The formal system for modal logics

and some facts we can rely on

The formal system $\mathbf{K}_n$ for of language with *n* agents:

1. **Propositional tautologies**
2. **Distribution Axiom** (denoted as **K**) $(K_i A \wedge K_i (A \rightarrow B)) \rightarrow K_i B$

Derivation rules:

R1. From the formulas $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \rightarrow \boldsymbol{\beta}$ derive $\boldsymbol{\beta}$ (**Modus Ponens**)

R2. From the formula $\boldsymbol{\alpha}$ derive $K_i\, \boldsymbol{\alpha}$ (**Knowledge Generalization**)

can be complemented by additional axioms corresponding to properties of the used admissibility relations - reflexivity, transitivity or symmetry

3. **Knowledge Axiom** (denoted as **T**) $K_i A \rightarrow A$ — r
4. **Positive Introspection Axiom** (den.as **4**) $K_i A \rightarrow K_i K_i A$ — t
5. **Negative Introspection Axiom** (den.as **5**) $\neg K_i A \rightarrow K_i \neg K_i A$ — s+t
6. **Consistency Axiom** (den.as **D**) $\neg K_i\, false$

Propositional logics: its formal system

Prop-Ax1: $\varphi \to (\psi \to \varphi)$

Prop-Ax2: $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))$

Prop-Ax3: $(\neg \psi \to \neg \varphi) \to (\varphi \to \psi)$

Modus ponens:

$$\frac{\varphi,\ (\varphi \to \psi)}{\psi}$$

Find tautologies among the following formulas

$(\varphi \to (\psi \to \tau)) \to (\psi \to (\varphi \to \tau))$

$(\alpha1 \to \beta) \to ((\alpha2 \to \beta) \to ((\alpha1 \vee \alpha2) \to \beta))$

$(\alpha \to \beta) \to (\neg \beta \to \neg \alpha)$

$\alpha \to (\alpha \to \tau)$

$\neg \alpha \vee \alpha$

$\alpha \to (\neg \alpha \to \tau)$

$((\alpha \to \beta) \to (\neg\alpha \vee \beta)) \,\&\, ((\neg\alpha \vee \beta) \to (\alpha \to \beta))$

$((\neg\alpha \vee \neg\beta) \to \neg(\alpha \,\&\, \beta)) \,\&\, (\neg(\alpha \,\&\, \beta) \to (\neg\alpha \vee \neg\beta))$

$\alpha \to (\alpha \vee \beta)$

$(\alpha \to \beta) \to ((\alpha \vee \varphi) \to (\beta \vee \varphi))$

Propositional logics: its formal system
and some of its useful tautologies

Prop-Ax1: $\varphi \rightarrow (\psi \rightarrow \varphi)$

Prop-Ax2: $(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$

Prop-Ax3: $(\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi)$

Modus ponens:

$$\frac{\varphi,\ (\varphi \rightarrow \psi)}{\psi}$$

Prop-T1: $(\varphi \rightarrow (\psi \rightarrow \tau)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \tau))$

Prop-T2: $(\alpha 1 \rightarrow \beta) \rightarrow ((\alpha 2 \rightarrow \beta) \rightarrow ((\alpha 1 \vee \alpha 2) \rightarrow \beta))$

Prop-T3: $(\alpha \rightarrow \beta) \rightarrow (\neg \beta \rightarrow \neg \alpha)$

Prop-T4a: $\neg$ false

Prop-T4b: $\neg \alpha \vee \alpha$

Prop-T4c: $\alpha \rightarrow (\neg \alpha \rightarrow \tau)$

Prop-T5: $((\alpha \rightarrow \beta) \rightarrow (\neg\alpha \vee \beta))\ \&\ ((\neg\alpha \vee \beta) \rightarrow (\alpha \rightarrow \beta))$

Prop-T6: $((\neg\alpha \vee \neg\beta) \rightarrow \neg(\alpha \ \&\ \beta))\ \&\ (\neg(\alpha \ \&\ \beta) \rightarrow (\neg\alpha \vee \neg\beta))$

Prop-T7a: $\alpha \rightarrow (\alpha \vee \beta)$

Prop-T7b: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \vee \varphi) \rightarrow (\beta \vee \varphi))$

We can prove the following formulas:

M-T1a: $\mathbf{K_n}$, $(\varphi \rightarrow \psi)$ |- $K_i\, \varphi \rightarrow K_i\, \psi$ (see the lecture)

M-T1b: Let φ ,ψ be two equiv. fomulas (ie. $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ is a tautology and this is denoted as $\varphi \equiv \psi$) then $\mathbf{K_n}$ |- $K_i\, \varphi \equiv K_i\, \psi$

M-T2a: $\mathbf{K_n}$ |- $K_i(\boldsymbol{\alpha}\ \&\ \boldsymbol{\beta}) \rightarrow K_i\, \boldsymbol{\alpha}$, **M-T2b:** $\mathbf{K_n}$ |- $K_i(\boldsymbol{\alpha}\ \&\ \boldsymbol{\beta}) \equiv K_i\, \boldsymbol{\alpha}\ \&\ K_i\, \boldsymbol{\beta}$
(see the lecture)

