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## Part III

# Computing with a Single Camera

- 9 Calibration: Internal Camera Parameters from Vanishing Points and Lines
- 10 Resectioning: Projection Matrix from 6 Known Points
- 11 Exterior Orientation: Camera Rotation and Translation from 3 Known Points

### covered by

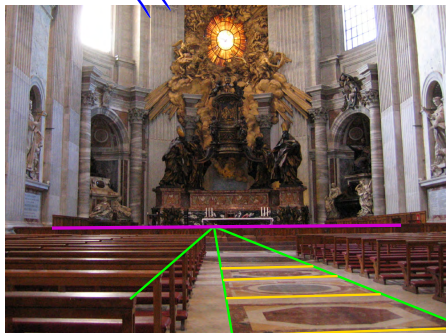
- [1] [H&Z] Secs: 8.6, 7.1, 22.1
- [2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981
- [3] [Golub & van Loan 1996, Sec. 2.5]

# Obtaining Vanishing Points and Lines

- orthogonal pairs can be collected from more images by camera rotation

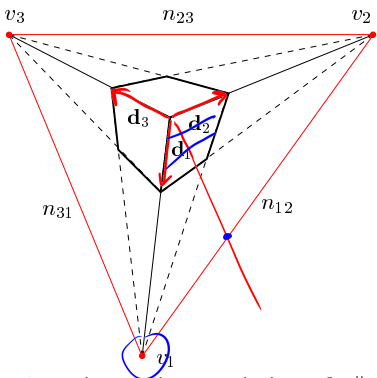


- vanishing line can be obtained without vanishing points (see Slide 46)



# ► Camera Calibration from Vanishing Points and Lines

**Problem:** Given finite vanishing points and/or vanishing lines, compute  $\mathbf{K}$



$$\mathbf{d}_i = \mathbf{Q}^{-1} \mathbf{v}_i, \quad i = 1, 2, 3 \quad \text{Slide 33}$$

$$\mathbf{p}_{ij} = \mathbf{Q}^\top \mathbf{n}_{ij}, \quad i, j = 1, 2, 3, i \neq j \quad \text{Slide 36}$$

## Constraints

1. orthogonal rays  $\mathbf{d}_1 \perp \mathbf{d}_2$  in space then

$$0 = \mathbf{d}_1^\top \mathbf{d}_2 = \mathbf{v}_1^\top \mathbf{Q}^{-\top} \mathbf{Q}^{-1} \mathbf{v}_2 = \mathbf{v}_1^\top \underbrace{(\mathbf{K}\mathbf{K}^\top)^{-1}}_{\omega \text{ (IAC)}} \mathbf{v}_2$$

2. orthogonal planes  $\mathbf{p}_{ij} \perp \mathbf{p}_{ik}$  in space

$$0 = \mathbf{p}_{ij}^\top \mathbf{p}_{ik} = \mathbf{n}_{ij}^\top \mathbf{Q}\mathbf{Q}^\top \mathbf{n}_{ik} = \mathbf{n}_{ij}^\top \omega^{-1} \mathbf{n}_{ik}$$

3. orthogonal ray and plane  $\mathbf{d}_k \parallel \mathbf{p}_{ij}, k \neq i, j$  normal parallel to optical ray

$$\underbrace{\mathbf{p}_{ij}}_{\mathbf{n}_2} \cong \underbrace{\mathbf{d}_k}_{\mathbf{3}} \Rightarrow \mathbf{Q}^\top \mathbf{n}_{ij} = \lambda \mathbf{Q}^{-1} \mathbf{v}_k \Rightarrow \mathbf{n}_{ij} = \lambda \mathbf{Q}^{-\top} \mathbf{Q}^{-1} \mathbf{v}_k = \lambda \omega \mathbf{v}_k, \quad \lambda \neq 0$$

- $n_{ij}$  may be constructed from non-orthogonal  $v_i$  and  $v_j$ , e.g. using the cross-ratio

