Part VI

3D Structure and Camera Motion

6.1 Introduction

6.2 Reconstructing Camera Systems

6.3 Bundle Adjustment

covered by

[1] [H&Z] Secs: 9.5.3, 10.1, 10.2, 10.3, 12.1, 12.2, 12.4, 12.5, 18.1

[2] Triggs, B. et al. Bundle Adjustment—A Modern Synthesis. In *Proc ICCV Workshop on Vision Algorithms.* Springer-Verlag. pp. 298–372, 1999.

additional references

D. Martinec and T. Pajdla. Robust Rotation and Translation Estimation in Multiview Reconstruction. In *Proc CVPR*, 2007

M. I. A. Lourakis and A. A. Argyros. SBA: A Software Package for Generic Sparse Bundle Adjustment. ACM Trans Math Software 36(1):1–30, 2009.

▶Constructing Cameras from the Fundamental Matrix

Given $\mathbf{F}$, construct some cameras $\mathbf{P}_1$, $\mathbf{P}_2$ such that $\mathbf{F}$ is their fundamental matrix.

Solution See [H&Z, p. 256]

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \, \underline{\mathbf{v}}^\top & \lambda \, \underline{\mathbf{e}}_2 \end{bmatrix}$$

where

- $\underline{\mathbf{v}}$ is any 3-vector, e.g. $\underline{\mathbf{v}} = \underline{\mathbf{e}}_1 = \mathrm{null}(\mathbf{F})$, i.e. $\mathbf{F}\,\mathbf{e}_1 = 0$, to make the camera finite
- $\lambda \neq 0$ is a scalar,
- $\underline{\mathbf{e}}_2 = \mathrm{null}(\mathbf{F}^\top)$, i.e. $\underline{\mathbf{e}}_2^\top \mathbf{F} = 0$

Proof

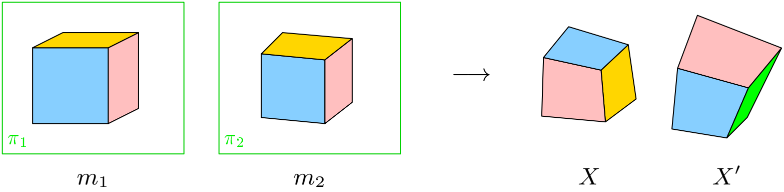
1. $\mathbf{S}$ is skew-symmetric iff $\mathbf{x}^\top \mathbf{S} \mathbf{x} = 0$ for all $\mathbf{x}$ — look-up the proof!
2. we have $\underline{\mathbf{x}} \simeq \mathbf{P} \underline{\mathbf{X}}$
3. a non-zero $\mathbf{F}$ is a f.m. of $(\mathbf{P}_1, \mathbf{P}_2)$ iff $\mathbf{P}_2^\top \mathbf{F} \mathbf{P}_1$ is skew-symmetric
4. if $\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$ and $\mathbf{P}_2 = \begin{bmatrix} \mathbf{S}\mathbf{F} & \underline{\mathbf{e}}_2 \end{bmatrix}$ then $\mathbf{F}$ corresponds to $(\mathbf{P}_1, \mathbf{P}_2)$ by Step 3
5. we can write $\mathbf{S} = [\mathbf{s}]_\times$
6. a suitable choice is $\mathbf{s} = \underline{\mathbf{e}}_2$ [Luong96]
7. for the full the class including $\underline{\mathbf{v}}$, see [H&Z, Sec. 9.5]

▶The Projective Reconstruction Theorem

Observation: Unless $\mathbf{P}_i$ are constrained, then for any number of cameras $i = 1, \ldots, k$

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_i \underline{\mathbf{X}} = \underbrace{\mathbf{P}_i \mathbf{H}^{-1}}_{\mathbf{P}'_i} \underbrace{\mathbf{H}\underline{\mathbf{X}}}_{\underline{\mathbf{X}}'} = \mathbf{P}'_i \underline{\mathbf{X}}'$$

