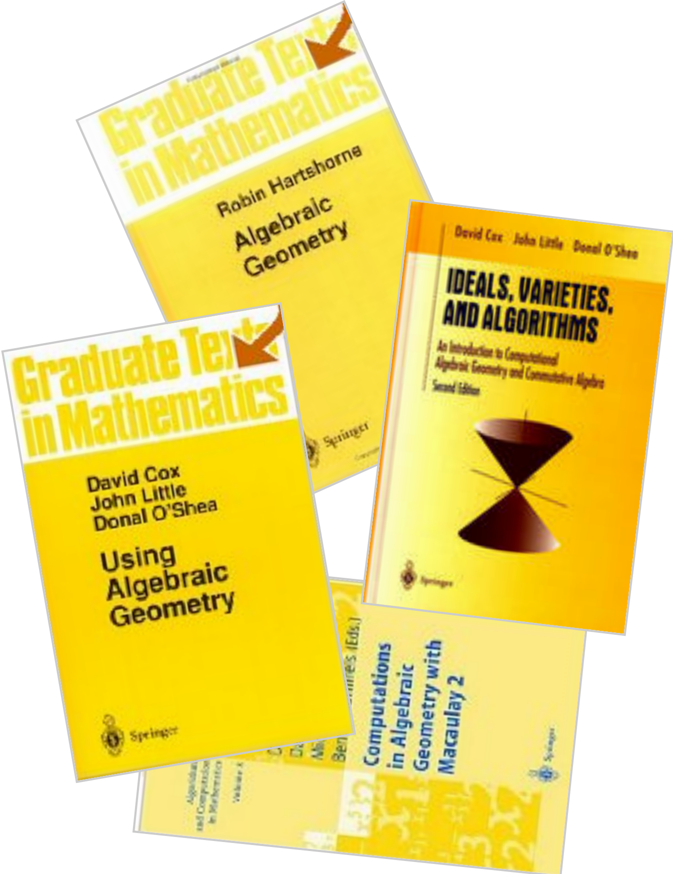
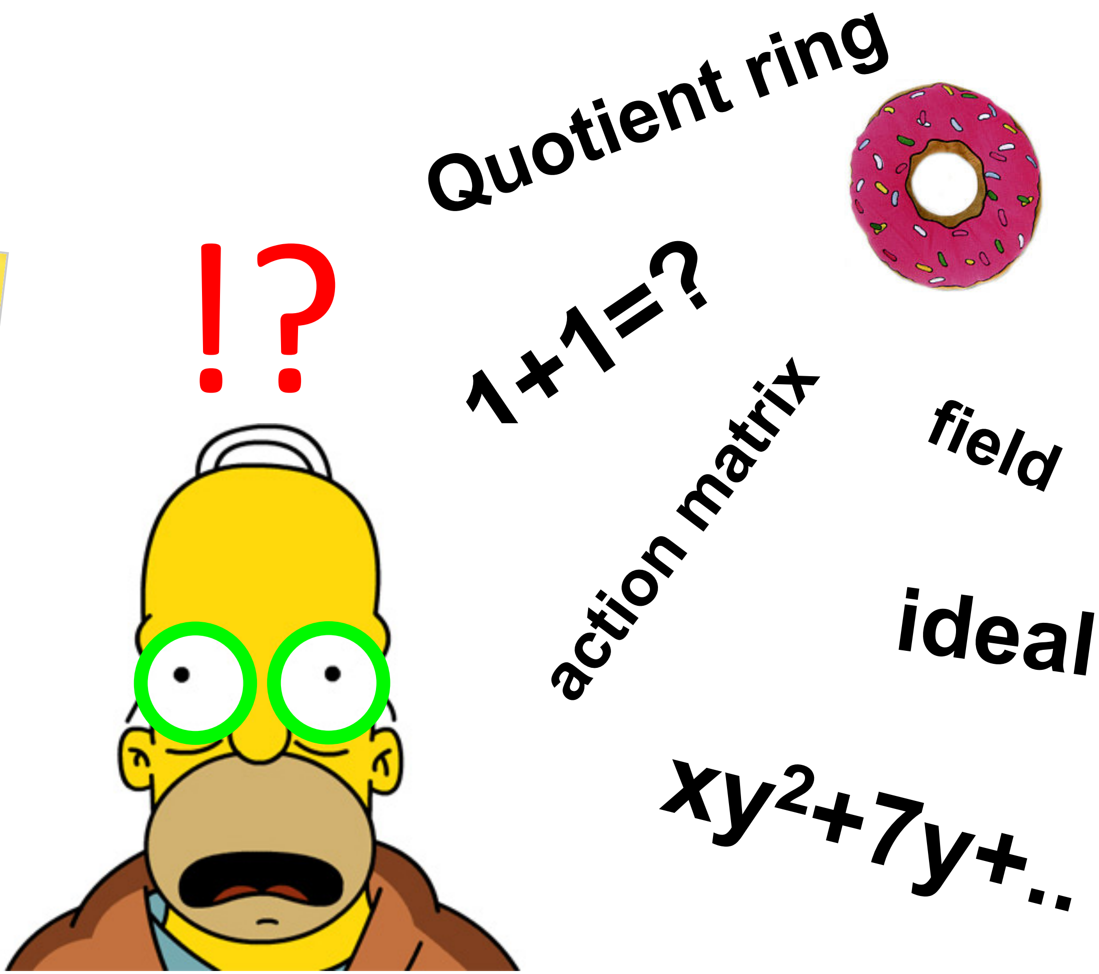
Advanced Robotics