- $\omega$  is a symmetric, positive definite  $3 \times 3$  matrix

IAC = Image of Absolute Conic

► cont'd

condition	constraint	# constraints on $\omega$
(2) orthogonal v.p.	$\mathbf{v}_i^\top \omega \mathbf{v}_j = 0$	1
(3) orthogonal v.l.	$\mathbf{n}_{ij}^\top \omega^{-1} \mathbf{n}_{ik} = 0$	1
(4) v.p. orthogonal to v.l.	$\mathbf{n}_{ij} = \lambda \omega \mathbf{v}_k$	2
(5) orthogonal raster $\theta = \pi/2$	$\omega_{12} = \omega_{21} = 0$	1
(6) unit aspect $a = 1$ when $\theta = \pi/2$	$\omega_{11} = \omega_{22}$	1
(7) known principal point $u_0 = v_0 = 0$	$\omega_{13} = \omega_{31} = \omega_{23} = \omega_{32} = 0$	2

- these are homogeneous linear equations for the 5 parameters in  $\omega$  in the form  $\mathbf{D}\mathbf{w} = \mathbf{0}$   
 $\lambda$  can be eliminated from (4)  
 we will come to solving overdetermined homogeneous equations later → Slide 97
- we need at least 5 constraints for full  $\mathbf{K}$
- we get  $\mathbf{K}$  from  $\omega^{-1} = \mathbf{K}\mathbf{K}^\top$  by Choleski decomposition  $\mathbf{K} = \text{chol}(\omega)$   
 this trick avoids solving a set of quadratic equations for the parameters in  $\mathbf{K}$

```
In[1]:= K = {{f, s, u[0]}, {0, a*f, v[0]}, {0, 0, 1}};
K // MatrixForm
```

```
Out[2]/MatrixForm=
```

$$\begin{pmatrix} f & s & u[0] \\ 0 & a f & v[0] \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[23]:= ω = Inverse[K.Transpose[K]] * Det[K] ^ 2;
ω // Simplify // MatrixForm
```

```
Out[24]/MatrixForm=
```

$$\begin{pmatrix} a^2 f^2 & -a f s & a f (-a f u[0] + s v[0]) \\ -a f s & f^2 + s^2 & a f s u[0] - (f^2 + s^2) v[0] \\ a f (-a f u[0] + s v[0]) & a f s u[0] - (f^2 + s^2) v[0] & a^2 f^2 (f^2 + u[0]^2) - 2 a f s u[0] v[0] + (f^2 + s^2) v[0]^2 \end{pmatrix}$$

```
In[29]:= ω / f^2 /. {s -> 0} // Simplify // MatrixForm
```

```
Out[29]/MatrixForm=
```

$$\begin{pmatrix} a^2 & 0 & -a^2 u[0] \\ 0 & 1 & -v[0] \\ -a^2 u[0] & -v[0] & a^2 (f^2 + u[0]^2) + v[0]^2 \end{pmatrix}$$

```
In[30]:= % /. a -> 1 // MatrixForm
```

```
Out[30]/MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & -u[0] \\ 0 & 1 & -v[0] \\ -u[0] & -v[0] & f^2 + u[0]^2 + v[0]^2 \end{pmatrix}$$

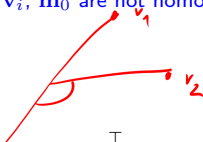
$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & f^2 \end{bmatrix}$$

# Examples

## Ex 1:

Assuming known  $m_0 = (u_0, v_0)$ , two finite orthogonal vanishing points suffice to get  $f$   
in this formula,  $\mathbf{v}_i, \mathbf{m}_0$  are not homogeneous!

$$- \left| (\mathbf{v}_1 - \mathbf{m}_0)^\top (\mathbf{v}_2 - \mathbf{m}_0) \right| + f^2 = 0$$



## Ex 2:

Non-orthogonal vanishing points  $\mathbf{v}_i, \mathbf{v}_j$ , known angle  $\phi$ :  $\cos \phi = \frac{\mathbf{v}_i^\top \omega \mathbf{v}_j}{\sqrt{\mathbf{v}_i^\top \omega \mathbf{v}_i} \sqrt{\mathbf{v}_j^\top \omega \mathbf{v}_j}}$

- leads to polynomial equations
- e.g. assuming orthogonal raster, unit aspect (ORUA):  $a = 1, \theta = \pi/2$

$$\omega = \frac{1}{f^2} \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ -u_0 & -v_0 & f^2 + u_0^2 + v_0^2 \end{bmatrix}$$

- ORUA and  $u_0 = v_0 = 0$  gives

$$(f^2 + \mathbf{v}_i^\top \omega \mathbf{v}_j)^2 = (f^2 + \|\mathbf{v}_i\|^2) \cdot (f^2 + \|\mathbf{v}_j\|^2) \cdot \cos^2 \phi$$

## ► Camera Orientation from Vanishing Points

**Problem:** Given  $\mathbf{K}$  and two vanishing points corresponding to two known orthogonal directions  $\mathbf{d}_1$ ,  $\mathbf{d}_2$ , compute camera orientation  $\mathbf{R}$  with respect to the plane.