- when $\mathbf{P}_i$ and $\underline{\mathbf{X}}$ are both determined from correspondences (including calibrations $\mathbf{K}_i$), they are given up to a common 3D homography $\mathbf{H}$
  (translation, rotation, scale, shear, pure perspectivity)

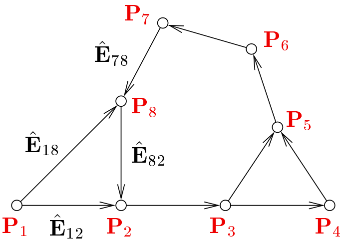


- when cameras are internally calibrated ($\mathbf{K}_i$ known) then $\mathbf{H}$ is restricted to a similarity since it must preserve the calibrations $\mathbf{K}_i$ [H&Z, Secs. 10.2, 10.3], [Longuet-Higgins 1981]
  (translation, rotation, scale)

▶Reconstructing Camera Systems

Problem: Given a set of p decomposed pairwise essential matrices $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_\times \hat{\mathbf{R}}_{ij}$ and calibration matrices $\mathbf{K}_i$ reconstruct the camera system $\mathbf{P}_i$, $i = 1, \ldots, k$

→81 and →146 on representing $\mathbf{E}$



We construct calibrated camera pairs $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$ →129

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1}\hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1}\hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons i, j correspond to graph nodes — k nodes
- pairs ij correspond to graph edges — p edges

$\hat{\mathbf{P}}_{ij}$ are in different coordinate systems but these are related by similarities $\hat{\mathbf{P}}_{ij}\mathbf{H}_{ij} = \mathbf{P}_{ij}$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\top & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \stackrel{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\mathbb{R}^{6,4}} \tag{29}$$

- (29) is a linear system of $24p$ eqs. in $7p + 6k$ unknowns — $7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij})$, $6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each $\mathbf{P}_i$ appears on the right side as many times as is the degree of node $\mathbf{P}_i$ — eg. P_5 3-times

▶cont'd

Eq. (29) implies $\begin{bmatrix} \mathbf{R}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{R}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i \\ \mathbf{R}_j \end{bmatrix}$ $\begin{bmatrix} \mathbf{t}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{t}_{ij} + s_{ij}\hat{\mathbf{t}}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{t}_j \end{bmatrix}$

- $\mathbf{R}_{ij}$ and $\mathbf{t}_{ij}$ can be eliminated:

$$\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j, \qquad \hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} = \mathbf{t}_j, \qquad s_{ij} > 0 \tag{30}$$

- note transformations that do not change these equations — assuming no error in $\hat{\mathbf{R}}_{ij}$

  1. $\mathbf{R}_i \mapsto \mathbf{R}_i\mathbf{R}$, 2. $\mathbf{t}_i \mapsto \sigma\,\mathbf{t}_i$ and $s_{ij} \mapsto \sigma s_{ij}$, 3. $\mathbf{t}_i \mapsto \mathbf{t}_i + \mathbf{R}_i\mathbf{t}$

- the global frame is fixed, e.g. by selecting

$$\mathbf{R}_1 = \mathbf{I}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \frac{1}{p}\sum_{i,j} s_{ij} = 1 \tag{31}$$

- rotation equations are decoupled from translation equations
- in principle, s_{ij} could correct the sign of $\hat{\mathbf{t}}_{ij}$ from essential matrix decomposition →81
  but $\mathbf{R}_i$ cannot correct the α sign in $\hat{\mathbf{R}}_{ij}$
  ⇒ therefore make sure all points are in front of cameras and constrain $s_{ij} > 0$; →83

