Knowledge in multi-agent systems

Olga Štěpánková

laboratory Gerstner

As a tradition, knowledge and reasoning used to be studied in case of a single individual (philosophers and logicians).

Is it sufficient for analysis of daily situations?

- NL discussion
- Business negotiation
- Decision about the next step in complex trafic situation ?

In all these cases we need to reason about interaction between agents.

Agents can be people, robots, complex computer systems, …, machines

Examples.

Mr. Bird and Mr. Ladybird.

Watergate case. *Dean does not know, if Nixon knows, that Dean knows, that Nixon knows, that McCord slipped in secretly into O'Brien's office ovy kanceláře ve Watergate.*

It is not easy to keep the track in such complex reasoning, namely if we are not familiar with the context.

When acting in real world, an agent must consider

- facts valid in the considered world,
- knowledge of his partners (other agents).

Common knowledge example

Often we assume that all actors in complex situations share meaning of some notions, e.g.

Each driver knows that red light means ”stop” and green “go”.

- Is it sufficient to feel safe on a crossroad?
- Consider “turning left”

To make traffic safe, we have to be sure, that everyone

- knows the rule (meaning of the signs)
- follows it
- **knows that all the others follow it as well**, …!

In many situations it is necessary to assume that all the following observations are true simultaneously:

- *Everyone knows the fact* **F**,
- *Everyone knows, that everyone knows the fact* **F**,
- *Everyone knows that, everyone knows, that everyone knows the fact* **F**,
- …

Such a fact **F** is referred to as **common knowledge**. It is a prerequisite for

- Meaningful discussion
- Rational decision making, …

Muddy children puzzle.

Imagine n children playing together. Their mother warned them that if they get dirty there will be severe consequences. During their play some of the children, say k of them exactly, get mud on their forehead.

Along comes their father, who says "*At least one of you has mud on your forehead.*"

Provided $k > 1$, this is no surprise for any child !

The father asks "*Does any of you know whether you have mud on your forehead*?"

over and over again.

Can the children come to some conclusion?

Assume that all the children are perceptive, intelligent, truthful and answer simultaneously. ***k*** dětí je ušmudaných.

Let us denote the father's claim "*At least one of you has mud on your forehead.*" by the symbol ***p*****.**

If $k > 1$, it may seem that father provides no new information. All over this information is useful - why?

Before the father says ***p***, no child can come to an answer to the question ***„Does any of you know if you have mud on your forehead?"***

By induction on k it can be proven that for every round of q questions where $q < k$, all the children have to answer NO.

THUS: ***k - 1*** times we will hear the answer NO and in the ***k***-th round all the children will answer YES.

Father mediates COMMON KMOWLEDGE !

Formal apparatus to work with knowledge

Kripke's idea of possible worlds semantics for modal logic:

- Besides the true state of affairs, there are a number of alternative states or "worlds" the agent can consider as possible.
- For example we cannot know what will be the weather tomorrow – we can exclude some states (minus 20^{o}C) but others have to be considered.

Definition. An agent *knows the fact* ***p*** , if ***p*** is true in all worlds the agent considers possible considering all information he has available.

Example.

Agent1 walks the streets of Prague, where it is sunny. He has no information about the weather in Berlin.

Thus

- **Agent1** has to consider only worlds where there is sunshine in Prague.
- But he can assume nothing about the sky in Berlin – it can be either gray or blue.

Agent1 knows in this case that there is sunny in Prague. But he does not know that there is sunny in Berlin.

Intuitive observation:

The number of possible worlds corresponds to vagueness!

The smaller is this number **(**of possible worlds the agent considers) the more accurate is his knowledge.

As soon as the agent gains some additional info from a reliable resource (e.g. it is sunny in Berlin), he can cancel all possible worlds contradicting the obtained fact.

We need **tools** that will help us **to do reasoning**.

Modal logics provides a language for such reasoning

Let us consider a group of ***n*** agents named *1, 2, . . . , n,* who want to reason in a context that can be described using a set primitive propositions **Φ** denoted as

$$p, p', q, q', \ldots$$

These primitive propositions express the basic facts about the intended context, e.g. „*it is raining in Prague*“, „*Mary has mud on her forehead*”.

