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►Computing with Perspective Camera Projection Matrix

$$\underline{\mathbf{m}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \simeq \begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

(u_0, v_0) – principal point in pixels

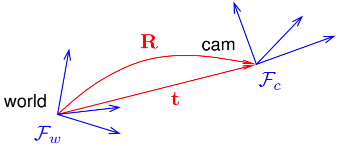
Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$ — since $\underline{\mathbf{m}} \simeq (x, y, z/f)$
   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π $(z = 0)$

▶Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:

$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$



$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ selects the first 3 rows of $\mathbf{T}$ and discards the last row

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

  $\mathbf{C}$ – camera position in the world reference frame — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

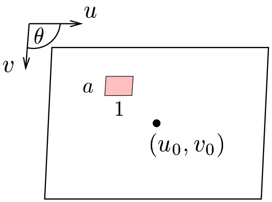
  $\mathbf{r}_3^\top$ – camera axis in the world reference frame — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}\underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

▸Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- digitization raster skew angle θ
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Derive this $\mathbf{K}$; hints: $u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u\mathbf{e}_u + v\mathbf{e}_v$, $\mathbf{K}$ maps from an orthogonal system to a skewed system $[w'u', w'v', w']^\top = \mathbf{K}[u, v, 1]^\top$; first skew then sampling
deadline LD+2 wk

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite projective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left hand 3×3 submatrix $\mathbf{Q}$ non-singular.

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$ full rank (if finite perspective cam.)

$\mathbf{K} \in \mathbb{R}^{3,3}$ upper triangular with positive diagonal entries

$\mathbf{R} \in \mathbb{R}^{3,3}$ rotation: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ see next
2. RQ decomposition of $\mathbf{Q} = \mathbf{K}\mathbf{R}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q}\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}}$$

3. $\mathbf{t} = -\mathbf{R}\mathbf{C}$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g.

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & s\\ 0 & -s & c\end{bmatrix}, \quad c^2 + s^2 = 1, \quad \text{gives} \quad c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{K}\mathbf{R} = \mathbf{K}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}$, where $\mathbf{T} = \mathrm{diag}(-1, -1, 1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive
  'skinny' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 1996, sec. 5.2]

RQ Decomposition Step

```
Q = Array[q, {3, 3}];
R32 = {{1, 0, 0}, {0, c, s}, {0, -s, c}};
R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

```
Q1 = Q.R32;
Q1 // MatrixForm
s1 = Solve[{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}];
s1 = s1[[2]]
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q[1,1] & c\,q[1,2] - s\,q[1,3] & s\,q[1,2] + c\,q[1,3] \\ q[2,1] & c\,q[2,2] - s\,q[2,3] & s\,q[2,2] + c\,q[2,3] \\ q[3,1] & c\,q[3,2] - s\,q[3,3] & s\,q[3,2] + c\,q[3,3] \end{pmatrix}$$

$$\left\{c \to \frac{q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}},\ s \to \frac{q[3,2]}{\sqrt{q[3,2]^2 + q[3,3]^2}}\right\}$$

$$\begin{pmatrix} q[1,1] & \frac{-q[1,3]\,q[3,2] + q[1,2]\,q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}} & \frac{q[1,2]\,q[3,2] + q[1,3]\,q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}} \\ q[2,1] & \frac{-q[2,3]\,q[3,2] + q[2,2]\,q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}} & \frac{q[2,2]\,q[3,2] + q[2,3]\,q[3,3]}{\sqrt{q[3,2]^2 + q[3,3]^2}} \\ q[3,1] & 0 & \sqrt{q[3,2]^2 + q[3,3]^2} \end{pmatrix}$$

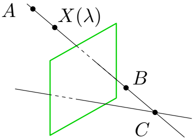
▶Center of Projection

Observation: finite $\mathbf{P}$ has a non-trivial right null-space   rank 3 but 4 columns

Theorem

Let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equal to the projection center $\underline{\mathbf{C}}$ (in world coordinate frame).

Proof.



1. Consider spatial line $A\,B$ (B is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \underline{\mathbf{A}} + \lambda\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$

2. it images to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \mathbf{P}\,\underline{\mathbf{A}} + \lambda\,\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{P}\,\underline{\mathbf{A}}$$

- the whole line images to a single point ⇒ it must pass through the optical center of $\mathbf{P}$
- this holds for all choices of A ⇒ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

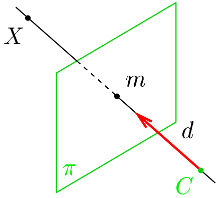
1. consider line ($\mathbf{d}$ line direction vector, $\lambda \in \mathbb{R}$)

$$\mathbf{X} = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the image of point X is

$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X} \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$



- optical ray line corresponding to image point m is

$$\mathbf{X} = \mathbf{C} + (\lambda\mathbf{Q})^{-1}\underline{\mathbf{m}}, \qquad \lambda \in \mathbb{R}$$

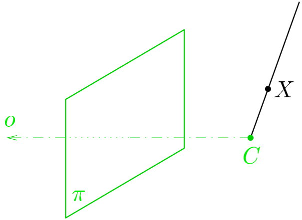
- optical ray may be represented by a point at infinity $(\mathbf{d}, 0)$

▶Optical Axis

Optical axis: The line through C that is perpendicular to image plane π

1. a line parallel to π images to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



2. point X in parallel to π iff $\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$
3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

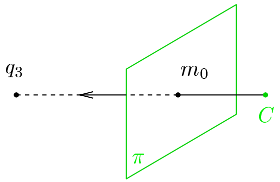
$$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$$

if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. we take point at infinity on the optical axis that must project to principal point m_0