\+ pairwise correspondences are sufficient

− suitable for well-distributed cameras only (dome-like configurations)

otherwise intractable or numerically unstable

►Solving Eq. (30) by Stepwise Gluing

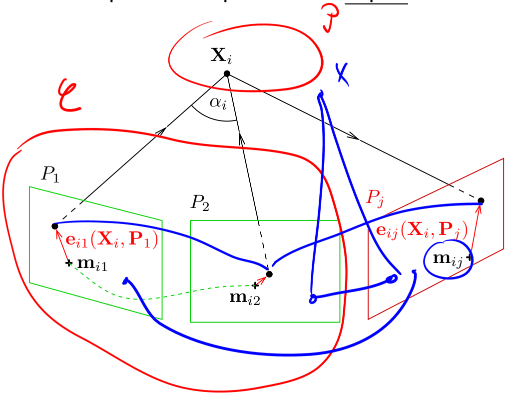
Given: Calibration matrices $\mathbf{K}_j$ and tentative correspondences per camera <u>triples</u>.

Initialization

1. initialize <u>camera cluster</u> $\mathcal{C}$ with P_1, P_2,
2. find essential matrix $\mathbf{E}_{12}$ and matches M_{12} by the 5-point algorithm →88
3. construct camera pair

$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

4. compute 3D reconstruction $\{X_i\}$ per match from M_{12} →105
5. initialize <u>point cloud</u> $\mathcal{X}$ with $\{X_i\}$ satisfying chirality constraint $z_i > 0$ and apical angle constraint $|\alpha_i| > \alpha_T$



Attaching camera $P_j \notin \mathcal{C}$

1. select points $\mathcal{X}_j$ from $\mathcal{X}$ that have matches to P_j
2. estimate $\mathbf{P}_j$ using $\mathcal{X}_j$, RANSAC with the 3-pt alg. (P3P), projection errors $\mathbf{e}_{ij}$ in $\mathcal{X}_j$ →68
3. reconstruct 3D points from all tentative matches from P_j to all P_l, $l \neq k$ that are <u>not</u> in $\mathcal{X}$
4. filter them by the chirality and apical angle constraints and add them to $\mathcal{X}$
5. add P_j to $\mathcal{C}$
6. perform <u>bundle adjustment</u> on $\mathcal{X}$ and $\mathcal{C}$

coming next →137

Finding The Rotation Component in Eq. (30): A Global Algorithm

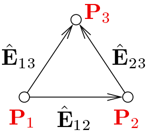
Task: Solve $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$, $i, j \in V$, $(i, j) \in E$ where $\mathbf{R}$ are a 3×3 rotation matrix each.
Per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$[r_i^1, r_i^2, r_i^3]$

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \qquad \text{for all } i,\ j \tag{32}$$

- fix c and denote $\mathbf{r}^c = \left[\mathbf{r}_1^c, \mathbf{r}_2^c, \ldots, \mathbf{r}_k^c\right]^\top$ c-th columns of all rotation matrices stacked; $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (32) becomes $\mathbf{D}\,\mathbf{r}^c = \mathbf{0}$ $\quad\mathbf{D} \in \mathbb{R}^{3p,3k}$
- $3p$ equations for $3k$ unknowns $\to p \geq k$ $\quad$ in a 1-connected graph we have to fix $\mathbf{r}_1^c = [1, 0, 0]$

Ex: $(k = p = 3)$



$\rightarrow$

$$\begin{aligned}
\hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c &= \mathbf{0}\\
\hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c &= \mathbf{0}\\
\hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c &= \mathbf{0}
\end{aligned}$$

$\rightarrow$

$$\mathbf{D}\,\mathbf{r}^c = \underbrace{\begin{bmatrix} \hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I} \\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \mathbf{r}_1^c \\ \mathbf{r}_2^c \\ \mathbf{r}_3^c \end{bmatrix}}_{r^c} = \mathbf{0}$$

- must hold for any c

Idea: [Martinec & Pajdla CVPR 2007]

1. find the space of all $\mathbf{r}^c \in \mathbb{R}^{3k}$ that solve (32) $\quad\mathbf{D}$ is sparse, use `[V,E] = eigs(D'*D,3,0);` (Matlab)
2. choose 3 unit orthogonal vectors in this space $\quad$ 3 smallest eigenvectors
3. find closest rotation matrices per cam. using SVD $\quad$ because $\|\mathbf{r}^c\| = 1$ is necessary but insufficient
   $\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$, where $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$