- coordinate system choice, e.g.:

$$\mathbf{d}_1 = (1, 0, 0), \quad \mathbf{d}_2 = (0, 1, 0)$$

- we know that

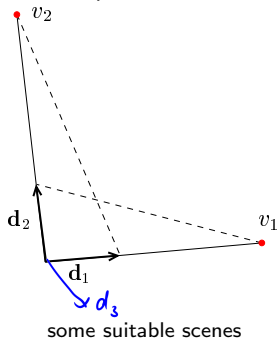
$$\mathbf{d}_i \cong \mathbf{Q}^{-1} \underline{\mathbf{v}}_i = (\mathbf{KR})^{-1} \underline{\mathbf{v}}_i = \mathbf{R}^{-1} \underbrace{\mathbf{K}^{-1} \underline{\mathbf{v}}_i}_{\underline{\mathbf{w}}_i}$$

$$\mathbf{R} \mathbf{d}_i \cong \underline{\mathbf{w}}_i$$

- then  $\underline{\mathbf{w}}_i / \|\underline{\mathbf{w}}_i\|$  is the  $i$ -th column  $\mathbf{r}_i$  of  $\mathbf{R}$
- the third column is orthogonal:

$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{R} = \begin{bmatrix} \frac{\underline{\mathbf{w}}_1}{\|\underline{\mathbf{w}}_1\|} & \frac{\underline{\mathbf{w}}_2}{\|\underline{\mathbf{w}}_2\|} & \frac{\underline{\mathbf{w}}_1 \times \underline{\mathbf{w}}_2}{\|\underline{\mathbf{w}}_1 \times \underline{\mathbf{w}}_2\|} \end{bmatrix}$$



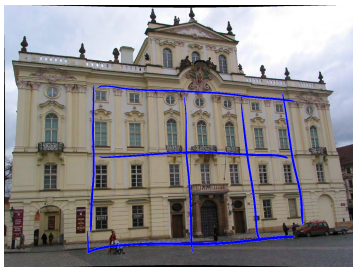
some suitable scenes





# Application: Planar Rectification

**Principle:** Rotate camera parallel to the plane of interest.



$$\underline{\mathbf{m}} \simeq \mathbf{KR} [\mathbf{I} \quad -\mathbf{C}] \underline{\mathbf{X}}$$

$$\underline{\mathbf{m}}' \simeq \mathbf{K} [\mathbf{I} \quad -\mathbf{C}] \underline{\mathbf{X}}$$

$$\mathbf{H} = \mathbf{K} \mathbf{R}^T \mathbf{K}^{-1}$$

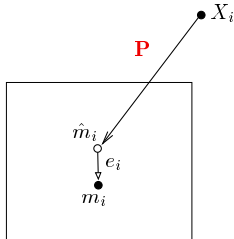
$$\underline{\mathbf{m}}' \simeq \mathbf{K}(\mathbf{KR})^{-1} \underline{\mathbf{m}} = \mathbf{KR}^T \mathbf{K}^{-1} \underline{\mathbf{m}} = \mathbf{H} \underline{\mathbf{m}}$$

- $\mathbf{H}$  is the rectifying homography
- both  $\mathbf{K}$  and  $\mathbf{R}$  can be calibrated from two finite vanishing points
- not possible when one (or both) of them are infinite

## ► Camera Resectioning

Camera calibration and orientation from a known set of  $k \geq 6$  reference points and their images  $\{(X_i, m_i)\}_{i=1}^6$ .

