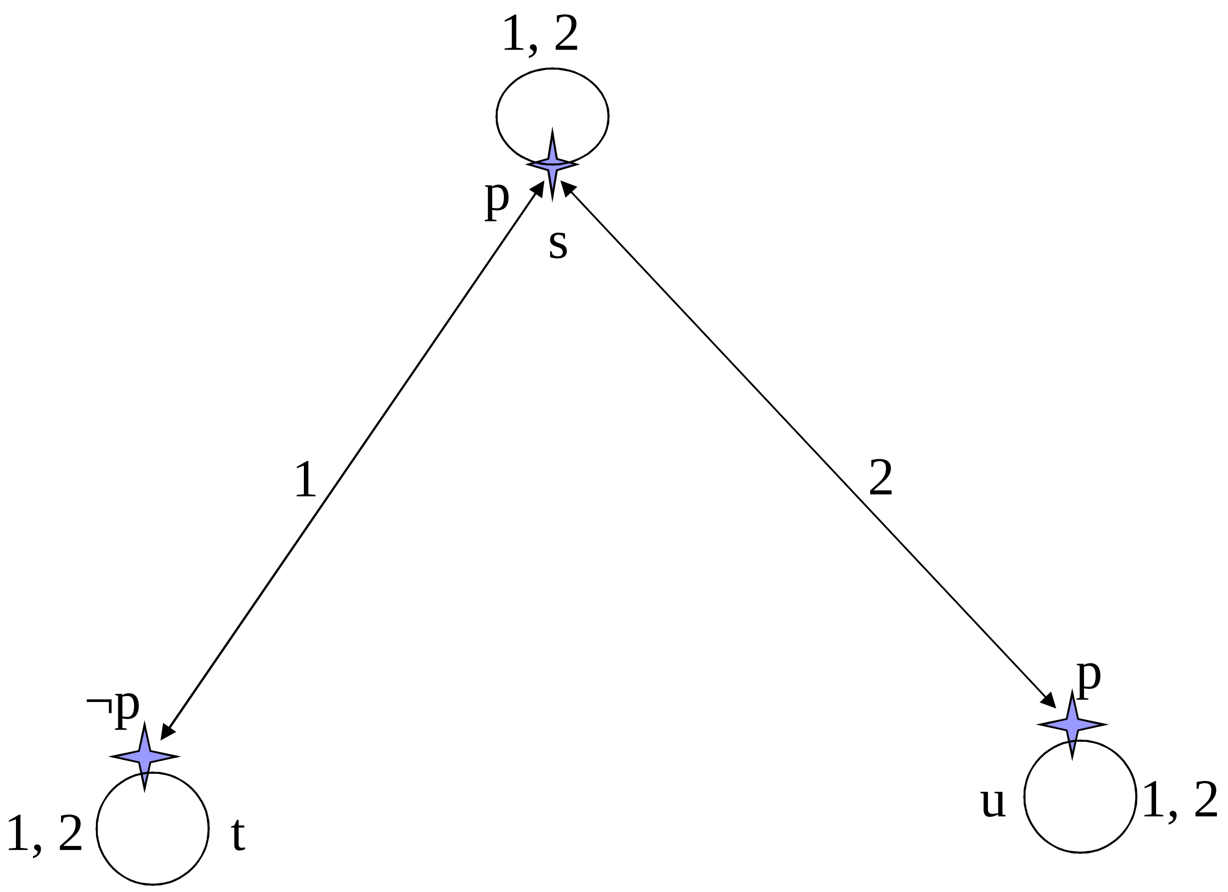
Knowledge in multi-agent systems

laboratory Gerstner


1, 2
p
s
1
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p
