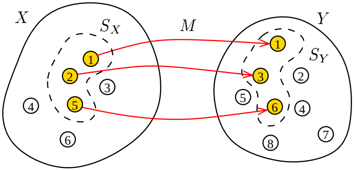
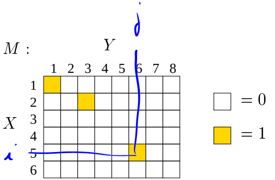
▶The Full Problem of Matching and Fundamental Matrix Estimation

Problem: Given two sets of image points $X = \{x_i\}_{i=1}^m$ and $Y = \{y_j\}_{j=1}^n$ and their descriptors D, find the most probable

1. inliers $S_X \subseteq X$, $S_Y \subseteq Y$
2. one-to-one perfect matching $M : S_X \to S_Y$ — perfect matching: 1-factor of the bipartite graph
3. fundamental matrix $\mathbf{F}$ such that $\operatorname{rank} \mathbf{F} = 2$
4. such that for each $x_i \in S_X$ and $y_j = M(x_i)$ it is probable that
   a. the image descriptor $D(x_i)$ is similar to $D(y_j)$, and
   b. the total geometric error $\sum_{ij} e_{ij}^2(\mathbf{F})$ is small — note a slight change in notation: e_{ij}
5. inlier-outlier and outlier-outlier matches are improbable





$$(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(M, \mathbf{F} \mid X, Y, D) \tag{17}$$

- probabilistic model: an efficient language for task formulation
- the (17) is a p.d.f. for all the involved variables (there is a constant number of variables!)
- binary matching table $M_{ij} \in \{0, 1\}$ of fixed size $m \times n$
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point x_i / y_j

Deriving A Robust Matching Model by Marginalization

For algorithmic efficiency, instead of $(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(M, \mathbf{F} \mid X, Y, D)$ we will solve

$$\mathbf{F}^* = \arg\max_{\mathbf{F}} p(\mathbf{F} \mid X, Y, D) \tag{18}$$

by marginalization of $p(M, \mathbf{F} \mid X, Y, D)$ over M — this simplification changes the problem!

$$p(M, \mathbf{F} \mid X, Y, D) \simeq p(M, \mathbf{F}, X, Y, D) = p(X, Y, D, M \mid \mathbf{F}) \cdot p(\mathbf{F})$$

assuming correspondence-wise independence:

$$p(X, Y, D, M \mid \mathbf{F}) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_i, y_j, D, m_{ij} \mid \mathbf{F}) \stackrel{\text{def}}{=} \prod_{i=1}^{m} \prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, m_{ij} \mid \mathbf{F})$$

- e_{ij} represents geometric error for match $x_i \leftrightarrow y_i$: $e_{ij}(x_i, y_i \mid \mathbf{F})$
- d_{ij} represents descriptor similarity for match $x_i \leftrightarrow y_i$: $d_{ij} = \|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|$

Marginalization:

$$\sum_{m_{11} \in \{0,1\}} \sum_{m_{12}} \cdots \sum_{m_{mn}} p(X, Y, D, M \mid \mathbf{F}) = \sum_{m_{11}} \sum_{m_{12}} \cdots \sum_{m_{mn}} \prod_{i=1}^{m} \prod_{j=1}^{n} p_e(e_{ij}, d_{ij}, m_{ij} \mid \mathbf{F}) =$$

$$= \cdots = \prod_{i=1}^{m} \prod_{j=1}^{n} \underbrace{\sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, m_{ij} \mid \mathbf{F})}_{\text{we will continue with this term}} = p(X, Y, D \mid \mathbf{F})$$

Robust Matching Model (cont'd)

$$\sum_{m_{ij}\in\{0,1\}} p_e(e_{ij}, d_{ij}, m_{ij} \mid \mathbf{F}) = \sum_{m_{ij}\in\{0,1\}} p_e(e_{ij}, d_{ij} \mid m_{ij}, \mathbf{F}) \cdot p(m_{ij} \mid \mathbf{F}) =$$

$$= \underbrace{p_e(e_{ij}, d_{ij} \mid m_{ij} = 1, \mathbf{F})}_{p_1(e_{ij}, d_{ij} \mid \mathbf{F})} \cdot \underbrace{p(m_{ij} = 1 \mid \mathbf{F})}_{1-\alpha_0} + \underbrace{p_e(e_{ij}, d_{ij} \mid m_{ij} = 0, \mathbf{F})}_{p_0(e_{ij}, d_{ij} \mid \mathbf{F})} \cdot \underbrace{p(m_{ij} = 0 \mid \mathbf{F})}_{\alpha_0} =$$

$$= (1 - \alpha_0)\, p_1(e_{ij}, d_{ij} \mid \mathbf{F}) + \alpha_0\, p_0(e_{ij}, d_{ij} \mid \mathbf{F}) \qquad (19)$$