To express the claim “*Karin knows that it is raining in Berlin*” we need to enrich the language by modal operators

$$K_1, K_2, \ldots, K_n,$$

(each agent *i* has his own operator K_i), where the expression K_n ***p*** is read as „ *the agent **i** knows **p***“. Further, we use connectives ¬ (*negation*) and conjunction **&** (often denoted **as** ∧)

Set of *Formulas* that can be constructed in the modal logics language.

$$p \in \Phi \quad \Rightarrow \quad p \in \boldsymbol{Formulas}$$

$$A, B \in \boldsymbol{Formulas} \quad \Rightarrow \quad \neg A, (A \wedge B) \in \boldsymbol{Formulas}$$

$$A \in \boldsymbol{Formulas} \text{ and } 1 \leq i \leq n \quad \Rightarrow \quad K_i A \in \boldsymbol{Formulas}$$

Standard abbreviations from propositional logics

$$A \vee B \quad \text{for} \quad \neg(\neg A \wedge \neg B)$$

$$A \rightarrow B \quad \text{for} \quad \neg A \vee B$$

$$A \leftrightarrow B \quad \text{for} \quad ((A \rightarrow B) \wedge (B \rightarrow A))$$

$$true \text{ for } p \vee \neg p \qquad false \text{ for } \neg true$$

Expressivity.

a) $$K_1 K_2 p \wedge \neg K_2 K_1 K_2 p$$

Agent1 knows, that **Agent2** knows ***p*** , but **Agent2** does not know, that **Agent1** knows, that **Agent2** knows ***p*** .

b) How do we express **possibility**?

It is understood as a **dual notion to knowldege**.

Agent1 considers ***A*** to be possible if the Agent 1 does not know $\neg$ ***A*** for sure, **ie**.

$$\neg K_1 \neg A$$

Semantics of modal logics

Kripke's semantics of possible worlds.

Kripke's structure M for n agents over a set $\mathbf{\Phi}$ of primitive propositions is the $(n+2)$-tuple $(S, \pi, K_1, K_2, \dots, K_n)$

- ***S*** is a **set** of all worlds (that can be considered in the given context) or **states**,
- $\boldsymbol{\pi}$ is interpretation of states corresponding to a truth function evaluating all primitive propositions from $\mathbf{\Phi}$ for each state ***s*** separately, ie. $\pi(s) : \Phi \to \{true,\ false\}$
- binary possibility relations $K_1, K_2, \dots, K_n$, on ***S*** interpreting the modal operators (relation K_i connects those pair of states the agent ***i*** considers to be possible alternatives)

Let us assume that the **possibility relations** have the properties of **equivalence**, namely

$$(s,t) \in K_i \quad \Leftrightarrow \quad (t,s) \in K_i$$

Suppose the agent i considers in the state s the state t possible. What does it mean?

The agent i has in both states s and t the same sets of possible alternative worlds (due to symmetry and tranzitivity).

Both sets s and t are for the agent i in this case undistinguishable!

Possible worlds semantics. Given a formula $\boldsymbol{A}$**,** we are interested in the notion **$(M, s) \models A$** , that is read as

- „the formula A is true in the structure M and the state s" or
- "A holds in (M, s)"nebo "(M, s) satisfies A ".

$(M, s) \models A$ is defined by **induction by the structure** of **A :**

$$(i)\quad (M,s) \models p \quad \text{iff} \quad \pi(s)(p) = true \;\; \{p \in \Phi\}$$

$$(ii)\quad (M,s) \models \neg A \quad \text{iff} \quad (M,s) \not\models A$$

$$(iii)\quad (M,s) \models A \wedge B \quad \text{iff} \quad (M,s) \models A \text{ a } (M,s) \models B$$

$$(iv)\quad (M,s) \models K_i A \quad \text{iff} \quad (M,t) \models A$$

$$\text{for all } t, (s,t) \in K_i$$

Kripke's structure can be depicted as a **labeled oriented graph**:

Its nodes are the states ***s*** from ***S***. Each node ***s*** is labeled by the set of primitive propositions that are true in the state ***s*.**

Oriented edges are labeled by sets of the agents as follows:

An edge from the node ***s*** to ***t*** is labeled by the index ***i*** iff the possibility relation K_i of the agent ***i*** contains the pair (***s***, ***t***).

