
- Affine combination
  = linear combination of the points $\{p_1, p_2, \ldots, p_n\}$
  whose coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ sum to 1, and $\lambda_i \in R$

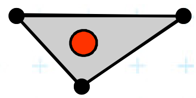
$$\sum_{i=1}^{n} \lambda_i p_i$$





- Affine independent points
  = no one point can be expressed as affine combination of the others





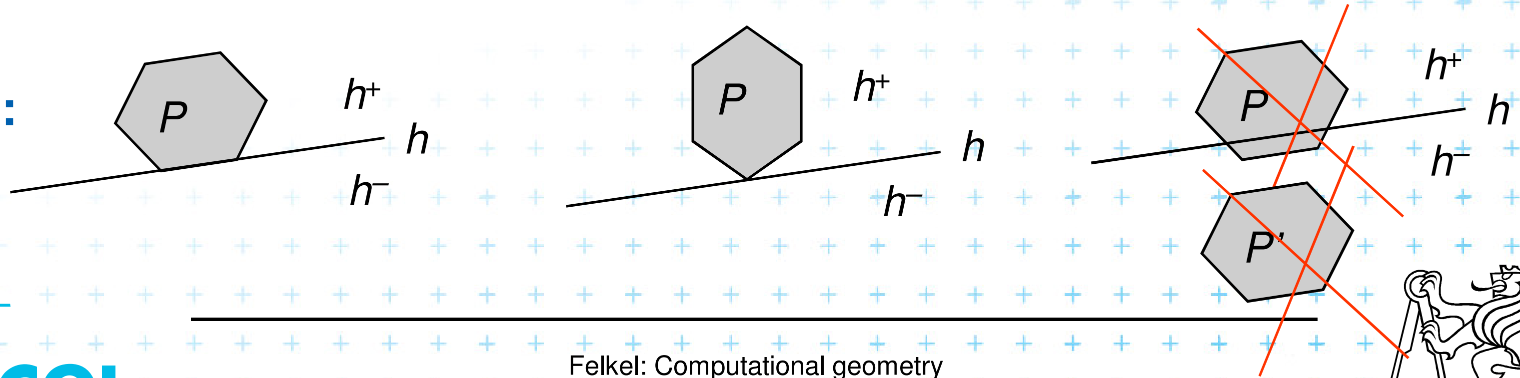
- Convex combination
  = linear combination of the points $\{p_1, p_2, \ldots, p_n\}$
  whose coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ sum to 1, and $\lambda_i \in R^+_0$
  (i.e., $\forall i \in \{1,\ldots,k\}, \lambda_i \geq 0$)

Terminology (3)

- Any (d-1)-dimensional hyperplane h divides the space into (open) halfspaces h^+ *and* h^-, *so that* $E^n = h^+ \cup h \cup h^-$
- *Def:* $\overline{h^+} = h^+ \cup h$, $\overline{h^-} = h^- \cup h$ (closed halfspaces)

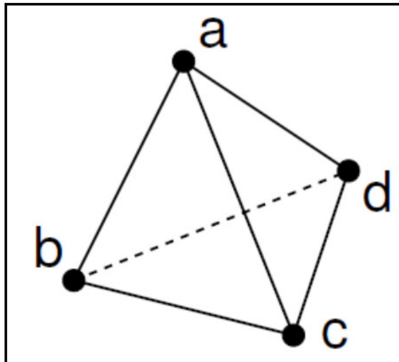
- Hyperplane supports a polytope P (Supporting hyperplane)
  – if $h \cap P$ is not empty and
  – if P is entirely contained within either $\overline{h^+}$ or $\overline{h^-}$

In 2D:



Faces and facets

- Face of the polytope
  = Intersection of polytope *P* with a supporting hyperplane *h*
  - Faces are convex polytops of dimension *d* ranging from 0 to d – 1
  - 0-face = vertex
  - 1-face = edge
  - (d – 1)-face = facet



Proper faces:
Vertices: a,b,c,d
Edges: ab, ac, ad, bc, bd, cd
Facets: abc, abd, acd, bcd

In 3D we often say *face*, but more precisely a *facet*

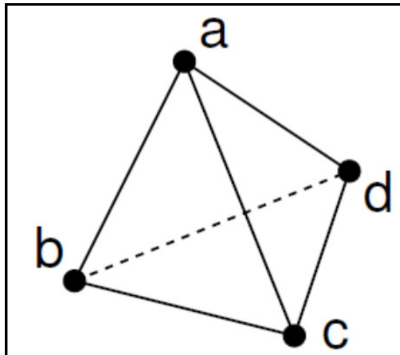
(In 3D a 2-face = facet)

Proper faces

- Proper faces

  = Faces of dimension d ranging from 0 to $d-1$
- Improper faces

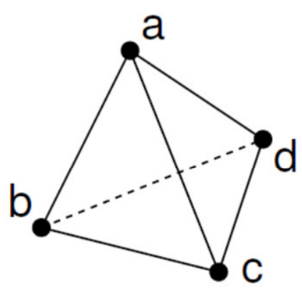
  = proper faces + two additional faces:
  - {} = Empty set = face of dimension -1
  - Entire polytope = face of dimension d