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t
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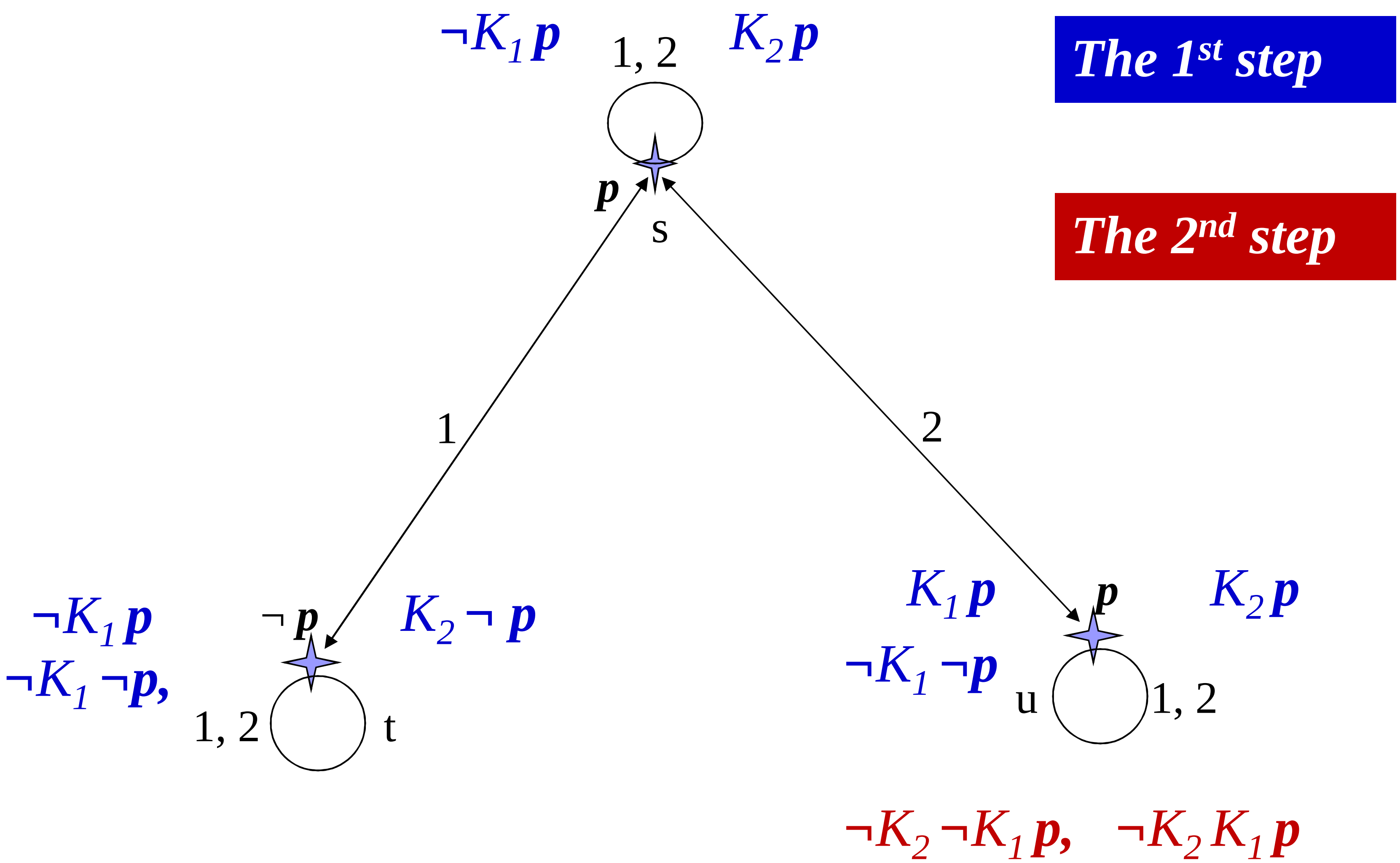
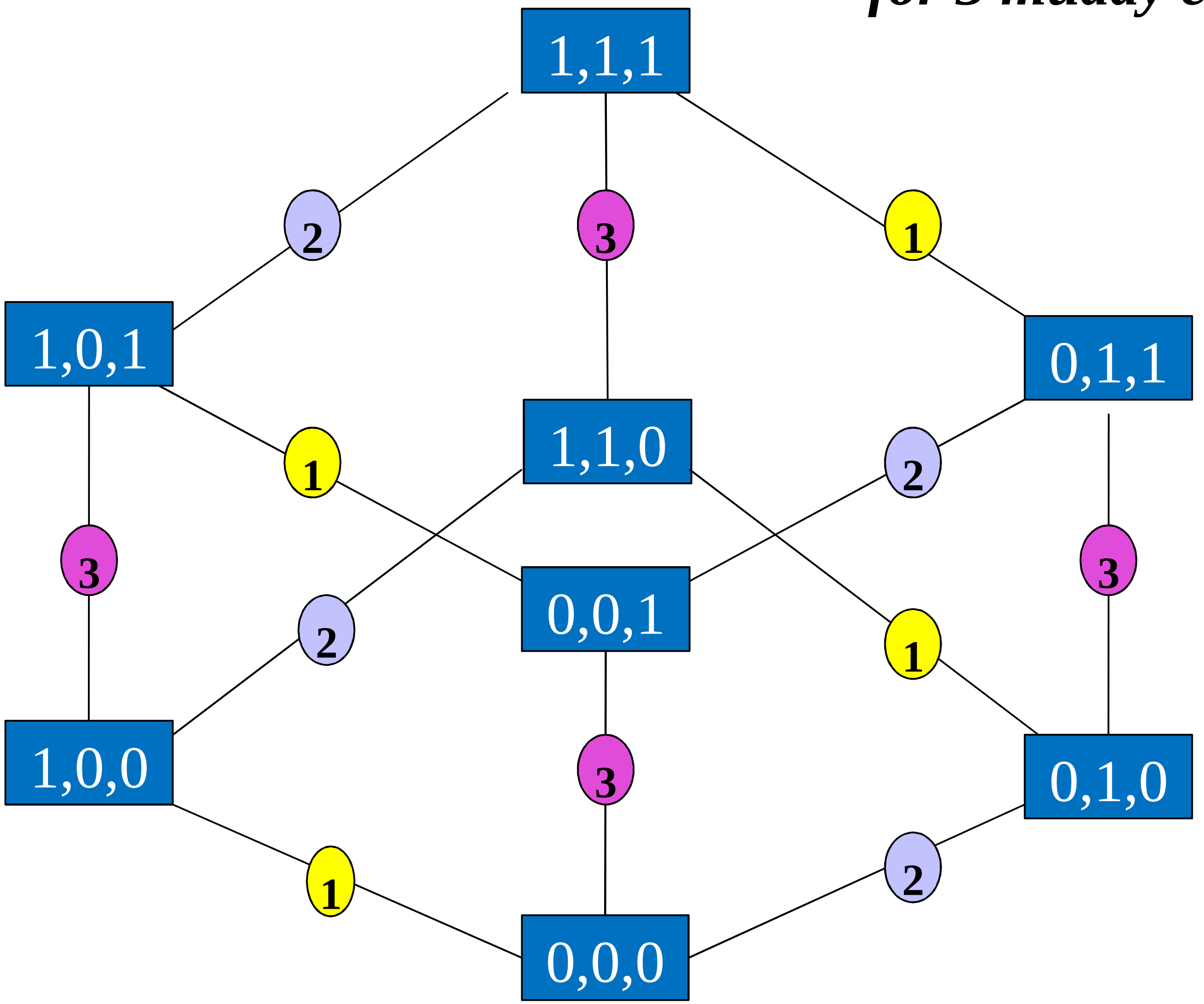