Example.

Let us consider $\Phi = \{p\}$, $n = 2$ and the following Kripke structure

$$M = (S, \pi, K_1, K_2)$$

(i) $S = \{s, t, u\}$

(ii) $\boldsymbol{p}$ is true in the states s and u , not in t, ie.

$$\pi(s)(\boldsymbol{p}) = \pi(u)(\boldsymbol{p}) = true \quad \text{a} \quad \pi(t)(\boldsymbol{p}) = false$$

(iii) The agent1 cannot distinguish the state s from t , ie.

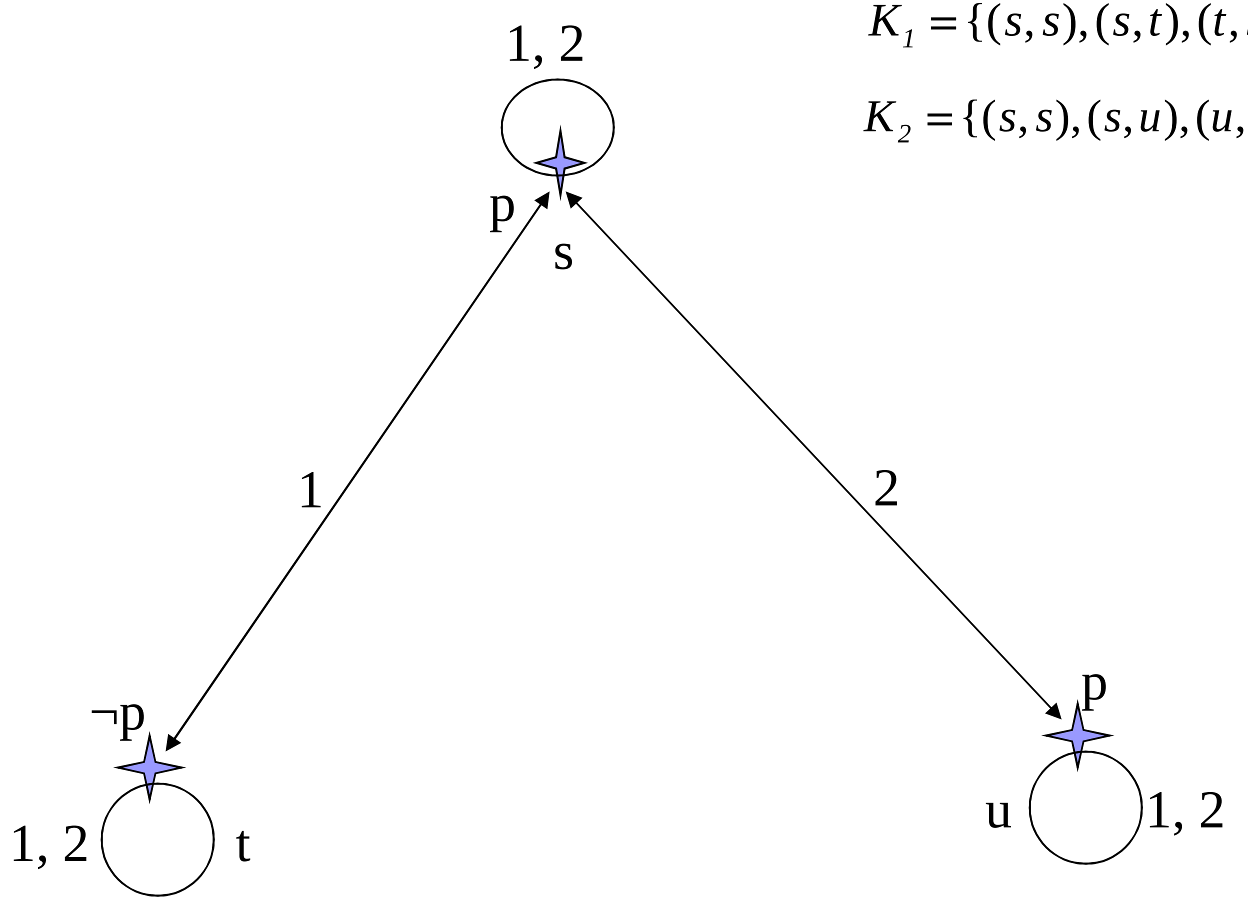
$$K_1 = \{(s,s),(s,t),(t,s),(t,t),(u,u)\}$$