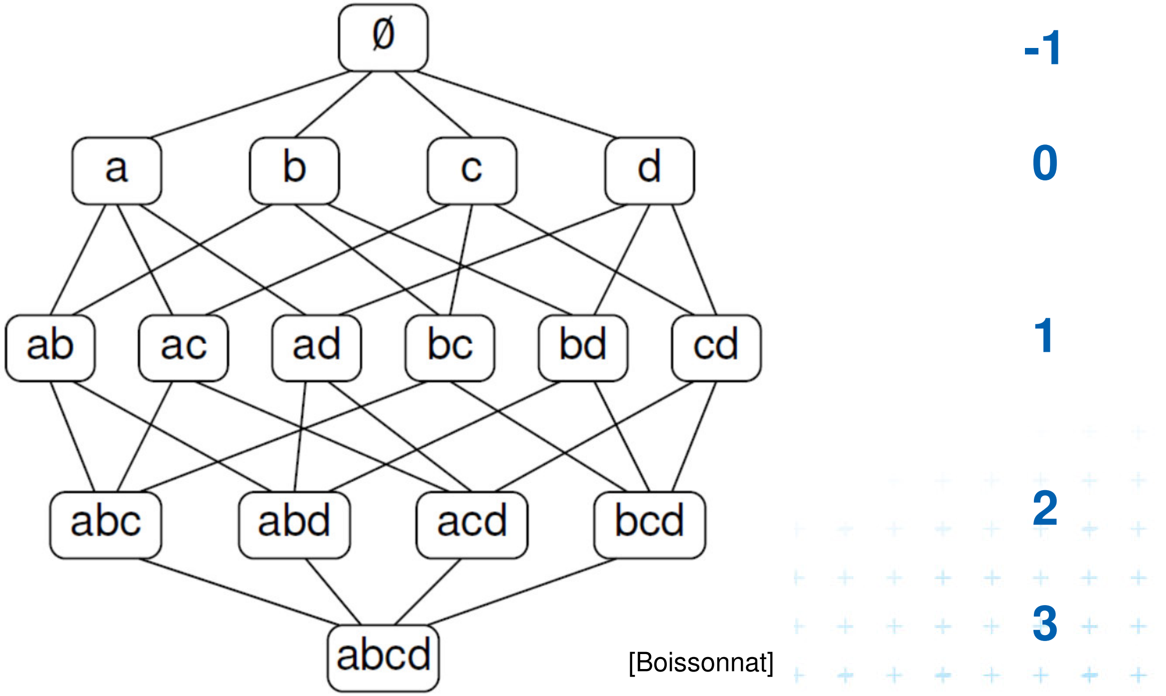


Improper faces:
- Empty set {}
- Vertices: a,b,c,d
- Edges: ab, ac, ad, bc, bd, cd
- Facets: abc, abd, bcd
- Entire polytope: abcd

Incident graph

- Stores topology of the polytope
- Ex: 3-simplex:





[Boissonnat]

- D-simplex is very regular face structure:
  – 1-face for each pair of vertices
  – 2-face for each triple of vertices

Facts about polytopes

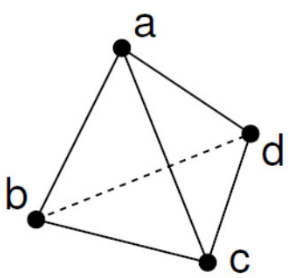
- Boundary o polytope is *union of its proper faces*
- Polytope has *finite number of faces (next slide)*. Each face is a polytope
- Polytope is *convex hull of its vertices (the def)* (its bounded)
- Polytope is the *intersection of finite number of closed halfspaces* $\overline{h^+}$ (conversely not: intersection of closed halfspaces may be unbounded => called polyhedron or unbounded polytope)

Number of faces on a d-simplex

- Number of *j*-dimensional faces on a *d*-simplex = number of (*j*+1)-element subsets from domain of size (*d*+1)

$$\binom{d+1}{j+1} = \frac{(d+1)!}{(j+1)!(d-j)!}$$

- Ex.: Tetrahedron = 3-simplex:
  - facets (2-dim. faces) $\binom{3+1}{2+1} = \frac{4!}{3!1!} = 4$
  - edges (1-dim. faces) $\binom{3+1}{1+1} = \frac{4!}{2!2!} = 6$
  - vertices (0-dim faces) $\binom{3+1}{0+1} = \frac{4!}{1!3!} = 4$



Complexity of 3D convex hull is O(n)

- The worst case complexity → if all *n* points on CH

=> use simplical 3-polytop for complexity derivation

1. has all points on its surface – on the Convex Hull
2. has usually more edges E and faces F than 3-polytope
3. has triangular facets, each generates 3 edges, shared by 2 triangles => $3F = 2E$ 2-manifold