Lecture 10

pajdla@cmp.felk.cvut.cz

SOLVING ALGEBRAIC EQUATIONS





LOOKS AS "the MATHEMATICS" !!!

LET' S TAKE AN ENGINEERING APPROACH

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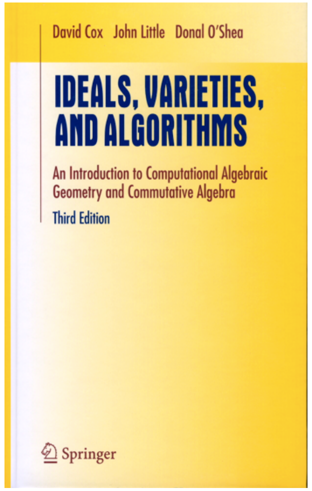
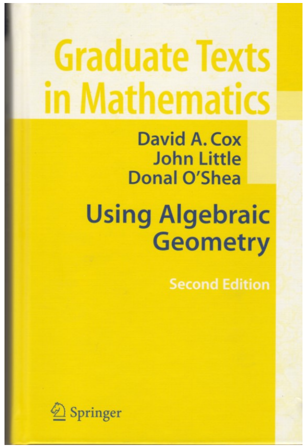
LITERATURE





Mathematics

1. Cox et al. Ideal Varieties and Algorithm. Springer 2015.
   (An Introduction to Algebraic Geometry … New edition!)
2. Cox et al. Using Algebraic Geometry. Springer 1998.
   (More advanced Algebraic Geometry)
3. F. Kubler, P. Renner, K. Schmedders.
   Computing All Solutions to Polynomial Equations in Economics
   (Very lightweight intro to polynomial system solving).

Minimal problem papers

cmp.felk.cvut.cz/minimal

SINGLE UNKNOWN → EIGENVALUES

1 equation, 1 variable → companion matrix → eigenvalues

$$f(x) = x^3 + 4\,x^2 + x - 6 \;= -6 + 1\,x + 4\,x^2 + 1\,x^3$$

$$\mathtt{M}_x = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -1 \\ 0 & 1 & -4 \end{bmatrix}$$

... a simple rule

```
>> e=eig(Mx)
```

$$\mathtt{e} \;=\; \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \qquad x_1 = 1,\; x_2 = -2,\; x_3 = -3$$

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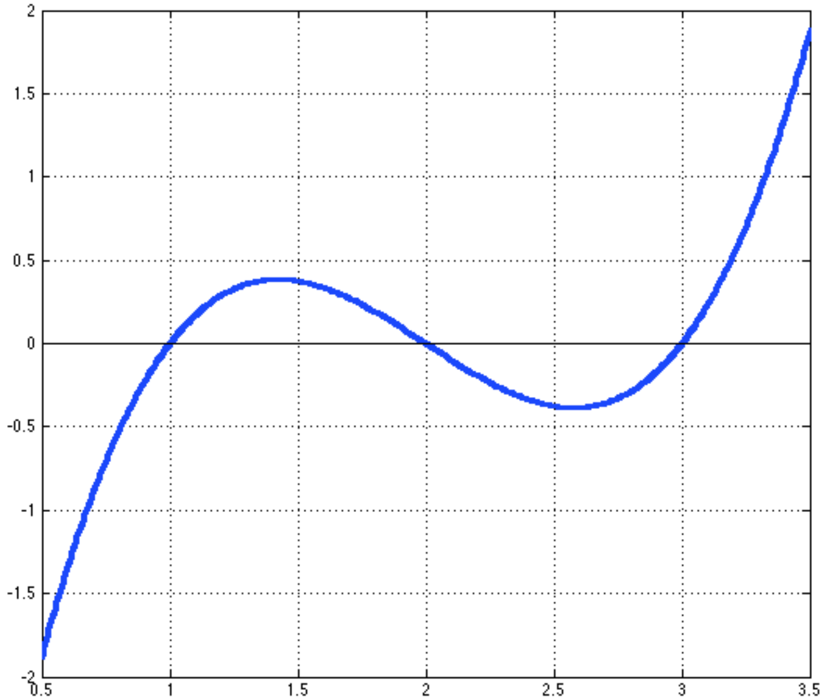
Towards the MULTIPLICATION MATRIX $\mathtt{M}_x$