For the agent2 theer holds $K_2 = \{(s,s),(s,u),(u,s),(t,t),(u,u)\}$

$\pi(s)(\boldsymbol{p}) = \pi(u)(\boldsymbol{p}) = true$ a $\pi(t)(\boldsymbol{p}) = false$

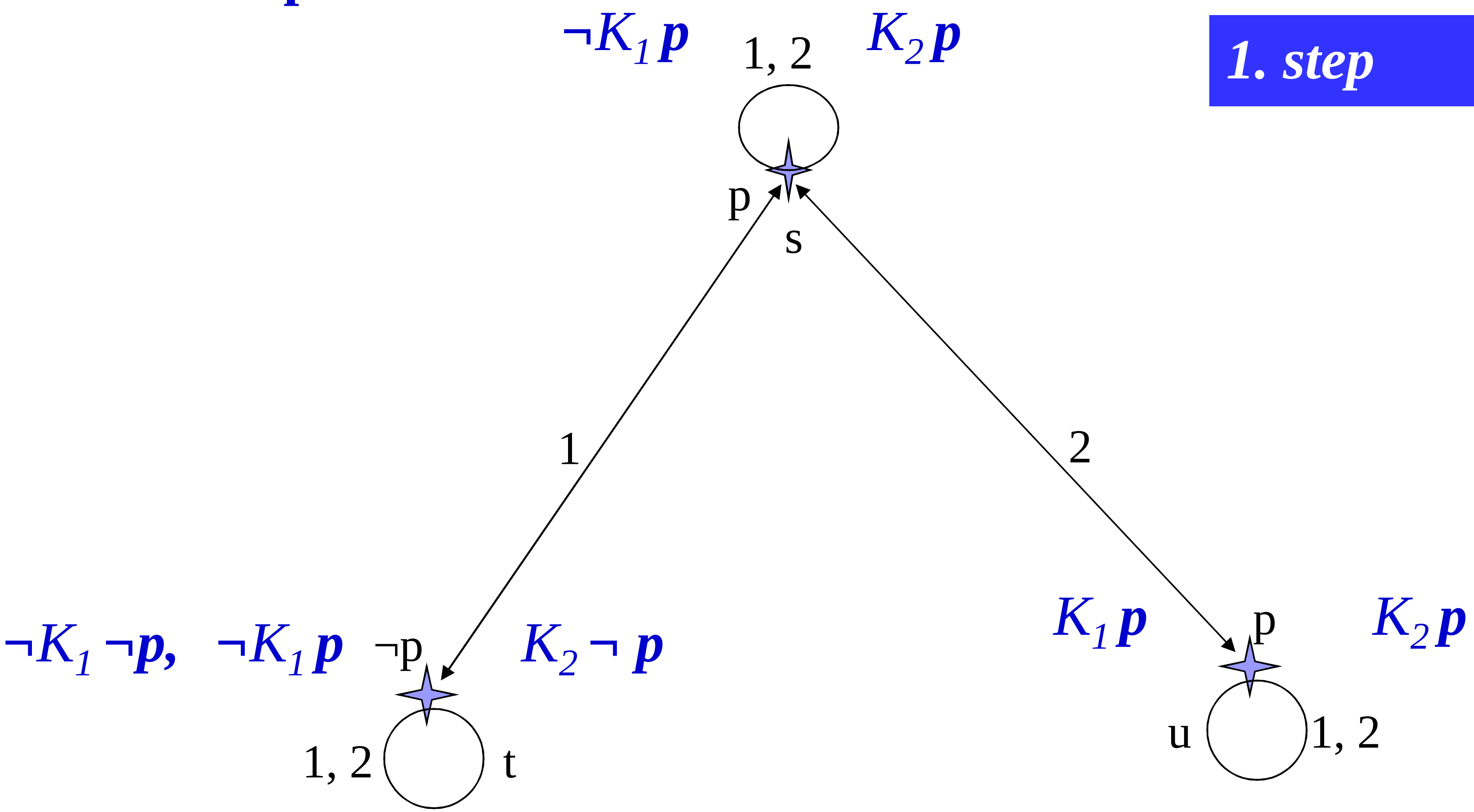
$K_1 = \{(s,s),(s,t),(t,s),(t,t),(u,u)\}$