- global world rotation is arbitrary

Finding The Translation Component in Eq. (30)

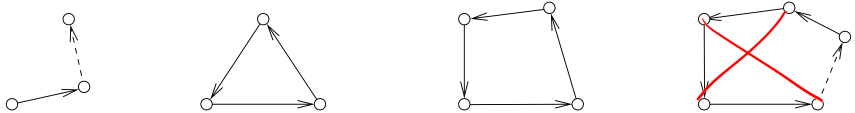
From (30) and (31): $d \le 3$ – rank of camera center set, p – #pairs, k – #cameras

$$\hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \sum_{i,j} s_{ij} = p, \qquad s_{ij} > 0, \qquad \mathbf{t}_i \in \mathbb{R}^d$$

- in rank d: $\quad d \cdot p + d + 1$ equations for $d \cdot k + p$ unknowns $\rightarrow p \ge \frac{d(k-1)-1}{d-1} \overset{\text{def}}{=} Q(d,k)$

Ex: Chains and circuits construction from sticks of known orientation and unknown length?

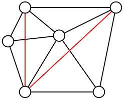
$p = k - 1$ | $k = p = 3$ | $k = p = 4$ | $k = p > 4$



$k \le 2$ for any d | $3 \ge d \ge 2$: non-collinear ok | $3 \ge d \ge 3$: non-planar ok | $3 \ge d \ge k-1$: impossible

- equations insufficient for chains, trees, or when $d = 1$ collinear cameras
- 3-connectivity implies sufficient equations for $d = 3$ cams. in general pos. in 3D
  - s-connected graph has $p \ge \lceil \frac{sk}{2} \rceil$ edges for $s \ge 2$, hence $p \ge \lceil \frac{3k}{2} \rceil \ge Q(3,k) = \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any k when $d = 2$ coplanar cams
  - since $p \ge \lceil 2k \rceil \ge Q(2,k) = 2k - 3$
  - maximal planar tringulated graphs have $p = 3k - 6$ and give a solution for $k \ge 3$

maximal planar triangulated graph example:



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Linear equations in (30) and (31) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \qquad \mathbf{t} = \left[\mathbf{t}_1^\top, \mathbf{t}_2^\top, \ldots, \mathbf{t}_k^\top, s_{12}, \ldots, s_{ij}, \ldots\right]^\top$$

for $d = 3$: $\mathbf{t} \in \mathbb{R}^{3k+p}$, $\mathbf{D} \in \mathbb{R}^{3p,3k+p}$ is sparse

$$\mathbf{t}^* = \arg\min_{\mathbf{t},\, s_{ij}>0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D}\, \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

▶Bundle Adjustment

Given:

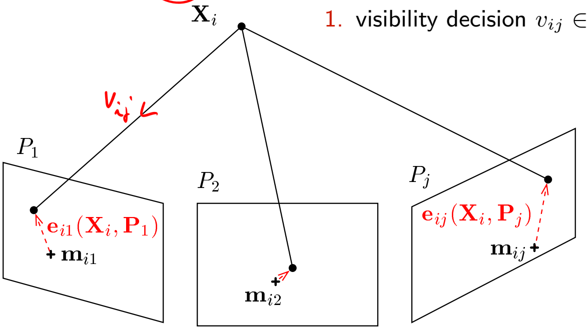
1. set of 3D points $\{\mathbf{X}_i\}_{i=1}^{p}$
2. set of cameras $\{\mathbf{P}_j\}_{j=1}^{c}$
3. fixed tentative projections $\mathbf{m}_{ij}$

Required:

1. corrected 3D points $\{\mathbf{X}'_i\}_{i=1}^{p}$
2. corrected cameras $\{\mathbf{P}'_j\}_{j=1}^{c}$