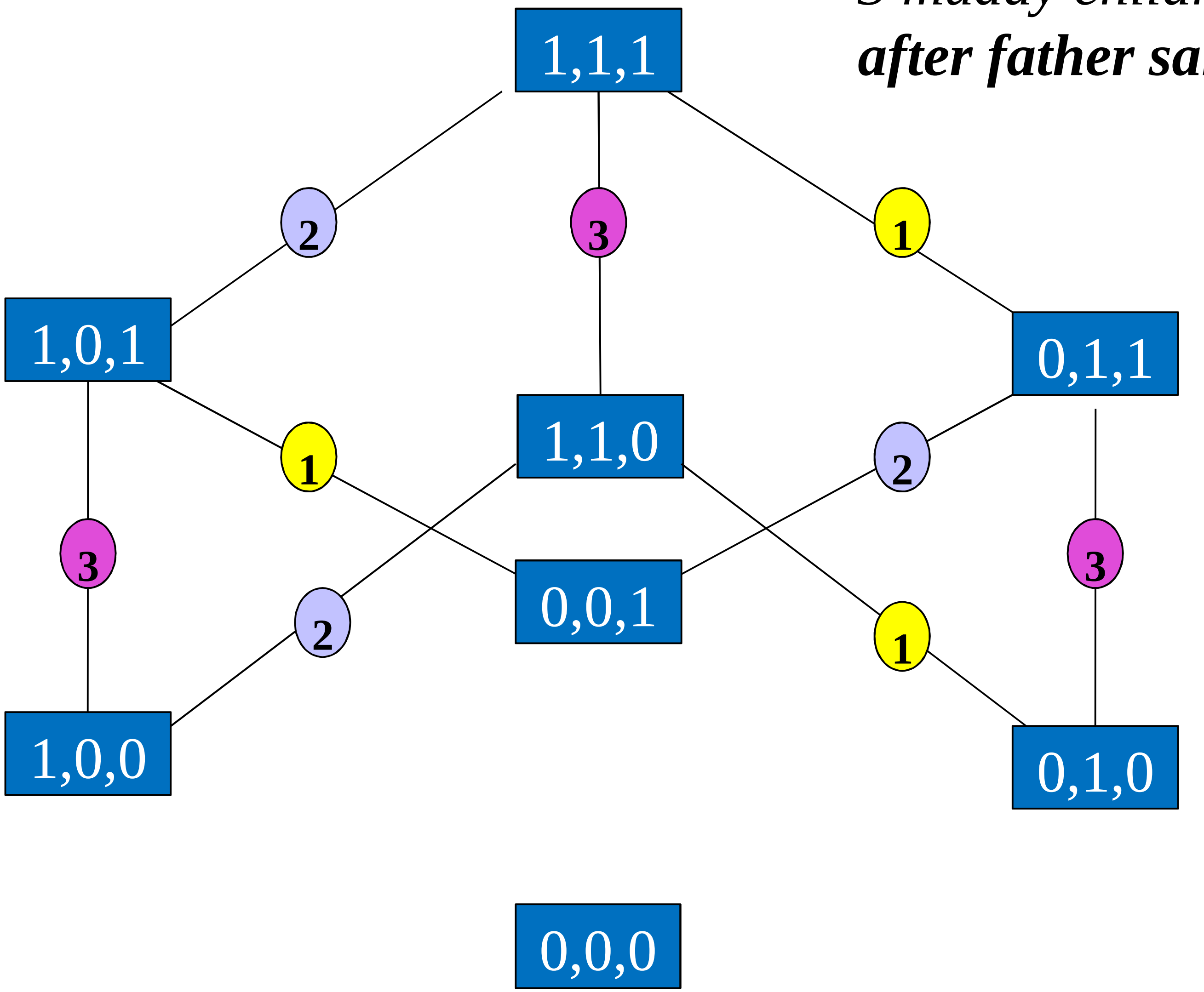
Caution! Can the formulas $\neg K_1 \neg p$ and $K_1 p$ **have the same meaning ?**