$K_2 = \{(s,s),(s,u),(u,s),(t,t),(u,u)\}$

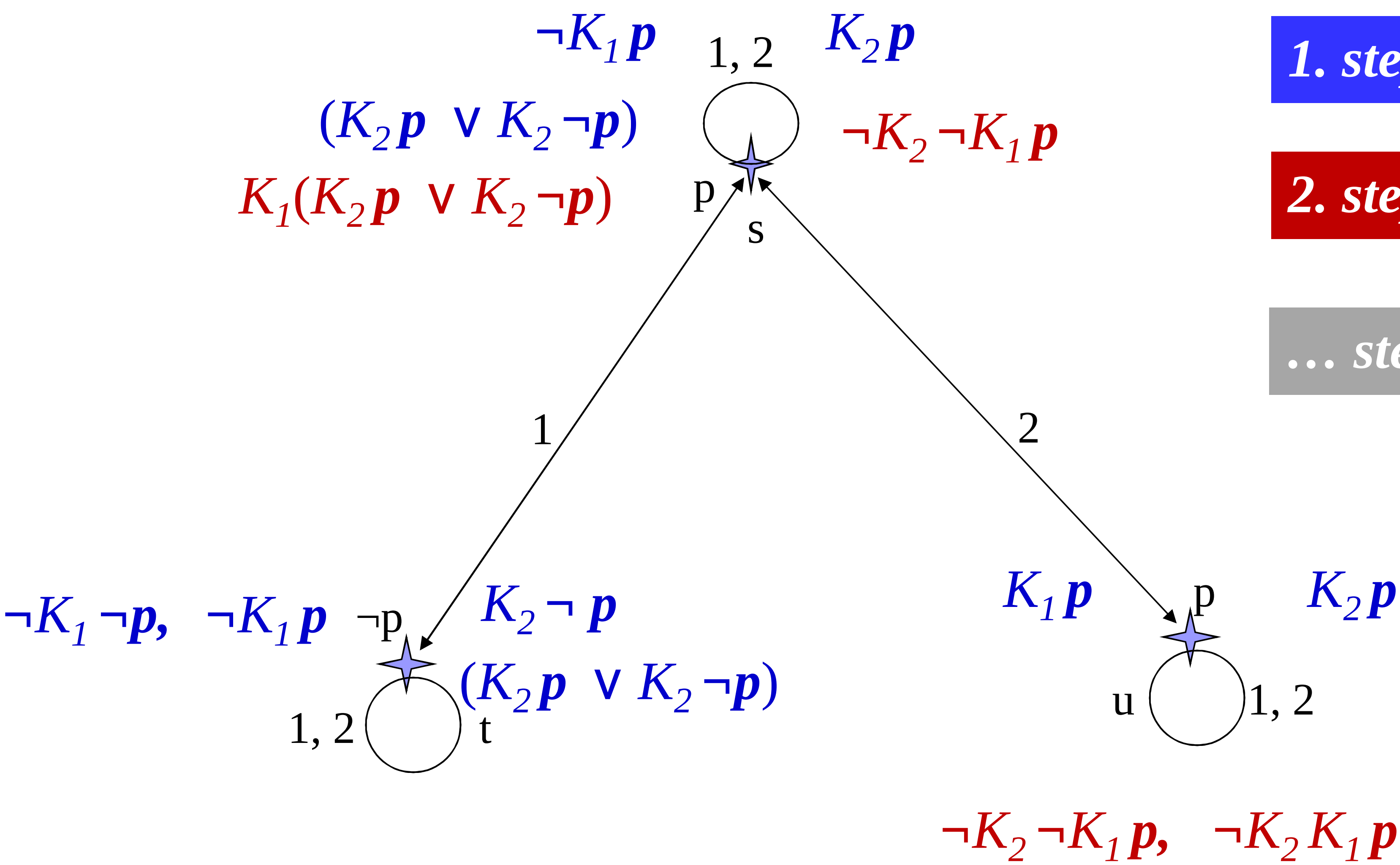


Truth values of some compound formulas **containing just 1 modal operator**

1. step



$$(M,s) \models p \wedge \neg K_1 p \wedge K_2 p \wedge K_1(K_2 p \vee K_2 \neg p) \wedge \neg K_2 \neg K_1 p$$

