►Gauge Freedom

1. The external frame is not fixed: See Projective Reconstruction Theorem →130

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.
   - $\mathbf{P}$ is 12 numbers for 11 parameters
   - we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
   - but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints

2a. either imposing constraints on projective entities
- cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
- points, e.g. $\|\underline{\mathbf{X}}_i\|^2 = 1$ — this way we can represent points at infinity

2b. or using minimal representations
- points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
- rotation matrix can be represented by axis-angle or the Cayley transform — see next

Implementing Simple Constraints

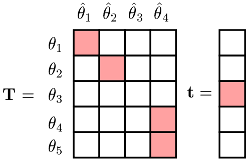
What for?

1. fixing external frame as in $\theta_i = \mathbf{t}_i$ 'trivial gauge'
2. representing additional knowledge as in $\theta_i = \theta_j$ e.g. cameras share calibration matrix $\mathbf{K}$

Introduce reduced parameters $\hat{\theta}$ and replication matrix $\mathbf{T}$:

$$\theta = \mathbf{T}\,\hat{\theta} + \mathbf{t}, \quad \mathbf{T} \in \mathbb{R}^{p,\hat{p}}, \quad \hat{p} \le p$$

then $\mathbf{L}_r$ in LM changes to $\mathbf{L}_r\,\mathbf{T}$ and everything else stays the same →107



these $\mathbf{T}$, $\mathbf{t}$ represent	
$\theta_1 = \hat{\theta}_1$	no change
$\theta_2 = \hat{\theta}_2$	no change
$\theta_3 = t_3$	constancy
$\theta_4 = \theta_5 = \hat{\theta}_4$	equality

- $\mathbf{T}$ deletes columns of $\mathbf{L}_r$ that correspond to fixed parameters it reduces the problem size
- consistent initialisation: $\theta^0 = \mathbf{T}\,\hat{\theta}^0 + \mathbf{t}$ or filter the init by pseudoinverse $\theta^0 \mapsto \mathbf{T}^\dagger \theta^0$
- no need for computing derivatives for θ_j corresponding to all-zero rows of $\mathbf{T}$ fixed θ

- constraining projective entities →145–146
- more complex constraints tend to make normal equations dense
- implementing constraints is safer than explicit renaming of the parameters, gives a flexibility to experiment
- other methods are much more involved, see [Triggs et al. 1999]
- **BA resource:** http://www.ics.forth.gr/~lourakis/sba/ [Lourakis 2009]

Matrix Exponential

- for any square matrix we define

$$\operatorname{expm} \mathbf{A} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \qquad \text{note: } \mathbf{A}^0 = \mathbf{I}$$

- some properties

$$\operatorname{expm} \mathbf{0} = \mathbf{I}, \quad \operatorname{expm}(-\mathbf{A}) = \left(\operatorname{expm} \mathbf{A}\right)^{-1}, \quad \operatorname{expm}(\mathbf{A} + \mathbf{B}) \neq \operatorname{expm}(\mathbf{A}) \operatorname{expm}(\mathbf{B})$$

$\operatorname{expm}(\mathbf{A}^\top) = (\operatorname{expm} \mathbf{A})^\top$ hence if $\mathbf{A}$ is skew symmetric then $\operatorname{expm} \mathbf{A}$ is orthogonal:

$$\left(\operatorname{expm}(\mathbf{A})\right)^\top = \operatorname{expm}(\mathbf{A}^\top) = \operatorname{expm}(-\mathbf{A}) = \left(\operatorname{expm}(\mathbf{A})\right)^{-1}$$

$$\operatorname{expm}(a\,\mathbf{A}) \operatorname{expm}(b\,\mathbf{A}) = \operatorname{expm}((a+b)\mathbf{A}), \quad \det \operatorname{expm} \mathbf{A} = \operatorname{expm}(\operatorname{tr} \mathbf{A})$$

Ex:

- homography can be represented via exponential map with 8 numbers e.g. as

$$\mathbf{H} = \operatorname{expm} \mathbf{Z} \quad \text{such that} \quad \operatorname{tr} \mathbf{Z} = 0, \text{ eg. } \mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & -(z_{11} + z_{22}) \end{bmatrix}$$

►Minimal Representations for Rotation

- $\mathbf{o}$ – rotation axis, $\|\mathbf{o}\| = 1$, φ – rotation angle
- **wanted**: simple mapping to/from rotation matrices

1. Matrix exponential. Let $\boldsymbol{\omega} = \varphi\,\mathbf{o}$, $0 < \varphi < \pi$, then

$$\mathbf{R} = \operatorname{expm}\,[\boldsymbol{\omega}]_\times = \sum_{n=0}^{\infty} \frac{[\boldsymbol{\omega}]_\times^n}{n!} \overset{\circledast 1}{=\cdots=} \mathbf{I} + \frac{\sin\varphi}{\varphi}\,[\boldsymbol{\omega}]_\times + \frac{1-\cos\varphi}{\varphi^2}\,[\boldsymbol{\omega}]_\times^2$$

   - for $\varphi = 0$ we take the limit and $\mathbf{R} = \mathbf{I}$
   - this is the Rodrigues' formula for rotation
   - inverse (the principal logarithm of $\mathbf{R}$) from

$$0 \le \varphi < \pi, \quad \cos\varphi = \frac{1}{2}(\operatorname{tr}\mathbf{R} - 1), \quad [\boldsymbol{\omega}]_\times = \frac{\varphi}{2\,\sin\varphi}(\mathbf{R} - \mathbf{R}^\top),$$

2. Cayley's representation; let $\mathbf{a} = \mathbf{o}\tan\frac{\varphi}{2}$, then

$$\mathbf{R} = (\mathbf{I} + [\mathbf{a}]_\times)(\mathbf{I} - [\mathbf{a}]_\times)^{-1}, \quad [\mathbf{a}]_\times = (\mathbf{R} + \mathbf{I})^{-1}(\mathbf{R} - \mathbf{I})$$

$$\mathbf{a}_1 \circ \mathbf{a}_2 = \frac{\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_1 \times \mathbf{a}_2}{1 - \mathbf{a}_1^\top \mathbf{a}_2} \qquad \text{composition of rotations } \mathbf{R} = \mathbf{R}_1\mathbf{R}_2$$

   - again, cannot represent rotations for $\phi \ge \pi$
   - no trigonometric functions
   - explicit composition formula

▶Minimal Representations for Other Entities

with the help of rotation we can minimally represent

1. fundamental matrix
$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top, \quad \mathbf{D} = \mathrm{diag}(1, d^2, 0), \quad \mathbf{U}, \mathbf{V} \text{ are rotations}, \quad 3 + 1 + 3 = 7 \text{ DOF}$$
2. essential matrix
$$\mathbf{E} = [-\mathbf{t}]_\times \mathbf{R}, \quad \mathbf{R} \text{ is rotation}, \quad \|\mathbf{t}\| = 1, \qquad 3 + 2 = 5 \text{ DOF}$$
3. camera
$$\mathbf{P} = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}, \qquad 5 + 3 + 3 = 11 \text{ DOF}$$

Interestingly, let [Eade 2017]

$$\mathbf{B} = \begin{bmatrix} [\boldsymbol{\omega}]_\times & \mathbf{u} \\ \mathbf{0}^\top & 0 \end{bmatrix}, \qquad \mathbf{B} \in \mathbb{R}^{4,4}$$

then, assuming $\|\boldsymbol{\omega}\| = \phi > 0$ — for $\phi = 0$ we take the limits

$$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \operatorname{expm} \mathbf{B} = \mathbf{I}_4 + \mathbf{B} + h_2(\phi)\,\mathbf{B}^2 + h_2(\phi)\,\mathbf{B}^3 = \begin{bmatrix} \operatorname{expm} [\boldsymbol{\omega}]_\times & \mathbf{V}\,\mathbf{u} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{V} = \mathbf{I}_3 + h_2(\phi)\,[\boldsymbol{\omega}]_\times + h_3(\phi)\,[\boldsymbol{\omega}]_\times^2, \quad \mathbf{V}^{-1} = \mathbf{I}_3 - \frac{1}{2}[\boldsymbol{\omega}]_\times + h_4(\phi)[\boldsymbol{\omega}]_\times^2$$