- $V - E + \boxed{F} = 2$ … Euler formula for V = n points

$V - E + 2E/3 = 2$

$V - 2 = E / 3$

$E = 3V - 6, \quad V = n$

$E = O(n)$

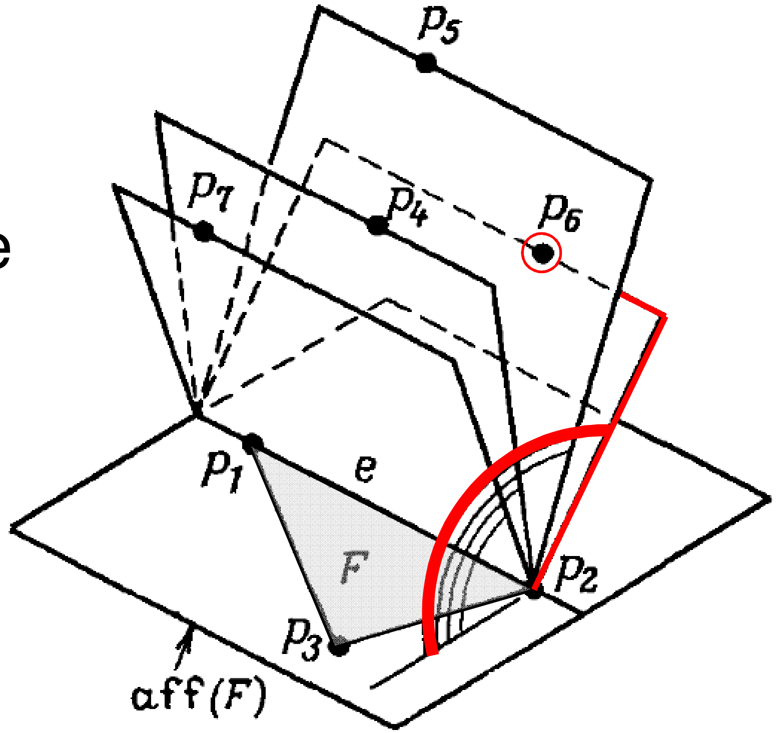
$F = 2\boxed{E} / 3$

$F = 2V - 4$

$F = O(n)$

1. Gift wrapping in higher dimensions

- First known algorithm for n-dimensions (1970)
- Direct extension of 2D alg.
- Complexity O(nF)
  - F is number of CH facets
  - Algorithm is output sensitive
  - Details on seminar, assignment [10]



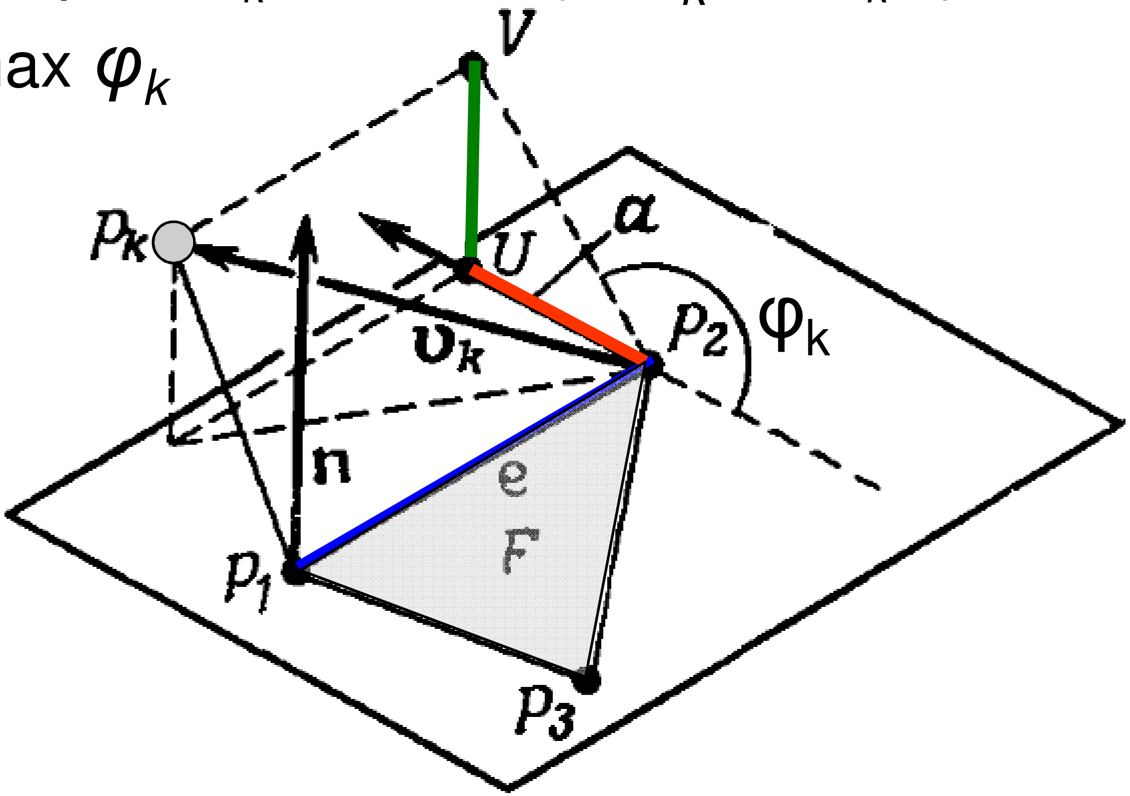
[Preparata]