a polynomial

$$f(x) = (x-1)(x-2)(x-3)$$
$$= x^3 - 6x^2 + 11x - 6$$

with roots

$$x_1 = 1,\ x_2 = 2,\ x_3 = 3$$

i.e.

$$f(x):\ f(x_1) = f(x_2) = f(x_3) = 0$$

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REMAINDERS are a linear space

$r(x) = h(x) \bmod f(x)$ … remainder on division by $f(x)$

$$h(x) = q(x)\, f(x) + r(x)$$

such that $\deg r(x) < \deg f(x)$ or $r(x) = 0$

- remainders have low degrees
- can be represented by finite vectors

e.g. for $f(x) = x^3 - 6\,x^2 + 11\,x - 6$ deg of $r(x)$ is smaller than 3:

$$r(x) = a_2\,x^2 + a_1\,x + a_0 \equiv \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^3$$

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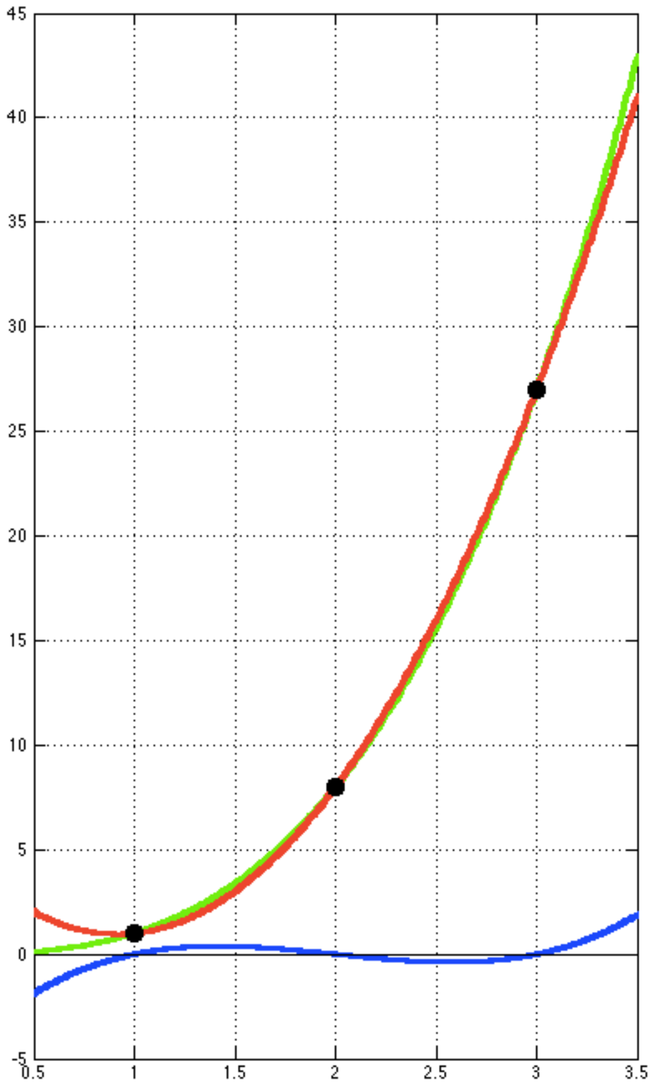
REMAINDERS evaluate



Evaluate a general polynomial $h(x)$ on

roots of $f(x)$: $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

using the the reminder $r(x)$

$$\begin{aligned} h(x_i) &= q(x_i)\, f(x_i) + r(x_i) \\ &= q(x_i)\, 0 + r(x_i) = r(x_i) \end{aligned}$$

$$h(x_i) = r(x_i) \quad i = 1, 2, 3$$

Example

$h(x) = x^3$ $\quad$ $f(x) = x^3 - 6\,x^2 + 11\,x - 6$

$$h(x) = 1\, f(x) + (6\,x^2 - 11\,x + 6)$$

$$r(x) = 6\,x^2 - 11\,x + 6$$

MAPPING by MULTIPLI CATION

Consider a mapping $\mathtt{M}$ of polynomials to polynomials generated by multiplication by x

$$h(x) \rightarrow x\,h(x)$$

It generates a mapping on reminders on division by $f(x)$

$$h(x) \bmod f(x) \xrightarrow{\mathtt{M}} (x\,h(x)) \bmod f(x)$$

$$a_2 x^2 + a_1 x_+ a_0 \;\rightarrow\; b_2 x^2 + b_1 x + b_0$$

… can be seen as a mapping in the linear space of vectors of coefficients

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \quad \ldots \quad \mathtt{M}\colon \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