No! Compare their truth values in the states **t** and **u.**

Kripke's structure for 3 muddy children



Kripke's structure for
3 muddy children
after father said** **p



p = " I can see that there is someone dirty, here."

Example 1. Card game “Aces and nines”

3 players have a deck consisting of **4 ACEs** a **4 NINEs**. Each gets 2 cards, 2 remaining are left face down. None of the players looks at his/her cards - instead he/she raises them to his/her forehead so that **the others** can see them. All the players take turns trying to determine their own cards. If a player does not know his/her cards he/she must say so. The first, who announces “I know!” is the winner!

Given **4 ACEs + 4 NINEs,** each of the players **1,2,3** can have **NN, NA** or **AA.**

Round a)

1. Both the Player**1** and Player**2** say “I cannot determine my cards.”
2. The Player**3** can see, that **1AA** and **2NN**.
3. What will be the claim of the Player**3**?

Round b)

1. You are the Player**1** and you can see, that there holds **2NN** and **3AN**.
2. In the first turn no one was able to determine what he or she is holding. Now is your turn.
3. What will you announce?

Round c)

1. You are the Player**2** and you can see **1AN** and **3AN**.
2. In the first turn no one was able to determine what he or she is holding.
3. Player**1** cannot determine her cards at her second turn either.
4. What about you at your second turn ?

Having **4 ACEs** and **4 NINEs** each player **1,2** or **3** can hold one of the three possibilities **NN, AN** or **AA.**

Φ = {**1AA, 1AN, 1NN, 2AA, 2AN, 2NN, …**}

S = { (**AA-AA-NN**), (**AA-AN-AN**), (**AA-NN-AA**), …}

π((**AA-AA-NN**))(**2AA** & **3NN**) = *true*

π((**AA-AA-NN**))(**1NN**) = *false* ...