$$h_1(\phi) = \frac{\sin\phi}{\phi}, \quad h_2(\phi) = \frac{1 - \cos\phi}{\phi^2}, \quad h_3(\phi) = \frac{\phi - \sin\phi}{\phi^3}, \quad h_4(\phi) = \frac{1}{\phi^2}\left(1 - \frac{1}{2}\,\phi \cot\frac{\phi}{2}\right)$$

Part VII

Stereovision

7.1 Introduction
7.2 Epipolar Rectification
7.3 Binocular Disparity and Matching Table
7.4 Image Similarity
7.5 Marroquin's Winner Take All Algorithm
7.6 Maximum Likelihood Matching
7.7 Uniqueness and Ordering as Occlusion Models

mostly covered by

Šára, R. How To Teach Stereoscopic Vision. Proc. ELMAR 2010 referenced as [SP]

additional references

C. Geyer and K. Daniilidis. Conformal rectification of omnidirectional stereo pairs. In *Proc Computer Vision and Pattern Recognition Workshop*, p. 73, 2003.

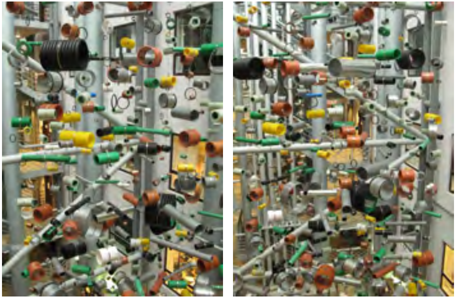
J. Gluckman and S. K. Nayar. Rectifying transformations that minimize resampling effects. In *Proc IEEE CS Conf on Computer Vision and Pattern Recognition*, vol. 1:111–117. 2001.

M. Pollefeys, R. Koch, and L. V. Gool. A simple and efficient rectification method for general motion. In *Proc Int Conf on Computer Vision*, vol. 1:496–501, 1999.

What Are The Relative Distances?

- monocular vision already gives a rough 3D sketch because we understand the scene

What Are The Relative Distances?



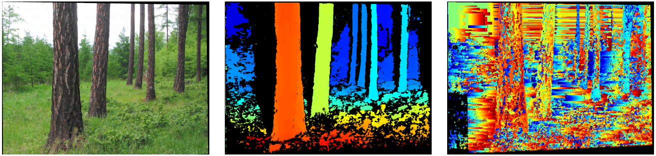
Centrum för teknikstudier at Malmö Högskola, Sweden

The Vyšehrad Fortress, Prague

- left: we have no help from image interpretation
- right: ambiguous interpretation due to a combination of missing texture and occlusion

▶How Difficult Is Stereo?

- when we do not recognize the scene and cannot use high-level constraints the problem seems difficult (right, less so in the center)
- most stereo matching algorithms do not require scene understanding prior to matching
- the success of a model-free stereo matching algorithm is unlikely:



left image a good disparity map disparity map from WTA

WTA Matching:

for every left-image pixel
find the most similar right-image pixel along the corresponding epipolar line [Marroquin 83]

A Summary of Our Observations and an Outlook

1. simple matching algorithms do not work
2. stereopsis requires image interpretation in sufficiently complex scenes

or another-modality measurement

we have a tradeoff: model strength $\leftrightarrow$ universality

Outlook:

1. represent the occlusion constraint: correspondences are not independent due to occlusions
   - epipolar rectification
   - disparity
   - uniqueness as an occlusion constraint
2. represent piecewise continuity the weakest of interpretations; piecewise: object boundaries
   - ordering as a weak continuity model
3. use a consistent framework
   - looking for the most probable solution (MAP)

Thank You