- the $p_0(e_{ij}, d_{ij} \mid \mathbf{F}) \approx \text{const}$ is a penalty for 'missing a correspondence' but it should be a p.d.f. (cannot be a constant) (see Slide 108 for a simplification)

$$\alpha_0 \to 1, \quad p_0 \to 0 \quad \text{so that} \quad \frac{\alpha_0}{1-\alpha_0}\, p_0 \approx \text{const}$$

- the $p_1(e_{ij}, d_{ij} \mid \mathbf{F})$ is typically an easy-to-design component: assuming independence of geometric error and descriptor similarity:

$$p_1(e_{ij}, d_{ij} \mid \mathbf{F}) = p_1(e_{ij} \mid \mathbf{F}) \cdot p_1(d_{ij})$$

- we choose, eg.

$$p_1(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_1, \mathbf{F})}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}}, \quad p_1(d_{ij}) = \frac{1}{T_d(\sigma_d, \dim \mathbf{d})}\, e^{-\frac{\|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|^2}{2\sigma_d^2}} \qquad (20)$$

- σ_1, σ_d, α_0 are 'hyper-parameters'
- the form of $T(\sigma_1, \mathbf{F})$ depends on error definition
- we will continue with the result from (19)

►Simplified Robust Energy (Error) Function

- assuming the choice of p_1 as in (20), we are simplifying

$$p(X,Y,D \mid \mathbf{F}) = \prod_{i=1}^{m}\prod_{j=1}^{n}\Big[(1-\alpha_0)\,p_1(e_{ij},d_{ij} \mid \mathbf{F}) + \alpha_0\,p_0(e_{ij},d_{ij} \mid \mathbf{F})\Big] \qquad (21)$$

- we define 'energy' as: $V(x) = -\log p(x)$ — this helps simplify the formulas
- for simplicity, we omit d_{ij}
- we choose $\sigma_0 \gg \sigma_1$ and the missed-correspondence penalty function as

$H = -\int p(x) \log p(x)\, dx$

entropy = mean energy

$$p_0(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_0,\mathbf{F})}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}$$

$\sigma_0 \to \infty$

- then

$$V(X,Y,D \mid \mathbf{F}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\underbrace{-\log\frac{1-\alpha_0}{T_e(\sigma_1,\mathbf{F})}}_{\Delta(\mathbf{F})} - \log\Big(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \underbrace{\frac{\alpha_0}{1-\alpha_0}\,\frac{T_e(\sigma_1,\mathbf{F})}{T_e(\sigma_0,\mathbf{F})}\, e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}}_{t\,\approx\,\mathrm{const}}\Big)\right]$$

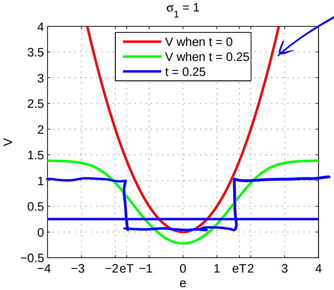
- by choosing representative of $\mathbf{F}$ such that $\Delta(\mathbf{F}) = \mathrm{const}$, we get

$$V(X,Y,D \mid \mathbf{F}) = m\,n\,\Delta + \sum_{i=1}^{m}\sum_{j=1}^{n}\underbrace{-\log\Big(e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t\Big)}_{\hat{V}(e_{ij})} \qquad (22)$$

note that m, n are fixed

▶The Action of the Robust Matching Model on Data

Example for $\hat{V}(e)$ from (22): $e_{ij}^2(F)$



red – the usual (non-robust) error when $t = 0$

blue – the rejected correspondence penalty t

green – 'robust energy' (22)