$$\hat{m}_i = \mathbf{P}X_i$$



$X_i$  is considered exact

$m_i$  is a measurement

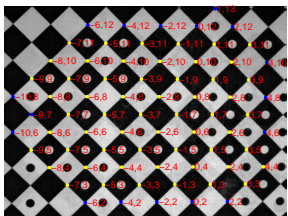
$$e_i^2 = \|\mathbf{m}_i - \hat{\mathbf{m}}_i\|^2$$

where  $\hat{\mathbf{m}}_i \simeq \mathbf{P}\mathbf{X}_i$

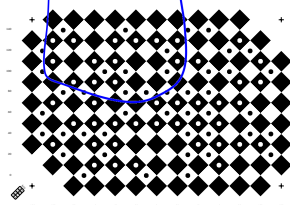
projection error



calibration target with translation stage



automatic calibration point detection



calibration chart

## ► The Minimal Problem for Resectioning

**Problem:** Given  $k = 6$  corresponding pairs  $\{(X_i, m_i)\}_{i=1}^k$ , find **P**

$$m_i \approx P X_i$$

$$\lambda_i \underline{m}_i = \mathbf{P} \underline{X}_i, \quad \mathbf{P} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \quad \underline{X}_i = (x_i, y_i, z_i, \mathbf{1}), \quad i = 1, 2, \dots, k, \quad k = 6$$

*finite points!  
easy to modify*

$$\underline{m}_i = (u_i, v_i, 1), \quad \lambda_i \in \mathbb{R}, \quad \lambda_i \neq 0$$

expanded:  $\lambda_i u_i = \mathbf{q}_1^\top \underline{X}_i + q_{14}, \quad \lambda_i v_i = \mathbf{q}_2^\top \underline{X}_i + q_{24}, \quad \lambda_i = \mathbf{q}_3^\top \underline{X}_i + q_{34}$

eliminating  $\lambda$  gives:  $(\mathbf{q}_3^\top \underline{X}_i + q_{34})u_i = \mathbf{q}_1^\top \underline{X}_i + q_{14}, \quad (\mathbf{q}_3^\top \underline{X}_i + q_{34})v_i = \mathbf{q}_2^\top \underline{X}_i + q_{24}$

Then

$$\mathbf{A} \mathbf{q} = \begin{bmatrix} \mathbf{X}_1^\top & 1 & \mathbf{0}^\top & 0 & -u_1 \mathbf{X}_1^\top & -u_1 \\ \mathbf{0}^\top & 0 & \mathbf{X}_1^\top & 1 & -v_1 \mathbf{X}_1^\top & -v_1 \\ \vdots & & & & & \\ \mathbf{X}_k^\top & 1 & \mathbf{0}^\top & 0 & -u_k \mathbf{X}_k^\top & -u_k \\ \mathbf{0}^\top & 0 & \mathbf{X}_k^\top & 1 & -v_k \mathbf{X}_k^\top & -v_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ q_{14} \\ \mathbf{q}_2 \\ q_{24} \\ \mathbf{q}_3 \\ q_{34} \end{bmatrix} = \mathbf{0} \quad (8)$$

*12 elements*

- we need 11 independent parameters for **P**
- $\mathbf{A} \in \mathbb{R}^{2k, 12}, \quad \mathbf{q} \in \mathbb{R}^{12}$
- 6 points in a general position give  $\text{rank } \mathbf{A} = 12$  and there is no non-trivial null space
- drop one row to get rank 11 matrix, then the basis of the null space of **A** gives **q**

## ► The Jack-Knife Solution for $k = 6$

- given the 6 correspondences, we have 12 equations for the 11 parameters
- can we use all the information present in data?

### Jack-knife estimation

1.  $n := 0$
2. for  $i = 1, 2, \dots, 2k$  do
  - a. delete  $i$ -th row from  $\mathbf{A}$ , this gives  $\mathbf{A}_i$
  - b. if  $\dim \text{null } \mathbf{A}_i > 1$  continue with the next  $i$
  - c.  $n := n + 1$
  - d. compute the right null-space  $\mathbf{q}_i$  of  $\mathbf{A}_i$
  - e. normalize  $\mathbf{q}_i$  to  $\hat{\mathbf{q}}_i = \mathbf{q}_i / q_{12}$
3. from all  $n$  vectors  $\hat{\mathbf{q}}_i$  collected in Step 1d compute

$$\mathbf{q} = \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{q}}_i, \quad \text{var}[\mathbf{q}] = \frac{n-1}{n} \text{diag} \sum_{i=1}^n (\hat{\mathbf{q}}_i - \mathbf{q})(\hat{\mathbf{q}}_i - \mathbf{q})^\top$$