The angle comparison [Preparata 3.4.1]

Cotangent of the agle φ_k between halfplanes F and ep_k
$= - |UP_2| \; / \; |UV|$, where $|UP_2| = \boldsymbol{v_k.a}$ and $|UV| = \boldsymbol{v_k.n}$

For each P_k compute $\varphi_k = \text{arcctan}(- \boldsymbol{v_k.a} \,/\, \boldsymbol{v_k.n})$,

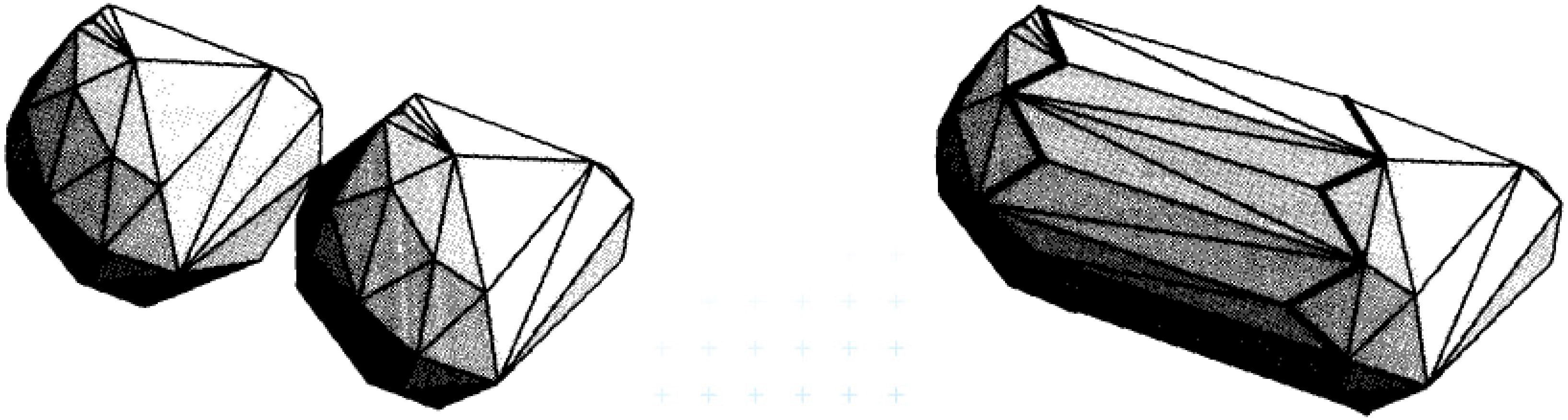
The angle is max φ_k



[Preparata]

2. Divide & conquer 3D convex hull [Preparata, Hong77]

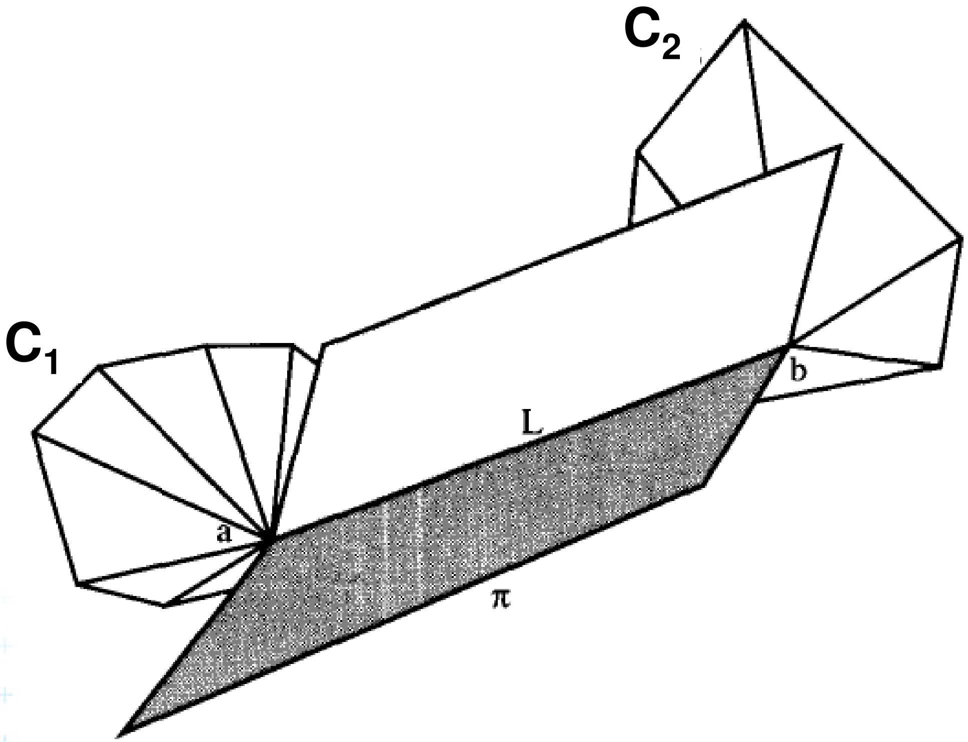
- Sort points in x-coord
- Recursively split, construct CH, merge
- Merge takes O(n) => O(*n* log *n*) total time