- if the error of a correspondence exceeds a limit, it is ignored
- then $\hat{V}(e) = \text{const}$ and we essentially count outliers in (22)
- t controls the 'turn-off' point
- the inlier/outlier threshold is e_T is the error for which $(1-\alpha_0)\,p_1(e_T) = \alpha_0\,p_0(e_T)$: note that $t \approx 0$

$$e_T = \sigma_1\sqrt{-\log t^2} \tag{23}$$

The full optimization problem is (18):

$$\mathbf{F}^* = \arg\max_{\mathbf{F}} p(\mathbf{F} \mid X, Y, D) = \arg\max_{\mathbf{F}} \frac{\overbrace{p(X,Y,D \mid \mathbf{F})}^{\text{likelihood}} \cdot \overbrace{p(\mathbf{F})}^{\text{prior}}}{\underbrace{p(X,Y,D)}_{\text{evidence}}} =$$

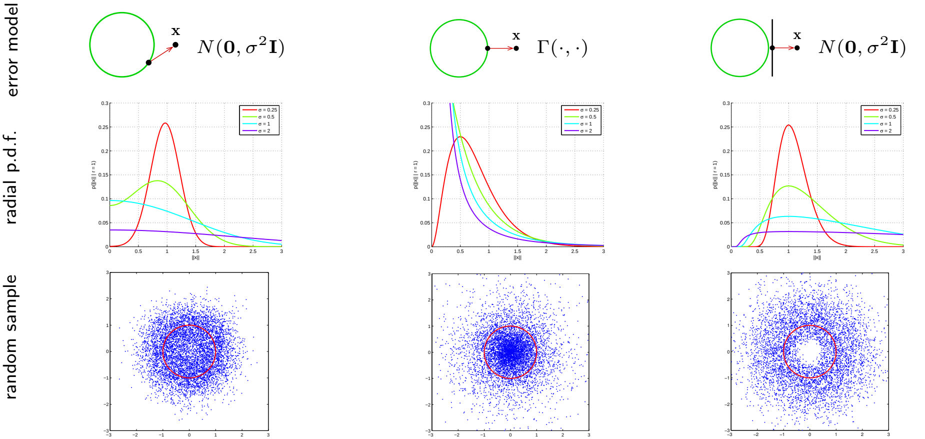
$$= \arg\min_{\mathbf{F}} \big\{ V(X,Y,D \mid \mathbf{F}) + V(\mathbf{F}) \big\}$$

- typically we take $V(\mathbf{F}) = 0$ unless we need to stabilize a computation, e.g. when video camera moves smoothly (on a high-mass vehicle) and we have a prediction for $\mathbf{F}$
- evidence is not needed unless we want to compare different models

Discussion: On The Art of Probabilistic Model Design...

- a few models for fitting zero-centered circle C of radius r to points in $\mathbb{R}^2$

marginalized over C | **orthogonal deviation from C** | **Sampson approximation**



$p(\mathbf{x} \mid r)$

Marginalized over C:

$$\approx \frac{1}{\sigma\sqrt{(2\pi)^3 r\,\|\mathbf{x}\|}}\, e^{-\frac{(\|\mathbf{x}\|-r)^2}{2\sigma^2}}$$

- mode inside the circle
- models the inside well
- tends to normal distrib.

Orthogonal deviation from C:

$$\frac{1}{2\pi\Gamma(\frac{r^2}{\sigma})}\,\frac{1}{\|\mathbf{x}\|^2}\left(\frac{r\|\mathbf{x}\|}{\sigma}\right)^{\frac{r^2}{\sigma}} e^{-\frac{r\|\mathbf{x}\|}{\sigma}}$$

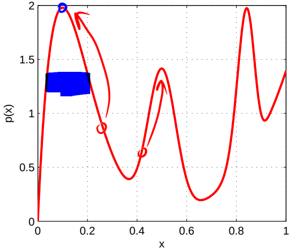
- peak at the center
- unusable for small radii
- tends to Dirac distrib.

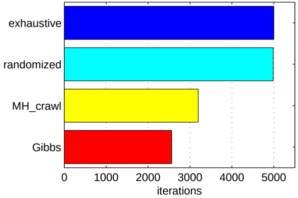
Sampson approximation:

$$\frac{1}{r\sigma\sqrt{(2\pi)^3}}\, e^{-\frac{e^2(\mathbf{x};r)}{2\sigma^2}}$$

- mode at the circle
- hole at the center
- tends to normal distrib.