…IS LINEAR

$$\mathtt{M}\colon \mathbb{R}^3 \to \mathbb{R}^3 \qquad h(x) \bmod f(x) \xrightarrow{\mathtt{M}} (x\,h(x)) \bmod f(x)$$

is a LINEAR MAPPING! … can be represented by a matrix

Example $f(x) = x^3 - 6\,x^2 + 11\,x - 6$

$$\begin{aligned} \mathtt{M}(1) &= x\,1 = x \\ \mathtt{M}(x) &= x\,x = x^2 \\ \mathtt{M}(x^2) &= x^3 \bmod f(x) \\ &= 6\,x^2 - 11\,x + 6 \end{aligned}$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}6\\-11\\6\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 6\\1 & 0 & -11\\0 & 1 & 6\end{bmatrix} = \begin{bmatrix}0 & 0 & 6\\1 & 0 & -11\\0 & 1 & 6\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix} \xrightarrow{\mathtt{M}} \begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix}$$

$$\begin{bmatrix}b_0\\b_1\\b_2\end{bmatrix} = \mathtt{M}_x \begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$$

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FINIS CORONAT OPUS

Putting it all together

Evaluate reminder

$$r(x_i) = a_2 x_i^2 + a_1 x_i + a_0$$

and its image

$$x_i r(x_i) = b_2 x_i^2 + b_1 x_i + b_0$$

$$(x_i a_2)\, x_i^2 + (x_i a_1)\, x_i + (x_i a_0) = b_2 x_i^2 + b_1 x_i + b_0$$

For all generic roots →

- use monic
- use LA

$$x_i \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x_i a_0 \\ x_i a_1 \\ x_i a_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \mathtt{M}_x \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\mathtt{M}_x \mathtt{a} = x_i \mathtt{a}$$

Roots of $f(x)$ are eigenvalues of $\mathtt{M}_x$

SOLVING ALGEBRAIC EQUATIONS & GB

$\longrightarrow$ generalize remainder on division to m equations & n vars

$$h(x, y, ...) = q_1(x, y, \ldots)\, f_1(x, y, ...) + q_2(x, y, ...)\, f_2(x, y, ...) + \cdots + r(x, y, ...)$$

divisors remainder

For general $f_i(x, y, ...)$ reminder $r(x, y, ...)$ not well defined

… depends on the ordering of divisors!

$\longrightarrow$ Groebner basis … divisors for which remainder well defined

$\longrightarrow$ does not depend on the ordering of divisors

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SOLVING ALGEBRAIC EQUATIONS & GB

Linear equations are algebraic equations

linear equations

$$\begin{aligned} 4\,x + 8\,y + 7\,z &= 0 \\ 6\,x + 3\,y + 2\,z &= 0 \\ 7\,x + 7\,y + 1\,z &= 0 \end{aligned}$$

Gaussian elimination

$$\begin{bmatrix} 4 & 8 & 7 \\ 6 & 3 & 2 \\ 7 & 7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 8 & 7 \\ 0 & 36 & 34 \\ 0 & 0 & -167 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Groebner basis

$$\begin{aligned} 4\,x + 8\,y + 7\,z &= 0 \\ 36\,y + 34\,z &= 0 \\ 167\,z &= 0 \end{aligned}$$

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SOLVING ALGEBRAIC EQUATIONS & GB

m equations, n variables (an example for m = 2 & n = 2)

$$\begin{aligned} 0 &= 25\,x\,y - 20\,y - 15\,x + 12 \\ 0 &= y^2 + x^2 - 1 \end{aligned}$$

$\rightarrow$ Groebner basis

(polynomials with the same solutions but easy to solve

lexicographical ordering of monomials)

$$\begin{aligned} f_1 \rightarrow \quad 0 &= 25\,x\,y - 20\,y - 15\,x + 12 \\ f_2 \rightarrow \quad 0 &= y^2 + x^2 - 1 \\ 0 &= 125\,y^3 - 75\,y^2 + 27 - 45\,y \quad \leftarrow \text{ how to get it ?} \\ &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2 \end{aligned}$$