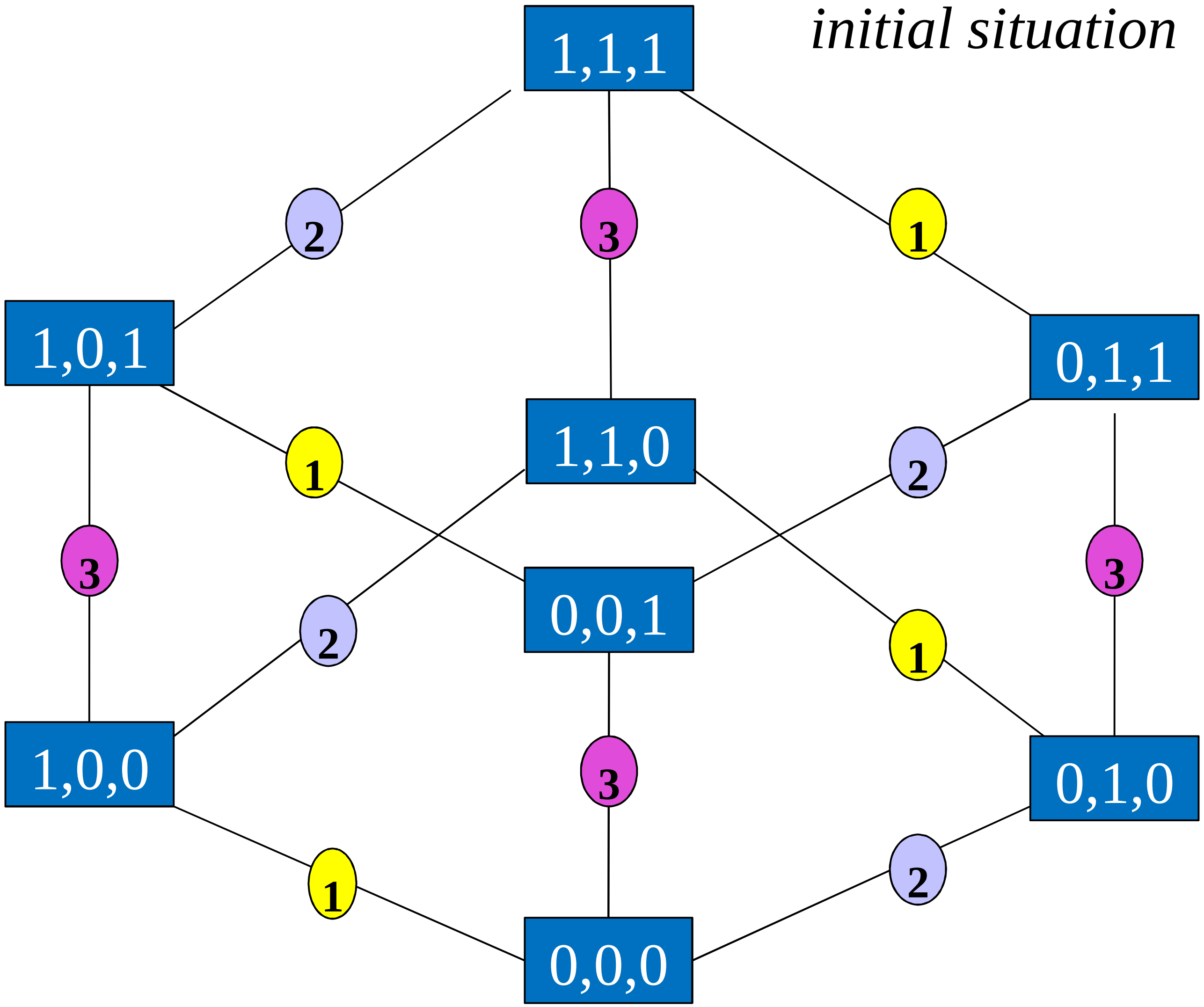


1. step

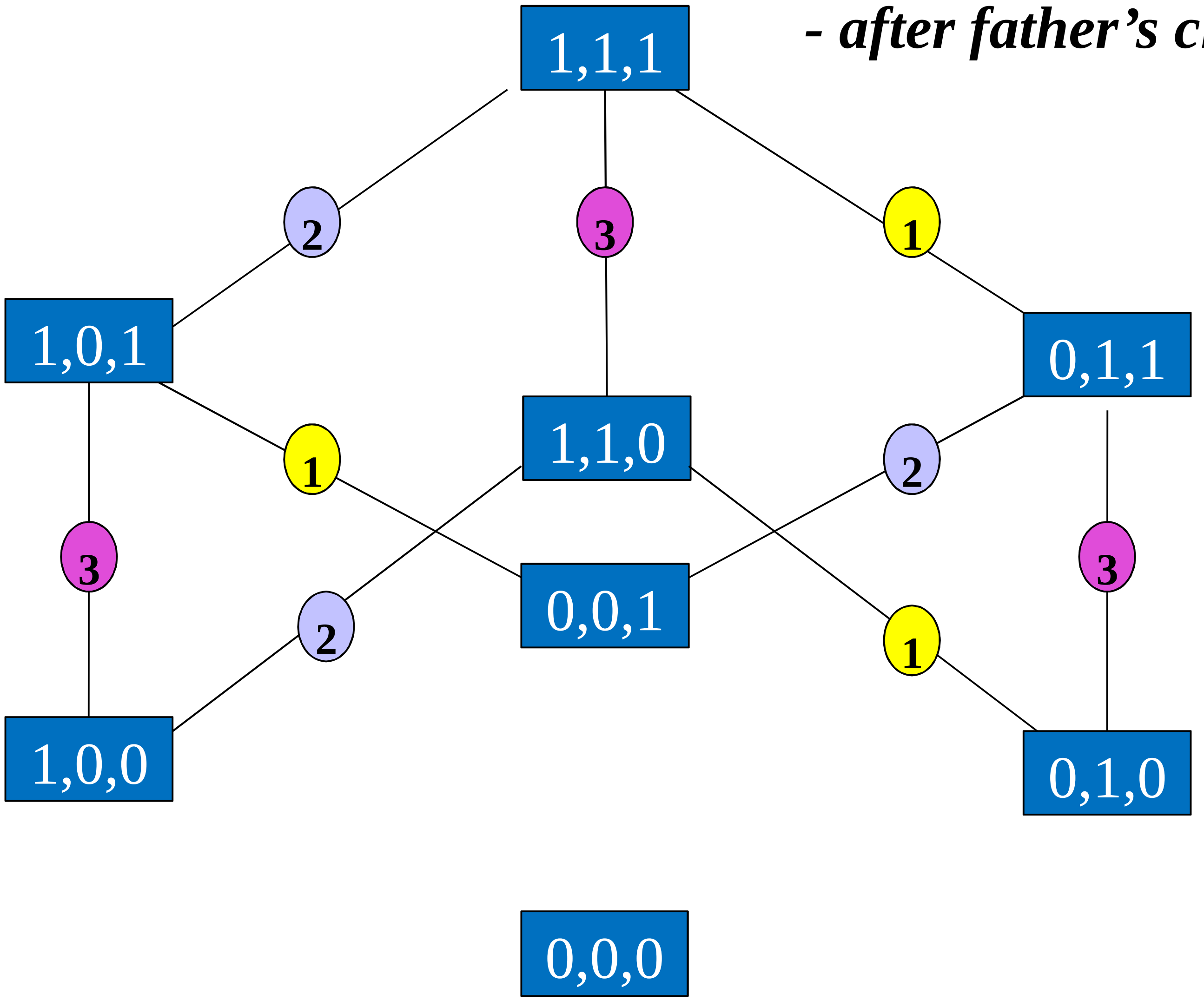
2. step

… steps

Kripke structure for 3 muddy children –
initial situation



Kripke structure for 3 muddy children - after father's claim p



Common and distributed knowledge

To express these notions three additional modal operators are introduced for any nonempty subset ***G*** of the set of all agents {1,2, …. *n*}, namely

E_G {"*everyone in the group G knows*"}

C_G {"*it is a common knowledge among the agents in G*"}

D_G {"*it is a distributed knowledge among the agents in G*"}

If *A* is a formula, then $\boldsymbol{E_G A}$ **,** $\boldsymbol{C_G A}$ and $\boldsymbol{D_G A}$ are formulas as well.