[Rourke]

Divide & conquer 3D convex hull [Preparata, Hong 77]

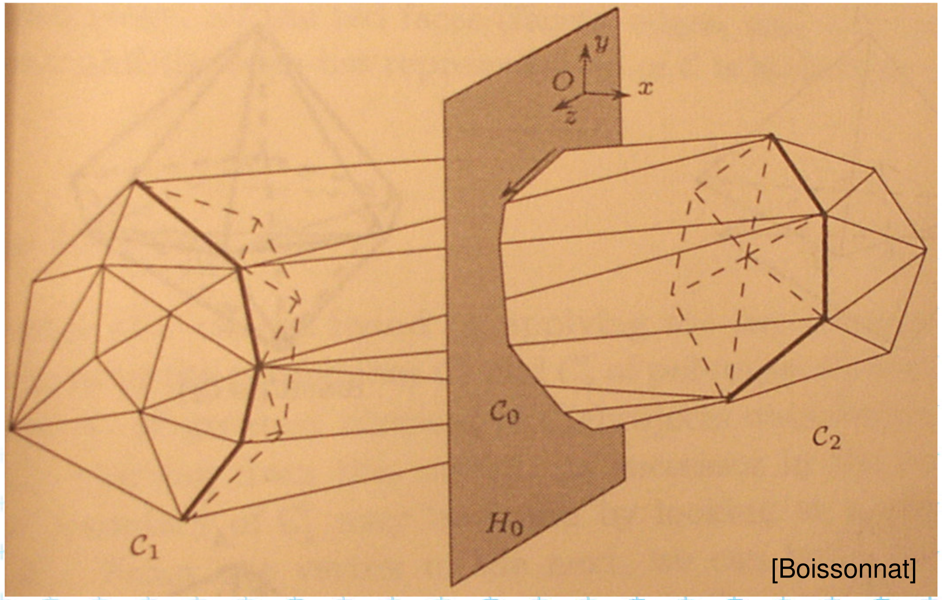
- Merge(C_1 with C_2) uses gift wrapping
  - Gift wrap plane around edge *e* – find new point *p* on C_1 or on C_2 (neighbor of *a* or *b*)
  - Search just the CW or CCW neighbors around *a, b*



[Rourke]

Divide & conquer 3D convex hull [Preparata, Hong 77]

- Performance O(*n* log *n*) rely on circular ordering
  – In 2D: Ordering of points around CH
  – In 3D: Ordering of vertices around 2-polytop C_0 (vertices on intersection of new CH edges with separating plane H_0) [ordering around horizon of C_1 and C_2 does not exist, both horizons may be non-convex and even not simple polygons]



  – In ≥ 4D: Such ordering does not exist

Divide & conquer 3D convex hull [Preparata, Hong 77]

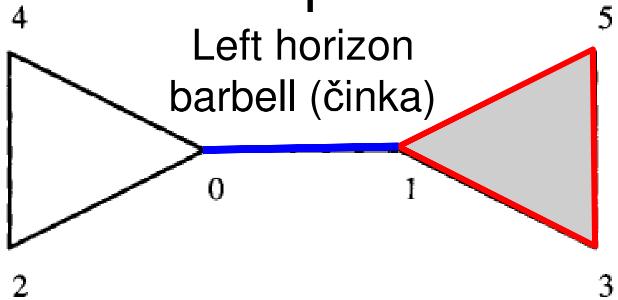
Merge(C_1 with C_2)