2. then

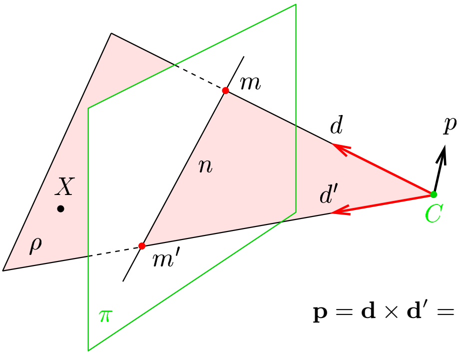
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{q}_3 \\ 0 \end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$

- principal point is also the center of radial distortion (see Slide 50)

▶Optical Plane

A spatial plane with normal p passing through optical center C and a given image line n.



optical ray given by m $\quad \mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' = \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} = \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^\top(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^\top\underline{\mathbf{n}}$$

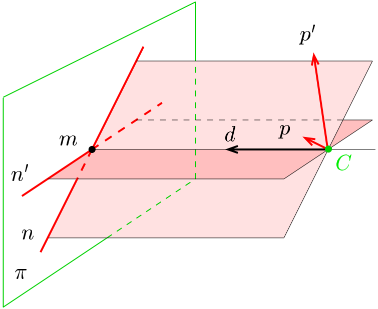
- note the factoring-out of $\mathbf{Q}$!

hence, $0 = \mathbf{p}^\top(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top\mathbf{Q}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^\top\underline{\mathbf{n}})^\top\underline{\mathbf{X}}$ for every X in plane ρ

see Slide 28

optical plane is given by n: $\quad \boldsymbol{\rho} \simeq \mathbf{P}^\top\underline{\mathbf{n}}$ $\qquad \rho_1\, x + \rho_2\, y + \rho_3\, z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\mathbf{m}}$$

▶Summary: Optical Center, Ray, Axis, Plane

General finite camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14}\\ \mathbf{q}_2^\top & q_{24}\\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P})$ optical center (world coords.)

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ optical ray direction (world coords.)

$\det(\mathbf{Q})\,\mathbf{q}_3$ outward optical axis (world coords.)

$\mathbf{Q}\,\mathbf{q}_3$ principal point (in image plane)

$\boldsymbol{\rho} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ optical plane (world coords.)

$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0\\ 0 & f/(a\sin\theta) & v_0\\ 0 & 0 & 1\end{bmatrix}$ camera (calibration) matrix (f, u_0, v_0 in pixels)

$\mathbf{R}$ camera rotation matrix (cam coords.)

$\mathbf{t}$ camera translation vector (cam coords.)

What Can We Do with An 'Uncalibrated' Perspective Camera?

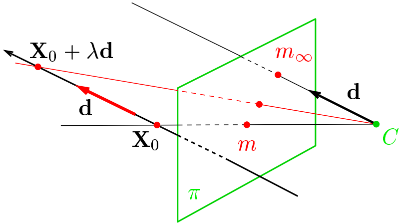
How far is the engine?

distance between sleepers 0.806m but we cannot count them, resolution is too low

We will review some life-saving theory. . .

▶Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



$$\underline{\mathbf{m}}_\infty = \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots = \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Derive or prove

- V.P. is independent on line position, it depends on its orientation only

  all parallel lines have the same V.P.
- the image of the V.P. of a spatial line with direction vector $\mathbf{d}$ is $\underline{\mathbf{m}} = \mathbf{Q}\,\mathbf{d}$
- V.P. m corresponds to spatial direction $\mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$ optical ray through m

- V.P. is the image of a point at infinity on any line, not just the optical ray as on Slide 33

Some Vanishing Point Applications


where is the sun?

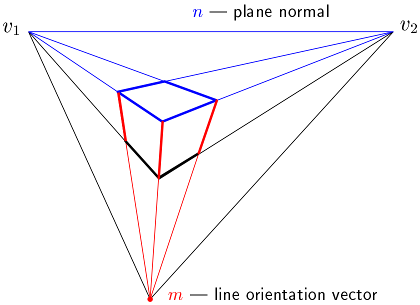

what is the wind direction?
(must have video)


fly above the lane,
at constant altitude!

▶Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane
and in all parallel planes

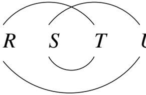


- V.L. n corresponds to space plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$
- a space plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top}\mathbf{p}$.

▶Cross Ratio

Four collinear space points R, S, T, U define cross-ratio

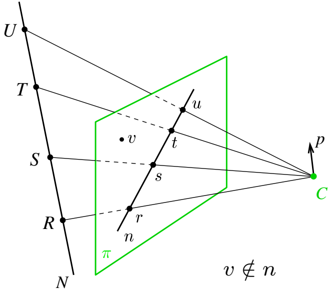
$$[RSTU] = \frac{|RT|}{|RU|}\,\frac{|SU|}{|ST|}$$



a mnemonic

$|RT|$ – signed distance from R to T
(w.r.t. a fixed line orientation)

$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU]$$



$v \notin n$

Obs: $$[RSTU] = \frac{|\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}|}{|\underline{\mathbf{r}},\ \underline{\mathbf{u}},\ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{s}},\ \underline{\mathbf{u}},\ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}|}, \qquad |\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}| = \det[\underline{\mathbf{r}},\ \underline{\mathbf{t}},\ \underline{\mathbf{v}}] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \qquad (1)$$

Corollaries:

- cross ratio is invariant under collineations (homographies) $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1)
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will “see” the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points $R,\ S,\ T,\ U$ may be at infinity

Thank You

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