- have a solution + an error estimate, per individual elements of  $\mathbf{P}$
- at least 5 points must be in a general position see Slide 67
- large error indicates near degeneracy
- computation not efficient with  $k > 6$  points, needs  $\binom{2k}{11}$  draws, e.g.  $k = 7 \rightarrow 364$  draws
- one needs  $k \geq 7$  for the full covariance matrix  $\boldsymbol{\Sigma}$
- better error estimation method: decompose  $\mathbf{P}_i$  to  $\mathbf{K}_i, \mathbf{R}_i, \mathbf{C}_i$  (Slide 30), represent  $\mathbf{R}_i$  with 3 parameters (e.g. Euler angles, or in Cayley representation, see Slide 144) and compute the errors for the parameters

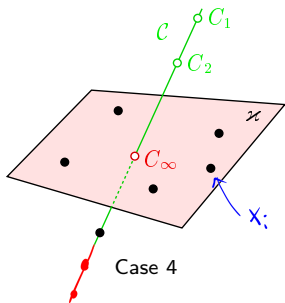


e.g. by 'economy-size' SVD  
this assumes finite camera with  $P_{3,3} = 1$

## ► Degenerate (Critical) Configurations for Resectioning

Let  $\mathcal{X} = \{X_i; i = 1, \dots\}$  be a set of points and  $\mathbf{P}_1 \neq \mathbf{P}_2$  be two regular (rank-3) cameras. Then two configurations  $(\mathbf{P}_1, \mathcal{X})$  and  $(\mathbf{P}_2, \mathcal{X})$  are image-equivalent if

$$\mathbf{P}_1 \underline{\mathbf{X}}_i \simeq \mathbf{P}_2 \underline{\mathbf{X}}_i \quad \text{for all } X_i \in \mathcal{X}$$



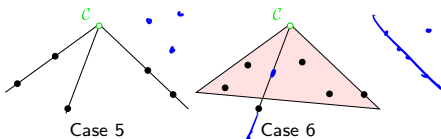
- if all calibration points  $X_i \in \mathcal{X}$  lie on a plane  $\varkappa$  the camera resectioning is non-unique and all image-equivalent camera centers lie on a spatial line  $\mathcal{C}$  with the  $C_\infty = \varkappa \cap \mathcal{C}$  excluded
- by adding points  $X_i \in \mathcal{X}$  to  $\mathcal{C}$  we gain nothing
- there are additional image-equivalent configurations, see next

see proof sketch in the notes or in [H&Z, Sec. 22.1.2]

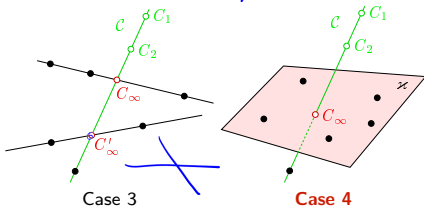
Note that if  $\mathbf{Q}$ ,  $\mathbf{T}$  are suitable non-singular homographies then  $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$ , where  $\mathbf{P}_0$  is canonical and

$$\underbrace{\mathbf{P}_0 \mathbf{T \underline{X}}_i}_{\underline{\mathbf{Y}}_i} \simeq \mathbf{P}_2 \underbrace{\mathbf{T \underline{X}}_i}_{\underline{\mathbf{Y}}_i} \quad \text{for all } Y_i \in \mathcal{Y}$$

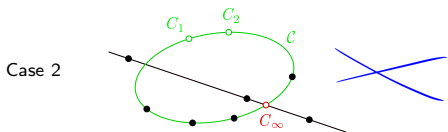
# cont'd (all cases)



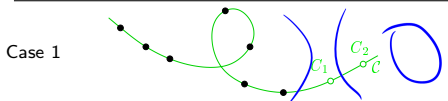
- cameras  $C_1, C_2$  co-located at point  $C$
- points on three optical rays or one optical ray and one optical plane
- Case 5: we see 3 isolated point images
- Case 6: we see a line of points and an isolated point



- cameras lie on a line  $C \setminus \{C_\infty, C'_\infty\}$
- points lie on  $C$  and
  1. on two lines meeting  $C$  at  $C_\infty, C'_\infty$
  2. or on a plane meeting  $C$  at  $C_\infty$
- Case 3: we see 2 lines of points