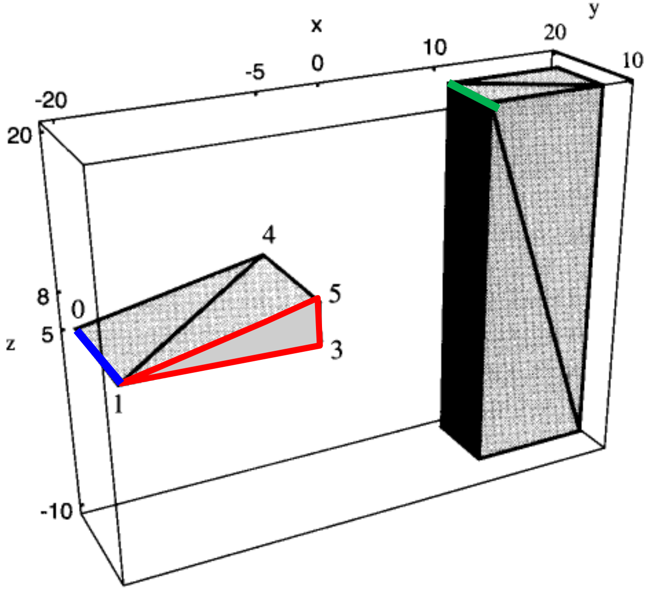
- Find the first CH edge *L* connecting C_1 with C_2
- *e* = *L*
- While not back at *L do*
  - store *e* to *C*
  - Gift wrap plane around edge *e* – find new point *P* on C_1 or on C_2 (neighbor of *a* or *b*)
  - *e* = new edge to just found end-point *P*
  - Store new triangle *eP* to *C*
- Discard hidden faces inside CH from *C*
- Report merged convex hull *C*

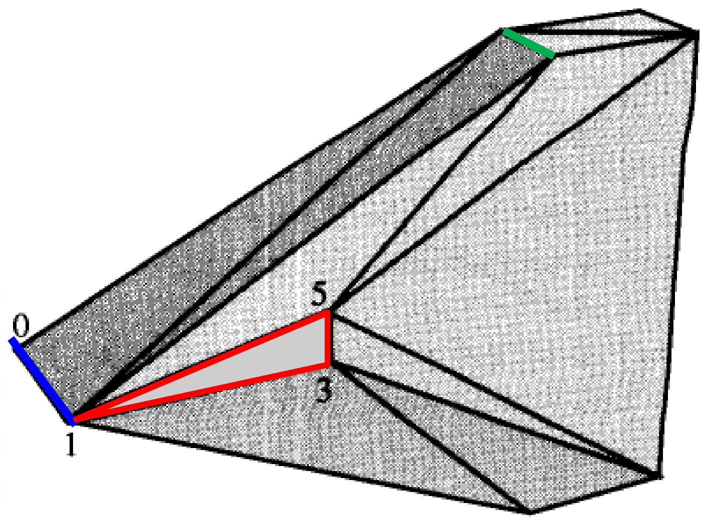
Divide & conquer 3D convex hull [Preparata, Hong 77]

- Problem of gift wrapping [Edelsbrunner 88]
  - The edges on horizon do not form simple circle but a “barbell” 0,2,4,0,1,3,5,1

Do not stop here!







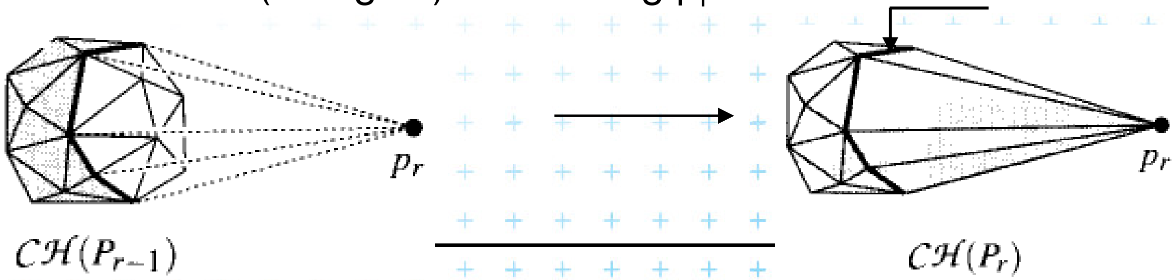
[Berg]

3. Randomized incremental alg. principle

1. Create tetrahedron (smallest CH in 3D)
   – Take 2 points p_1 and p_2
   – Search the 3rd point not lying on line p_1p_2
   – Search the 4th point not lying in plane $p_1p_2p_3$ …if not found, use 2D CH
2. Perform random permutation of remaining points $\{p_5,\ldots, p_n\}$
3. For p_r in $\{p_5,\ldots, p_n\}$ do add point p_r to CH(P_{r-1})

   Notation: for $r \geq 1$ let $P_r = \{p_1,\ldots, p_r\}$ is set of already processed pts

   – If p_r lies inside or on the boundary of CH(P_{r-1}) then do nothing
   – If p_r lies outside of CH(P_{r-1}) then
     - find and remove visible faces
     - create new faces (triangles) connecting p_r with lines of horizon



[Berg]