M-T3: $(\mathbf{K_n} + \mathbf{A3})$ |- $\neg K_i$ (false) **A6**

M-T4a: $(\mathbf{K2} + \mathbf{A3})$ |- $\neg K_i\, \boldsymbol{\alpha} \vee \neg K_i \neg K_i\, \boldsymbol{\alpha}$

M-T4b: $(\mathbf{K2} + \mathbf{A3})$ |- $\neg K_i(\boldsymbol{\alpha}\ \&\ \neg K_i\, \boldsymbol{\alpha})$

M-T5: $(\mathbf{K_n} + \mathbf{A6})$ |- $\neg (K_i\, \boldsymbol{\alpha}\ \&\ K_i \neg \boldsymbol{\alpha})$

M-T6: $\mathbf{K_n}$ |- $(K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})) \rightarrow (K_i \boldsymbol{p} \wedge K_i(\neg K_i \boldsymbol{p}))$

M-T7: In $(\mathbf{K_n} + \mathbf{A3})$, it is impossible to prove the formula $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$

M-T8: In $(\mathbf{K_n} + \mathbf{A3} + \mathbf{A5})$, $(K_i(\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ |- $K_i \boldsymbol{p} \vee K_i \neg \boldsymbol{p}$ and ***vice versa***

We know that $\mathbf{K}_n$, $(\alpha \to \beta) \vdash K_i\alpha \to K_i\beta$

$\mathbf{K}_n \vdash K_i(\alpha \,\&\, \beta) \equiv K_i\alpha \,\&\, K_i\beta$

Can one prove $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \equiv K_i\alpha \vee K_i\beta$?

Certainly not! Why?

Step 1. We can prove that $\mathbf{K}_n \vdash K_i\alpha \to K_i(\alpha \vee \beta)$ and $\mathbf{K}_n \vdash K_i\beta \to K_i(\alpha \vee \beta)$.
These too formulas can be combined using following prop. tautology

Prop-T2 : $(\alpha 1 \to \beta) \to ((\alpha 2 \to \beta) \to ((\alpha 1 \vee \alpha 2) \to \beta))$

and we get $\mathbf{K}_n \vdash (K_i\alpha \vee K_i\beta) \to K_i(\alpha \vee \beta)$

Step 2. To show that the inverse implication $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \to (K_i\alpha \vee K_i\beta)$ **is not valid** it is enough to construct simple Kripke structure with 3 states, where this formula is not valid.

M-T3: c1) K2, T(A3) $\vdash \neg K_i false$

1. $K_i false \rightarrow false$ [A3]
2. $\neg false \rightarrow (\neg K_i false)$ [eq.transf.of 1 based on Prop-T3]
3. $\neg false$ [Prop-T4]
4. $\neg K_i false$ [Modus Ponens: 2,3]

M-T4a: K2, T $\vdash \neg K_i \alpha \vee \neg K_i \neg K_i \alpha$

[eq.transf. based on Prop-T5 “replacement of $\rightarrow$ by $\vee$ “ in the axiom T $K_i (\neg K_i \alpha) \rightarrow \neg K_i \alpha$]

M-T4b: K2, T $\vdash \neg K_i (\alpha \, \& \, \neg K_i \alpha)$

1. $K_i \neg K_i \alpha \rightarrow \neg K_i \alpha$ [A3, the axiom T]
2. $\neg K_i \neg K_i \alpha \vee \neg K_i \alpha$ [eq.transf.of 1 based on Prop-T5]
3. $\neg (K_i \neg K_i \alpha \, \& \, K_i \alpha)$ [eq.transf.of 2 based on Prop-T6]
4. $\neg K_i (\neg K_i \alpha \, \& \, \alpha)$ [eq.transf.of 3 based on Mol-T2b], *viz* a2

Proof of M-T5: $(K_n + A6) \vdash \neg (K_i \alpha \wedge K_i \neg \alpha)$.

1. $(K_i \alpha \wedge K_i \neg \alpha) \equiv K_i(\alpha \wedge \neg \alpha)$ **M-T2b:**
2. $\neg K_i(\mathit{false})$ [A6]
3. $\mathit{false} \equiv (\alpha \wedge \neg \alpha)$ [tautology: A1]
4. $K_i\ \mathit{false} \equiv K_i(\alpha \wedge \neg \alpha)$ **M-T1b**
5. $\neg K_i(\alpha \wedge \neg \alpha)$ [MP for 2 and 4]
6. $\neg(K_i \alpha \wedge K_i \neg \alpha) \equiv \neg K_i(\alpha \wedge \neg \alpha)$ [taut.transcript of l. 1]
7. $\neg(K_i \alpha \wedge K_i \neg \alpha)$ [MP for 6 and 5]

M-T6: $\mathbf{K}_n \vdash (K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \;\&\; K_i (\neg K_i p))$

Crucial steps of the proof

a) $\neg (p \rightarrow K_i p) \equiv (p \wedge \neg K_i p)$ **[Taut]**

b) $K_i \neg (p \rightarrow K_i p) \equiv K_i(p \;\&\; \neg K_i p)$ **[M-T1b]**

c) $K_i(p \;\&\; \neg K_i p) \rightarrow (K_i p \;\&\; K_i (\neg K_i p))$ **[M-T2b]**

d) $(K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \;\&\; K_i (\neg K_i p))$ [lines b,c and transitivity of "$\rightarrow$"]

Mod_T7: The formula $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ is not provable in (K_n + Ax3).