$M = (S, \pi, K_1, K_2, K_3)$

Which formula expresses the fact that the Player2 does not know his cards?

Např. K_2 (**2AA** ∨ **2AN** ∨ **2NN)** & ¬ K_2 **AA** & ¬ K_2 **AN** & ¬ K_2 **NN**

Example 2. Card game for 2 players and 3 cards A,B, C

$G = \{ 1, 2 \}$ players **1** and **2**

$c = \{ A, B, C\}$ three cards **A**, **B**, **C**

Primitive propositions $\mathbf{\Phi} = \{ 1A, 1B, 1C, 2A, 2B, 2C\}$

1A means "Player**1** holds the card **A",** …

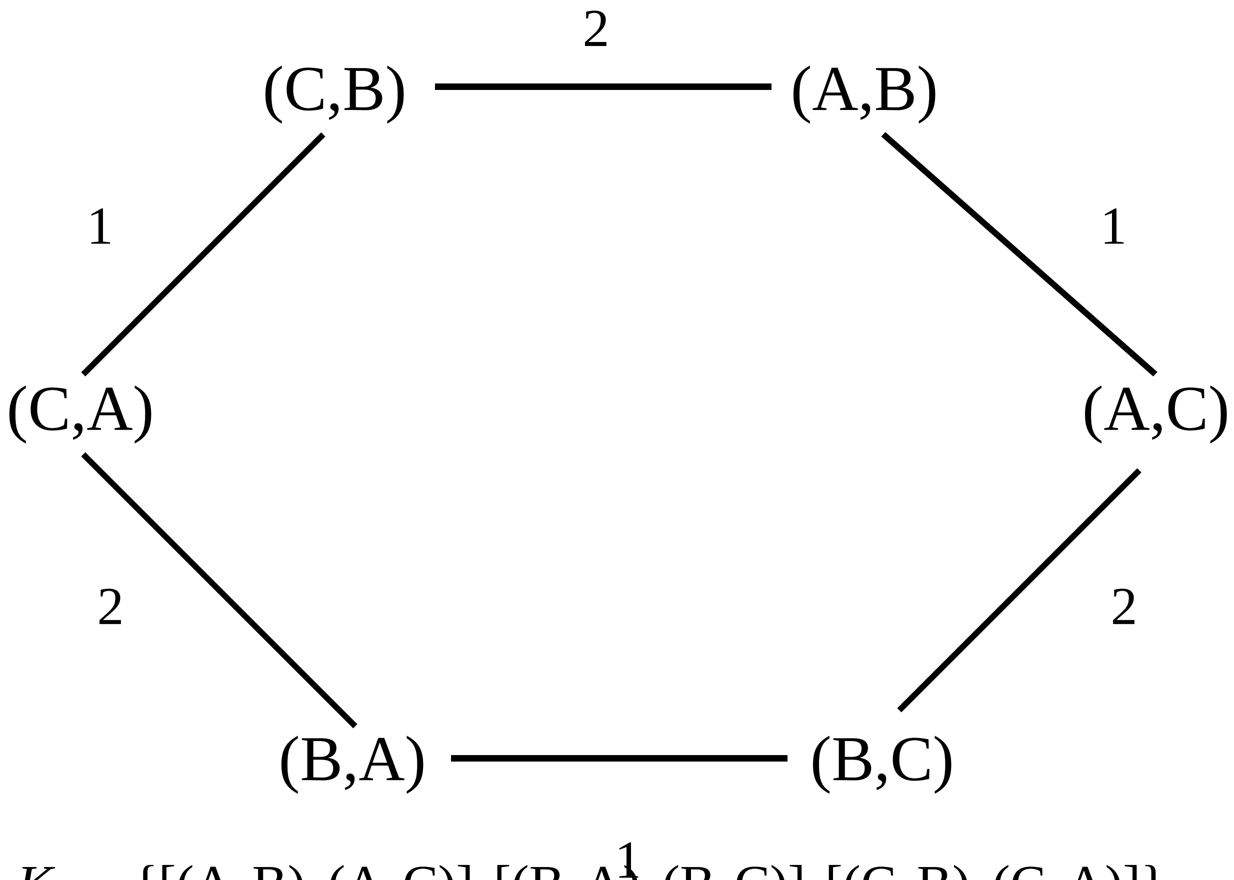
Possible states $\boldsymbol{S} = \{ (A,B), (A,C), (B, A), (B, C), (C, A), (C, B) \}$

(A,B): Player1 holds A and Player2 holds B, ...