How To Find the Global Maxima (Modes) of a PDF?





- averaged over 10^4 trials
- number of proposals before $|x - x_{\mathrm{true}}| \leq$ step
- uniform and Gibbs give the theoretical result

- consider the function $p(x)$ at left p.d.f. on $[0, 1]$, mode at 0.1
- consider several methods:
  1. exhaustive search

     $\arg\max_x p(x)$

     ```
     step = 1/(iterations-1);
     for x = 0:step:1
      if p(x) > bestp
       bestx = x; bestp = p(x);
      end
     end
     ```

     - slow algorithm (definite quantization); faster variants exist • fast to implement

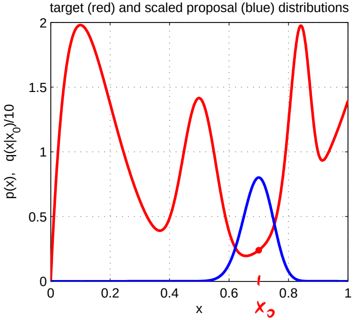
  2. randomized search with uniform sampling

     ```
     x = rand(1);  x= prnd;
     if p(x) > bestp
      bestx = x; bestp = p(x);
     end
     ```

     - slow algorithm but better convergence • fast to implement • how to stop it?

  3. random sampling from $p(x)$ (Gibbs sampler)
     - faster algorithm • fast to implement but often infeasible (e.g. when $p(x)$ is data dependent (our case))

  4. Metropolis-Hastings sampling
     - almost as fast (with care) • not so fast to implement • rarely infeasible • RANSAC belongs here



- red: probability density function $p(x)$ of a toy distribution on the unit interval target distribution

$$p(x) = \sum_{i=1}^{4} \alpha_i \,\mathrm{Be}(x; \alpha_i, \beta_i), \;\; \sum_{i=1}^{4} \alpha_i = 1, \;\; \alpha_i \geq 0$$

$$\mathrm{Be}(x; \alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $(0.1, 0.2, 0.5, 0.2)$ | $(0.1, 0.3, 0.8, 1.0)$

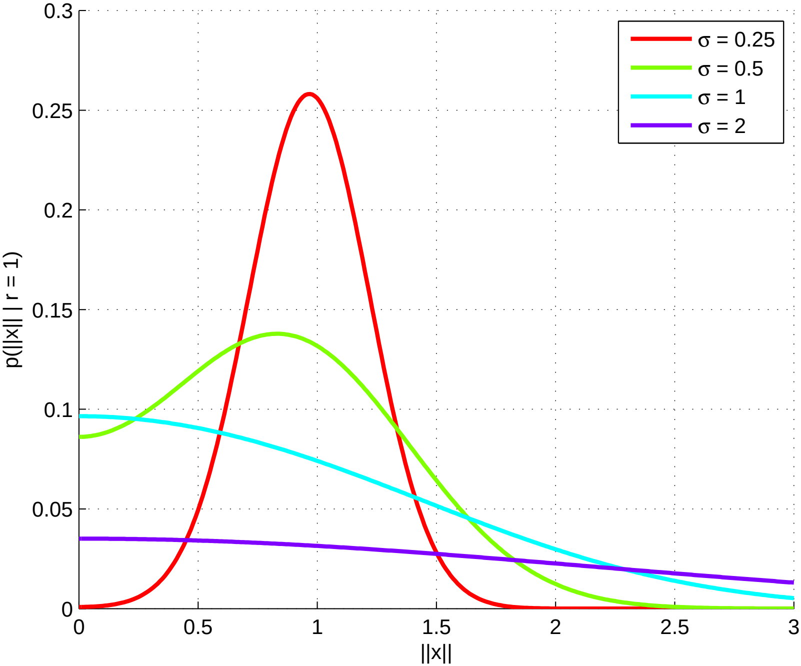
- note we can generate samples from this $p(x)$ how?