Latent:

1. visibility decision $v_{ij} \in \{0, 1\}$ per $\mathbf{m}_{ij}$



- for simplicity, $\mathbf{X}$, $\mathbf{m}$ are considered Cartesian (not homogeneous)
- we have projection error $\mathbf{e}_{ij}(\mathbf{X}_i, \mathbf{P}_j) = \mathbf{x}_i - \mathbf{m}_i$ per image feature, where $\underline{\mathbf{x}}_i = \mathbf{P}_j \underline{\mathbf{X}}_i$
- for simplicity, we will work with scalar error $e_{ij} = \|\mathbf{e}_{ij}\|$

Robust Objective Function for Bundle Adjustment

The data model is constructed by marginalization, as in Robust Matching Model →113

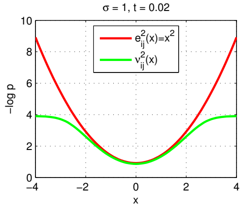
$$p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \prod_{\text{pts}:i=1}^{p} \prod_{\text{cams}:j=1}^{c} \Big((1 - P_0) p_1(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j) + P_0\, p_0(e_{ij} \mid \mathbf{X}_i, \mathbf{P}_j) \Big)$$

marginalized negative log-density is (→114)

$$Q = \quad -\log p(\{\mathbf{e}\} \mid \{\mathbf{P}, \mathbf{X}\}) = \sum_i \sum_j \underbrace{-\log\left(e^{-\frac{e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)}{2\sigma_1^2}} + t\right)}_{\rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)) = \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)} \stackrel{\text{def}}{=} \sum_i \sum_j \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)$$

- e_{ij} is the projection error (not Sampson error)
- ν_{ij} is a 'robust' error fcn.; it is non-robust ($\nu_{ij} = e_{ij}$) when $t = 0$
- $\rho(\cdot)$ is a 'robustification function' we often find in M-estimation
- the $\mathbf{L}_{ij}$ in Levenberg-Marquardt changes to vector

$$(\mathbf{L}_{ij})_l = \frac{\partial \nu_{ij}}{\partial \theta_l} = \underbrace{\frac{1}{1 + t\, e^{e_{ij}^2(\theta)/(2\sigma_1^2)}}}_{\text{small for big } e_{ij}} \cdot \frac{1}{\nu_{ij}(\theta)} \cdot \frac{1}{4\sigma_1^2} \cdot \frac{\partial e_{ij}^2(\theta)}{\partial \theta_l} \quad (33)$$



but the LM method stays the same as before →107–108

- outliers: almost no impact on $\mathbf{d}_s$ in normal equations because the red term in (33) scales contributions to both sums down for the particular ij

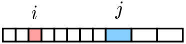
$$-\sum_{i,j}^{k} \mathbf{L}_{ij}^\top \nu_{ij}(\theta^s) = \Big(\sum_{i,j}^{k} \mathbf{L}_{ij}^\top \mathbf{L}_{ij}\Big) \mathbf{d}_s$$

▶Sparsity in Bundle Adjustment

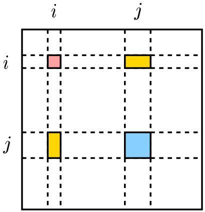
We have $q = 3p + 11k$ parameters: $\theta = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p;\ \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k)$ points, cameras
We will use a running index $r = 1, \ldots, z$, $z = p \cdot k$. Then each r corresponds to some i, j

$$\theta^* = \arg\min_{\theta} \sum_{r=1}^{z} \nu_r^2(\theta),\ \boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s,\ -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \left(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag} \mathbf{L}_r^\top \mathbf{L}_r \right) \mathbf{d}_s$$

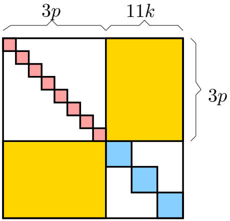
The block form of $\mathbf{L}_r$ in Levenberg-Marquardt ($\rightarrow$107) is zero except in columns i and j:
r-th error term is $\nu_r^2 = \rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j))$