Conflict graph

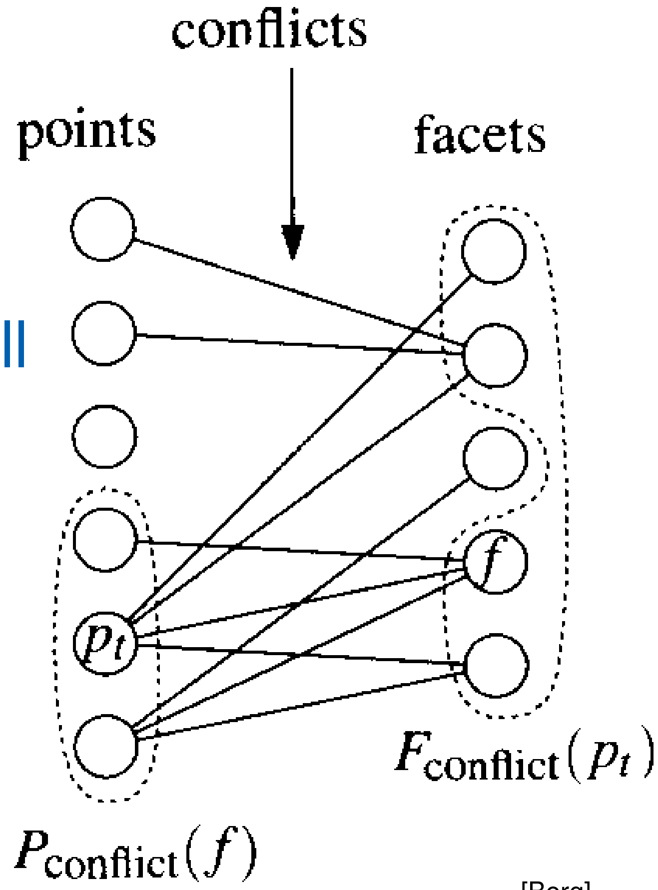
- Stores unprocessed points with facets of CH they see

- Bipartite graph

  points p_t, $t > r$ … unprocessed points

  facets of CH(P_r)… facets of convex hull

  conflict arcs … conflict, as visible facets cannot be in CH



[Berg]

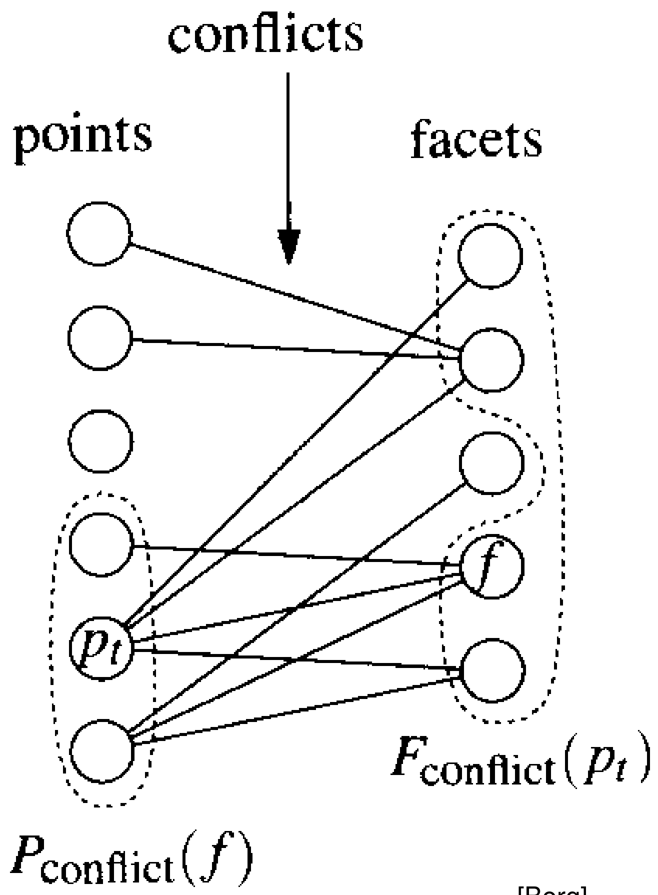
- Maintains sets:

  $\mathbf{P_{conflict}(f)}$ … points, that see f

  $\mathbf{F_{conflict}(p_r)}$… facets visible from p_r (visible region – deleted after insertion of p_r)

Conflict graph – init and final state

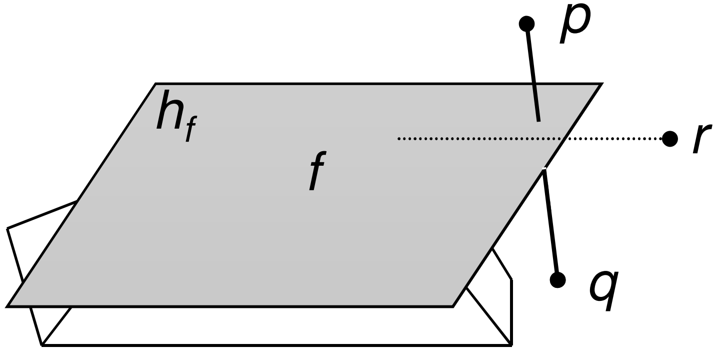
- Initialization
  - Points $\{p_5, \ldots, p_n\}$ (not in tetrahedron)
  - Facets of the tetrahedron (four)
  - Arcs – connect each tetrahedron facet with points visible from it
- Final state
  - Points – {} = empty set
  - Facets of the convex hull
  - Arcs - none



[Berg]