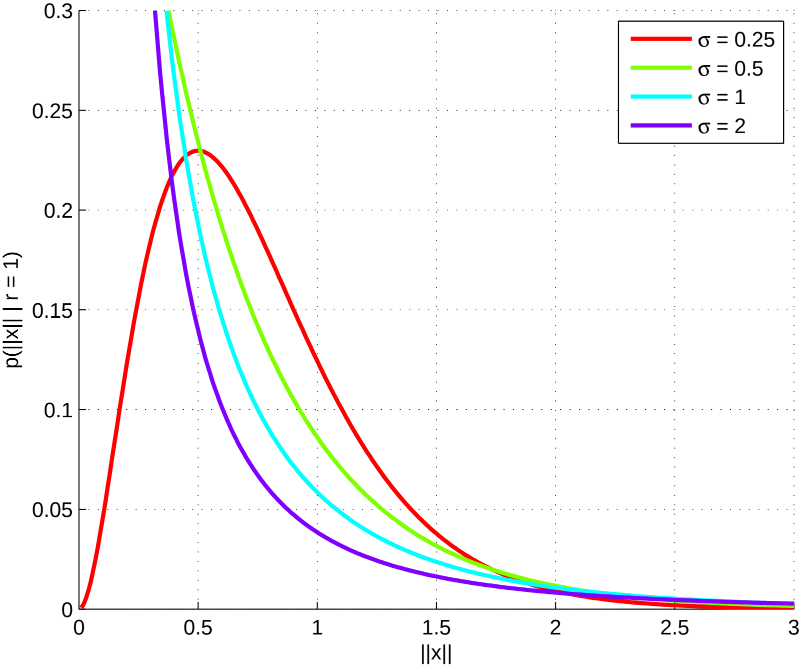
- suppose we cannot sample from $p(x)$ but we can sample from some 'simple' distribution, given the last sample x_0 (blue) proposal distribution

$$q(x \mid x_0) = \begin{cases} \mathrm{U}_{0,1}(x) & \text{(independent) uniform sampling} \\ \mathrm{Be}(x; \frac{x_0}{T}+1, \frac{1-x_0}{T}+1) & \text{'beta' diffusion (crawler)} \quad T \text{ – temperature} \\ p(x) & \text{(independent) Gibbs sampler} \end{cases}$$

- note we have unified all the random sampling methods on the previous slide
- how to transform proposal samples $q(x \mid x_0)$ to target distribution $p(x)$ samples?

Thank You


σ = 0.25
σ = 0.5
σ = 1
σ = 2
p(||x|| | r = 1)
||x||
0
0.05
0.1
0.15
0.2
0.25
0.3
0
0.5
1
1.5
2
2.5
3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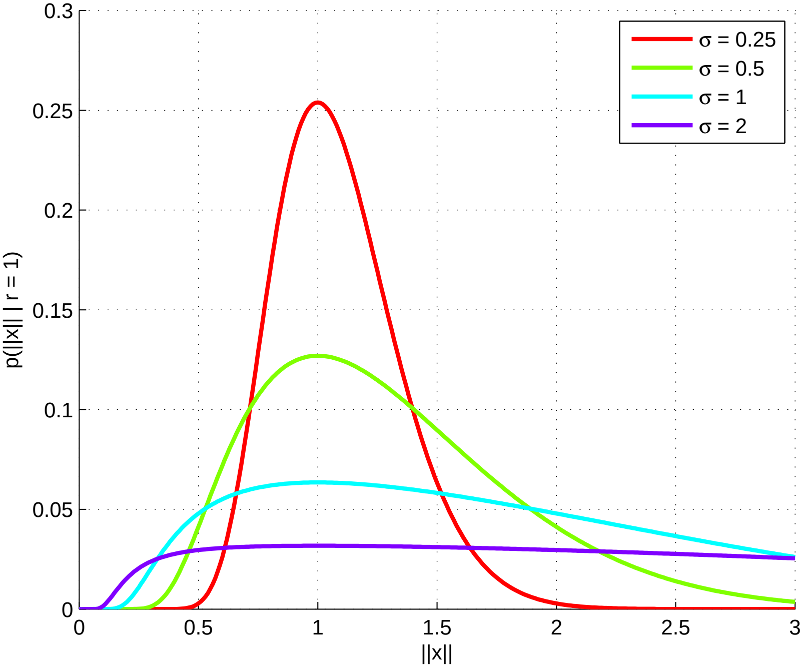

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