$\mathbf{L}_r =$ 

$r = (i, j)$ blocks:
■: $\mathbf{X}_i, 1 \times 3$
■: $\mathbf{P}_j, 1 \times 11$

$\mathbf{L}_r^\top \mathbf{L}_r =$ 

blocks:
■: $\mathbf{X}_i - \mathbf{X}_i$, 3×3
■: $\mathbf{X}_i - \mathbf{P}_j$, 3×11
■: $\mathbf{P}_j - \mathbf{P}_j$, 11×11

$\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r =$ 

- "points first, then cameras" scheme
- standard bundle adjustment eliminates points and solves cameras, then back-substitutes

▶Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{d}_s \text{ such that } \quad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \Big(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag} \mathbf{L}_r^\top \mathbf{L}_r\Big) \mathbf{d}_s$$

This is a linear set of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

- $\mathbf{A}$ is very large — approx. $3 \cdot 10^4 \times 3 \cdot 10^4$ for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$ is sparse and symmetric, $\mathbf{A}^{-1}$ is dense — direct matrix inversion is prohibitive

> Choleski: Every symmetric positive definite matrix $\mathbf{A}$ can be decomposed to $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is lower triangular. If $\mathbf{A}$ is sparse then $\mathbf{L}$ is sparse, too.

1. decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ — transforms the problem to solving $\mathbf{L}\underbrace{\mathbf{L}^\top\mathbf{x}}_{\mathbf{c}} = \mathbf{b}$

   $L\underset{\sim}{c} = b \qquad L^T x = c$

2. solve for $\mathbf{x}$ in two passes:

$$\mathbf{L}\,\mathbf{c} = \mathbf{b} \qquad \mathbf{c}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{b}_i - \sum_{j<i} \mathbf{L}_{ij}\mathbf{c}_j\Big) \qquad \text{forward substitution, } i = 1, \ldots, q$$

$c_1 = L_{11}^{-1} b_1 \qquad c_2 = L_{22}^{-1}(b_2 - L_{12}c_1)$

$$\mathbf{L}^\top\mathbf{x} = \mathbf{c} \qquad \mathbf{x}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{c}_i - \sum_{j>i} \mathbf{L}_{ji}\mathbf{x}_j\Big) \qquad \text{back-substitution}$$

- Choleski decomposition is fast (does not touch zero blocks)
  non-zero elements are $9p + 121k + 66pk \approx 3.4 \cdot 10^6$; ca. $250\times$ fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse $\mathbf{A}$ and diagonal pivoting for semi-definite $\mathbf{A}$ — [Triggs et al. 1999]
- λ controls the definiteness

Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%    L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%    for sparse square symmetric positive definite matrix A,
%    especially useful for arrowhead sparse matrices.

% (c) 2010 Radim Sara (sara@cmp.felk.cvut.cz)

 [p,q] = size(A);
 if p ~= q, error 'Matrix must be square'; end

 L = sparse(q,q);
 F = ones(q,1);
 for i=1:q
  F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
  for j = F(i):i-1
   k = max(F(i),F(j));
   a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
   L(i,j) = a/L(j,j);
  end
  a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
  if a < 0, error 'Matrix A must be positive definite'; end
  L(i,i) = sqrt(a);
 end
end
```

▶Gauge Freedom

1. The external frame is not fixed: See Projective Reconstruction Theorem →130

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.
   - $\mathbf{P}$ is 12 numbers for 11 parameters
   - we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
   - but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints

2a. either imposing constraints on projective entities
   - cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
   - points, e.g. $\|\underline{\mathbf{X}}_i\|^2 = 1$ — this way we can represent points at infinity

2b. or using minimal representations
   - points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
   - rotation matrix can be represented by axis-angle or the Cayley transform — see next

Thank You