Visibility between point and face

- Face *f* is visible from a point *p* if that point lies in the open half-space on the other side of h_f than the polytope



f is visible from *p* (*p* is *above* the plane)

f is not visible from *r* lying *in the plane* of *f* (this case will be discussed next)

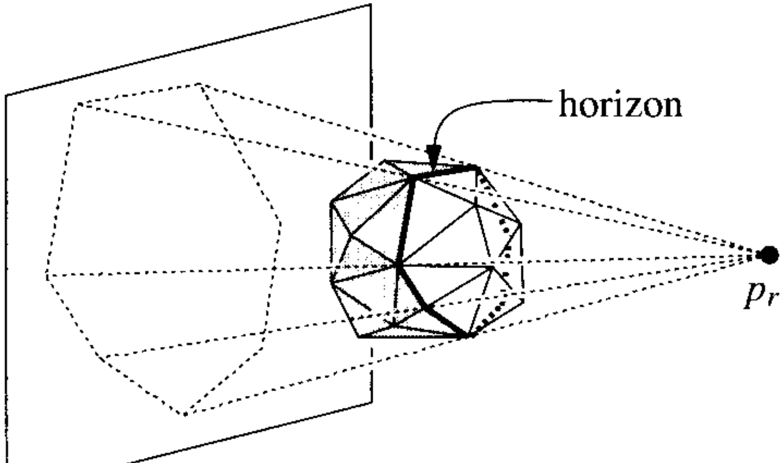
f is not visible from *q*

$p \in P_{conflict}(f)$, p is among the points that see the face f

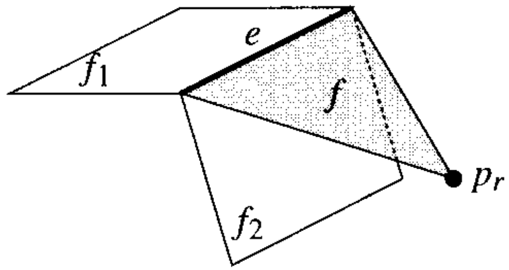
$f \in F_{conflict}(p)$ f is among the faces visible from point p

New triangles to horizon

- Horizon = edges *e* incident to visible and invisible facets



[Berg]



[Berg]

- New triangle *f* connects edge *e* on horizon and point p_r and
  - creates new node for facet f updates the conflict graph
  - add arcs to points visible from f (subset from $P_{coflict}(f_1) \cup P_{coflict}(f_2)$)
- Coplanar triangles on the plane ep_r are merged with new triangle. Conflicts are copied from the deleted triangle (same plane)

Incremental Convex hull algorithm

IncrementalConvexHull(P)

Input: Set of n points in general position in 3D space

Output: The convex hull C=CH(P) of P

1. Find four points that form an initial tetrahedron, C = CH($\{p_1, p_2, p_3, p_4\}$)
2. Compute random permutation $\{p_5, p_6, \ldots, p_n\}$ of the remaining points
3. Initialize the conflict graph with all visible pairs (p_t, f),
   where f is facet of C and p_t, $t > 4$, are non-processed points
4. **for** $r = 5$ to n **do** …insert p_r, into C
5. **if**($F_{conflict}(p_r)$ is not empty) **then** …p_r is outside, any facet is visible
6. Delete all facets $F_{conflict}(p_r)$ from C … only from hull C, not from G
7. Walk around visible region boundary, create list L of horizon edges
8. **for** all $e \in L$ **do**
9. connect e to p_r by a new triangular facet f
10. **if** f is coplanar with its neighbor facet f' along e
11. **then** merge f and f', take conflict list from f'
12. **else** … determine conflicts for new face f

… [continue on the next slide]

Incremental Convex hull algorithm (cont…)

else … not coplanar => determine conflicts for new face f
    Create node for f in G *//… new face in conflict graph G*
    Let f_1 and f_2 be the facets incident to e in the old $CH(P_{r-1})$
    $P(e) = P_{coflict}(f_1) \cup P_{coflict}(f_2)$
    for all points $p \in P(e)$ **do**
        if f is visible from p, **then** add(p, f) to G *… new edges*
Delete the node corresponding to p_r and the nodes corresponding to facets in $F_{coflict}(p_r)$ from G, together with their incident arcs
return C

Complexity: Convex hull of a set of points in E^3 can be computed incrementally in $O(n \log n)$ randomized expected time (process O(n) points, but number of facets and arcs depend on the order of inserting points – up to $O(n^2)$)

For proof see: [Berg, Section11.3]

Convex hull in higher dimensions

- Convex hull in *d* dimensions can have $\Omega(n^{\lfloor d/2 \rfloor})$
  Proved by [Klee, 1980]
- Therefore, 4D hull can have quadratic size
- No *O(n log n)* algorithm possible for d>3
- These approaches can extend to d>3
  - Gift wrapping
  - D&C
  - Randomized incremental
  - QuickHull

Conclusion

- Recapitulation of 2D algorithms
- >=3D algorithms
  - Gift wrapping
  - D&C
  - Randomized incremental
  - QuickHull

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