Proof. Suppose $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ is provable. We could apply modus ponens to **Mod_T6** and $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ to prove the formula $K_i \boldsymbol{p}$ & $K_i (\neg K_i \boldsymbol{p})$. Let us assume that $K_i \boldsymbol{p}$ & $K_i (\neg K_i \boldsymbol{p})$ holds:

1. $K_i \boldsymbol{p}$ & $K_i (\neg K_i \boldsymbol{p})$	[assumption]
2. $K_i \boldsymbol{p}$	[1 and property of conjunction $(\alpha \,\&\, \beta) \rightarrow \alpha$]
3. $K_i (\neg K_i \boldsymbol{p})$	[1 and property of conjunction $(\alpha \,\&\, \beta) \rightarrow \beta$]
4. $K_i (\neg K_i \boldsymbol{p}) \rightarrow \neg K_i \boldsymbol{p}$	**[Ax3]**
5. $\neg K_i \boldsymbol{p}$	[Modus ponens: 3,4]
6. ***false***	[definition of ***false*** and the lines 2 and 5]

But the axiom system (K_n + Ax3) has a model – that is why it cannot be contradictory and the formula ***false*** cannot be provable there. What is wrong with our proof? It must be based a wrong assumption! There were assumed $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ and its direct consequence (obtained through **MP** and **Mod_T6**) the formula $K_i \boldsymbol{p}$ & $K_i (\neg K_i \boldsymbol{p})$.

The original assumption „the formula $K_i \neg (\boldsymbol{p} \rightarrow K_i \boldsymbol{p})$ is provable" must be wrong!.

Mod_T8a: K_n , Ax3, Ax 5, $K_i(p \to K_i p) \vdash K_i p \lor K_i \neg p$

1. $K_i(p \to K_i p)$ — assumption
2. $K_i p \lor \neg K_i p$ — Prop_T4b $\alpha \lor \neg \alpha$
3. $K_i p \to K_i p \lor K_i \neg p$ — Prop-T7a $\alpha \to (\alpha \lor \beta)$
4. $K_i(p \to K_i p) \to (p \to K_i p)$ — Ax 3
5. $p \to K_i p$ — Modus Ponens for 4 and 1
6. $(p \to K_i p) \to (\neg K_i p \to \neg p)$ — Prop_T3
7. $\neg K_i p \to \neg p$ — Modus Ponens for 6 and 5
8. $\neg K_i p \to K_i \neg K_i p$ — Ax 5
9. $K_i \neg K_i p \to K_i \neg p$ — M-T1 for the line 7
10. $\neg K_i p \to K_i \neg p$ — transitivity of "$\to$" for lines 8 and 9
11. $(\alpha \to \beta) \to (\gamma \lor \alpha \to \gamma \lor \beta)$ — Prop-T7b
12. $(K_i p \lor \neg K_i p) \to (K_i p \lor K_i \neg p)$ — Modus Ponens for lines 10,11 and $\gamma = K_i p$
13. $(K_i p \lor K_i \neg p)$ — Modus Ponens for lines 12 and 2

Mod_T8b: $\mathbf{K}_n$, Ax 4 ,$(K_i p \lor K_i \neg p) \vdash K_i(\boldsymbol{p} \to K_i \boldsymbol{p})$

1. $K_i \boldsymbol{p} \to (\boldsymbol{p} \to K_i \boldsymbol{p})$ — propositional axiom
2. $K_i K_i \boldsymbol{p} \to K_i(\boldsymbol{p} \to K_i \boldsymbol{p})$ — Mod_T1 for the line 1
3. $K_i \boldsymbol{p} \to K_i K_i \boldsymbol{p}$ — **Ax4**
4. $K_i \boldsymbol{p} \to K_i(\boldsymbol{p} \to K_i \boldsymbol{p})$ — transitivity of "$\to$" for the lines 3 a 2
5. $\neg \boldsymbol{p} \to (\boldsymbol{p} \to K_i \boldsymbol{p})$ — Prop_4c
6. $K_i \neg \boldsymbol{p} \to K_i(\boldsymbol{p} \to K_i \boldsymbol{p})$ — Mod_T1 for the line 5
7. $(\alpha 1 \to \beta) \to ((\alpha 2 \to \beta) \to ((\alpha 1 \lor \alpha 2) \to \beta))$ Prop-T2 for
   $\alpha 1 = K_i p$, $\alpha 2 = K_i \neg p$, $\beta = K_i(p \to K_i p)$
8. $(K_i p \lor K_i \neg p) \to K_i(p \to K_i p)$ — Modus ponens applied first to 7 and 4 and later to this formula with 6
9. $(K_i p \lor K_i \neg p)$ — assumption
10. $K_i(p \to K_i p)$ — Modus ponens applied to 8 and 10