GRAPHICAL MARKOV MODELS (WS2011)
2. SEMINAR

We consider the following Markov chain model for all subsequent assignments. The p.d. for sequences $s = (s_1, \dots, s_n)$ of length n with states $s_i \in K$ is given by:

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1}).$$

The transition probabilities $p(s_i \mid s_{i-1})$ and the p.d. $p(s_1)$ for the state of the first element are assumed to be known.

Assignment 1.
a) Suppose that the marginal p.d.s $p(s_i)$ for the states of the i-th element of the sequence are known for all $i = 2, \dots, n$. Then it is easy to calculate all “inverse“ transition probabilities $p(s_{i-1} \mid s_i)$. How?
b) Describe an efficient algorithm for calculating $p(s_i)$ for all $i = 2, \dots, n$.

Assignment 2. Suppose that there is a special state $k^* \in K$. We want to know how often this state appears on average in a sequence generated by the model. Describe an efficient method for the calculation of this average.
Hint: You may use the fact that the mean value of a sum of random variables is equal to the sum of their means. The number of occurrences of the state k^* in a sequence s can be obviously written in the form

$$\delta_{s_1 k^*} + \delta_{s_2 k^*} + \ldots + \delta_{s_n k^*}.$$

Assignment 3. Let $A \subset K$ be a subset of states and let $\mathcal{A} = A^n$ denote the set of all sequences s with $s_i \in A$ for all $i = 1, \dots, n$. Find an efficient algorithm for calculating the probability $p(\mathcal{A})$ of the event $\mathcal{A}$.

Assignment 4. Suppose that the set of states K is completely ordered ($k = 1, 2, \dots, m$). The matrix of transition probabilities $p(s_i = k \mid s_{i-1} = k')$ (we assume a homogeneous model) is given by

$$p(k \mid k') = \begin{cases} a & \text{if } k = k',\, k' \neq m, \\ b & \text{if } k = k' + 1,\, k' \neq m, \\ 1 & \text{if } k = k' = m, \\ 0 & \text{else,} \end{cases}$$

where $a, b > 0$ and $a + b = 1$. The probability $p(s_1)$ for the state of the first element is 1 for $k = 1$ and 0 else.
a) Calculate the probability $p(s_i = 1)$.
b*) Calculate the probabilities $p(s_i = k)$ for $k \neq 1$.

Assignment 5. Suppose that $|K| = 2$ and that the matrix of transition probabilities (we assume a homogeneous model) is given by

$$p(k \mid k') = \begin{cases} 1-\alpha & \text{if } k = k', \\ \alpha & \text{else.} \end{cases}$$

Verify that the chain is irreducible and aperiodic. Calculate the n-th power of the matrix of transition probabilities and the stationary (marginal) distribution.