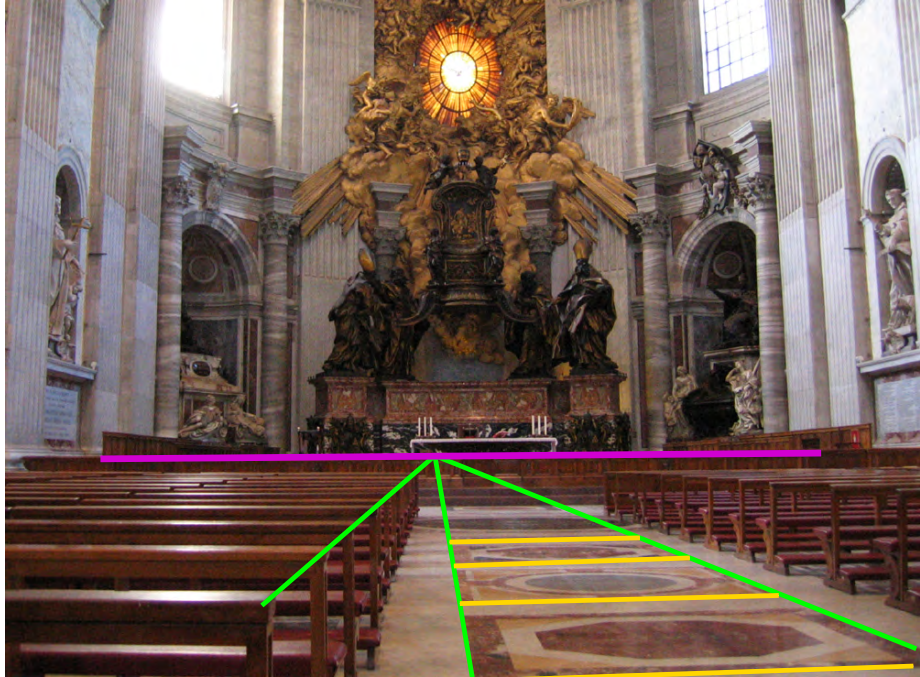


- cameras lie on a planar conic  $C \setminus \{C_\infty\}$   
not necessarily an ellipse
- points lie on  $C$  and an additional line meeting the conic at  $C_\infty$
- Case 2: we see 2 lines of points



- cameras and points all lie on a twisted cubic  $C$
- Case 1: we see a conic

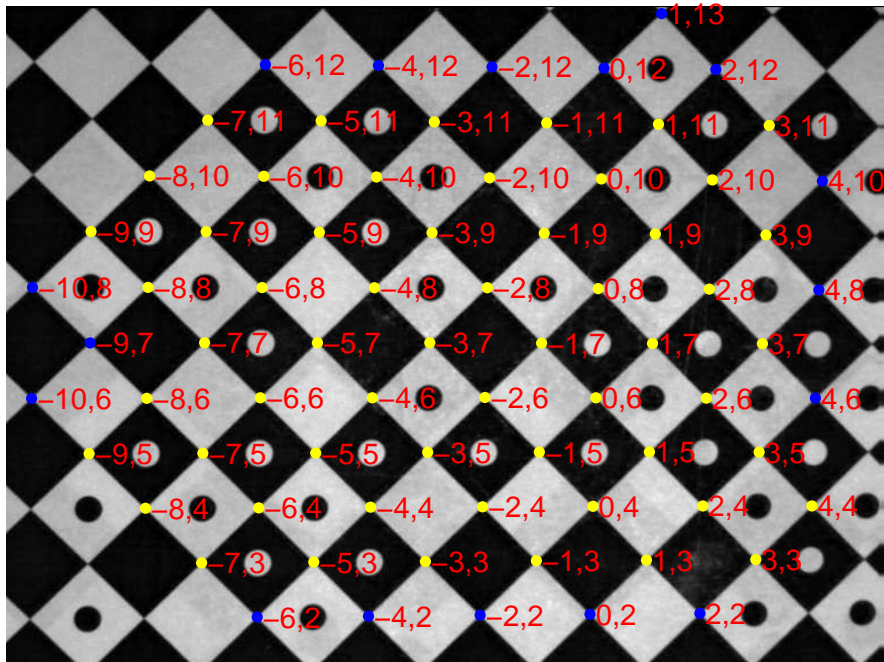
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