$\pi((A, B))(1A) = \mathit{true}$ $\pi((A, B))(1B) = \mathit{false}$ ...

$M = (S, \pi, K_1, K_2)$

Let us denote as *M* the Kripke structure given by this graph:



$$K_1 = \{[(A,B),(A,C)],[(B,A),(B,C)],[(C,B),(C,A)]\}$$

$$K_2 = \{[(C,A),(B,A)],[(A,B),(C,B)],[(A,C),(B,C)]\}$$

This example points to the fact, that the Kripke structure has to include even states the agent does not consider as possible.

For example in the state (***A,B***) the *Player1* knows, that the state (***B,C***) is not possible. (*Player1* knows the card it holds, namely the card *A*.)

All over it *Player1* considers it possible, that *Player2* considers the state (***B,C***) as one of the alternative possibilities – it has to be included in the Kripkeho structure. How is this depicted in the graph? There is no edge labeled by ***1*** from (***A,B***) to (***B,C***).

> There is an edge labeled by 1 from (***A,B***) to (***A,C***), and an edge labeled by 2 from (***A,C***) to (***B,C***).

It is easy o verify that

$$(M,(A,B)) \models K_1(2B \vee 2C)$$

$$(M,(B,C)) \models K_2(2C) \wedge K_2(1A \vee 1B)$$

Can we verify more complex claims?

$$(M,(A,B)) \models C_G(1A \vee 1B \vee 1C)$$

$$(M,(A,B)) \models C_G(1B \rightarrow (2A \vee 2C))$$

$$(M,(A,B)) \models D_G(1A \wedge 2B)$$

Let $\boldsymbol{M} = (S, \pi, \ldots K_1, K_2, K_3, \ldots, K_n)$ be any Kripke structure such that any K_i of its possibility relations is equivalence.

Let $\boldsymbol{s} \in S$ be any of ***M***'s states. Verify, that for any formulas ***A***, ***B*** **there must hold**

i. $(\boldsymbol{M}, \boldsymbol{s}) \models (K_i \boldsymbol{A} \,\&\, K_i (\boldsymbol{A} \rightarrow \boldsymbol{B})) \rightarrow K_i \boldsymbol{B}$

ii. $(\boldsymbol{M}, \boldsymbol{s}) \models K_i \boldsymbol{A} \rightarrow \boldsymbol{A}$

iii. $(\boldsymbol{M}, \boldsymbol{s}) \models K_i \boldsymbol{A} \rightarrow K_i K_i \boldsymbol{A}$

iv. $(\boldsymbol{M}, \boldsymbol{s}) \models \neg K_i \boldsymbol{A} \rightarrow K_i (\neg K_i \boldsymbol{A})$

Let us define

$$(M,s) \models E_G A \quad \Leftrightarrow \quad (M,s) \models K_i A \ \textit{for all} \ i \in G$$

$$(M,s) \models C_G A \quad \Leftrightarrow \quad (M,s) \models E_G^k A \ \textit{for all} \ 1 \leq k$$

Both notions have an interesting graphical interpretation:

Let ***G*** be a nonempty set of agents. We say that the state ***t*** is ***G*-reachable** from the state ***s*** in $0 < \boldsymbol{k}$ steps, if there is a sequence of states

$$s \equiv s_0, s_1, \ldots, s_k \equiv t$$

Such that, for any $j, 0 \leq j < k$ there exists $i \in \boldsymbol{G}$ such that

$$(s_j, s_{j+1}) \in K_i.$$

We say that ***t*** is **G**-reachable from ***s***, if ***t*** is ***G***-reachable in finite number of steps.

Lemma.

$$(i)\ (M,s) \models E_G^k A \quad \Leftrightarrow \quad (M,t) \models A \quad \textit{for any } t, \ G-\textit{reachable in } k \textit{ steps}$$

$$(ii)\ (M,s) \models C_G A \quad \Leftrightarrow \quad (M,t) \models A \quad \textit{for any } t, \ G-\textit{reachable from } s.$$

Proof.

(i) By induction on k, (ii) is a consequence of (i).

Both claims are valid for any admissibility relations K_i (Here, there is no need to limit our attention to equivalence relations, because the proof does not require anything special from admissibility relations).