Examples of formulas with operators $\boldsymbol{E_G}$, $\boldsymbol{C_G}$ and $\boldsymbol{D_G A}$ and their reading

$K_3 \neg C_{[1,2]} p$ Agent3 knows, that p is not common knowledge between the agents 1 a 2.

$D_G q \wedge \neg C_G q$ q is distributed knowledge of agents in the group $\boldsymbol{G}$, but it is not a common knowledge there.

In order to define semantics of these operators we have to introduce (nested) iteration of the operator $\boldsymbol{E_G}$:

$$E_G^0 A \equiv A \qquad\qquad E_G^{n+1} A \equiv E_G E_G^n A$$

Let us define

$$(M,s) \models E_G A \quad \Leftrightarrow \quad (M,s) \models K_i A \ \textit{for all} \ \ i \in G$$

$$(M,s) \models C_G A \quad \Leftrightarrow \quad (M,s) \models E_G^k A \ \textit{for all} \ 1 \le k$$

Both notions have an interesting graphical interpretation:

Let $\boldsymbol{G}$ be a nonempty set of agents. We say that the state $\boldsymbol{t}$ is ***G*-reachable** from the state ***s*** in $0 < \boldsymbol{k}$ steps, if there is a sequence of states

$$s \equiv s_0, s_1, \ldots, s_k \equiv t$$

Such that, for any $j, 0 \le j < k$ there exists $i \in \boldsymbol{G}$ such that

$$(s_j, s_{j+1}) \in K_i.$$

We say that $\boldsymbol{t}$ is **G**-reachable from ***s***, if $\boldsymbol{t}$ is ***G***-reachable in finite number of steps.

Lemma.

$$(i)\ (M,s) \models E_G^k A \quad \Leftrightarrow \quad (M,t) \models A \quad \text{for any } t\text{, that is } G-\text{reachable in } k \text{ steps from } s$$

$$(ii)\ (M,s) \models C_G A \quad \Leftrightarrow \quad (M,t) \models A \quad \text{for any } t \text{ that is } G-\text{reachable from } s.$$

Proof.

(i) Can be proved by induction on k, (ii) is a consequence of (i).

Both claims are valid for any possibility relations K_i - no specific property (e.g. equivalence) is requested from K_i .

Let us denote by $\boldsymbol{p}$ the statement „one of the children is muddy“

If we consider just the set of 2 children ($k = 2$) and the situation when both children are muddy. It is not hard to show, that each child is in a state, where it is true that „this child knows p“, but the claim „everyone knows that everyone knows p“ is not true.

Similarly, if $k = 3$ and all are muddy, the claim „everyone knows, that everyone knows p“ holds. But the following statement does not hold „ everyone knows, that everyone knows, that everyone knows p“

Excercise. Suppose there are exactly k muddy children. Before the father´s first claim each child is in a state where $\boldsymbol{E}^{(k-1)}\boldsymbol{p}$ holds but where $\boldsymbol{E}^{k}\boldsymbol{p}$ does not hold.

Wise men puzzle

There are three wise men. It is common knowledge that there are three red hats and two white hats. The king puts a hat on the head of each of the three wise men and asks them sequentially if they know the color of the hat on their head.

- **The wise man1** says that he does not know.
- **The wise man2** says that he does not know.
- **The wise man3** says that he knows.

a) What color is the third wise man's hat?

b) We have implicitly assumed that all the wise men can see. Suppose that the wise man3 is blind and it is common knowledge that the others can see. How will answer the wise man3 in this case?

Recommended resources

- Ronald Fagin, Joseph Y. Halpern, Yoram Moses, Moshe Y. Vardi: *Reasoning About Knowledge*, MIT Pres 1995, 2003
- Chapter 8 in the volume *Umělá inteligence*(6), Academia 2012