… a polynomial combination of f_1, f_2

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MATRIX FORM OF POLYNOMIALS (F4-Like Approach)

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2
\end{aligned}
$$

$$
\begin{matrix} f_1 \rightarrow \\ f_2 \rightarrow \end{matrix}
\qquad
\begin{bmatrix} 0 & 25 & -15 & 0 & -20 & 12 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}
\begin{bmatrix} x^2 \\ x\,y \\ x \\ y^2 \\ y \\ 1 \end{bmatrix}
$$

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ADDING MULTIPLES

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2 = a\,\boxed{x\,f_1} + b\,\boxed{y\,f_2} + c
\end{aligned}
$$

$$
\begin{array}{ll} f_1 & \rightarrow \\ f_2 & \rightarrow \\ x\,f_1 & \rightarrow \\ y\,f_2 & \rightarrow \end{array}
\quad
\begin{bmatrix}
0 & 0 & 25 & -15 & 0 & 0 & -20 & 12 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
25 & -15 & -20 & 12 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\boxed{x^2\,y} \\ x^2 \\ x\,y \\ x \\ \boxed{y^3} \\ y^2 \\ y \\ 1
\end{bmatrix}
$$

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GAUSSIAN ELIMINATION

$$
\begin{aligned}
f_1 &= 25\,x\,y - 20\,y - 15\,x + 12 \\
f_2 &= y^2 + x^2 - 1 \\
f_3 &= (-5\,x - 4)\,f_1 + (125\,y - 75)\,f_2 = a\,x\,f_1 + b\,y\,f_2 + c \\
&= 125\,y^3 - 75\,y^2 - 45\,y + 27
\end{aligned}
$$

$$
\begin{bmatrix}
5 & -3 & -4 & \frac{12}{5} & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 5 & 0 & -5 \\
0 & 0 & 125 & -75 & 0 & 0 & -100 & 60 \\
0 & 0 & 0 & 0 & 125 & -75 & -45 & 27
\end{bmatrix}
\begin{bmatrix}
x^2\,y \\ x^2 \\ x\,y \\ x \\ y^3 \\ y^2 \\ y \\ 1
\end{bmatrix}
$$

Gaussian elimination

SOLVING POLYNOMIAL EQUATIONS BY CONSTRUCTING GROEBNER BASIS

A generalization of the Gaussian elimination

multiplication by scalars

↓

multiplication by scalars & variables

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PRACTICAL CASES – (10 eqns, 10 vars, deg 3)

m equations, n variables

$$
\begin{aligned}
f_1(x,y) &= 25\,x\,y - 15\,x - 20\,y + 12 \\
f_2(x,y) &= x^2 + y^2 - 1 \\
&\vdots
\end{aligned}
$$

$\rightarrow$ Groebner basis $\rightarrow$ generalized companion matrix (multiplication in $\mathbb{Q}[x_1, \ldots]/I(f_1, \ldots)$)

(e.g. in grevlex ordering)

$$
\mathtt{M}_{x+y} = \begin{bmatrix} 0 & 125 & 0 & 125 \\ -60 & 100 & 125 & 75 \\ -63 & 45 & 175 & 45 \\ 65 & 100 & -125 & 75 \end{bmatrix}
$$

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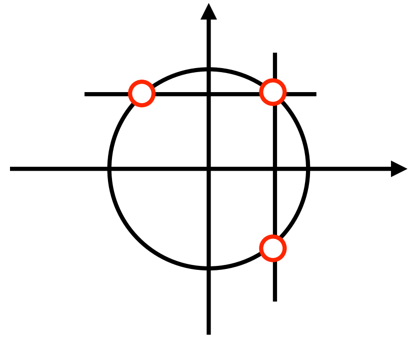
SOLVING ALGEBRAIC EQUATIONS

m equations, n variables



$$
\begin{aligned}
f_1(x,y) &= 25\,x\,y - 15\,x - 20\,y + 12 \\
f_2(x,y) &= x^2 + y^2 - 1
\end{aligned}
$$

$\rightarrow$ Groebner basis $\rightarrow$ generalized companion matrix $\rightarrow$ eigenvectors

$$
\mathrm{v} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{4}{5} & \frac{4}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & \frac{3}{5} \\ \frac{16}{25} & \frac{16}{25} & \frac{16}{25} & \frac{16}{25} \end{bmatrix}
$$

$$
\begin{aligned}
x_1 &= -\frac{4}{5}, & y_1 &= \frac{3}{5} \\
x_2 &= \frac{4}{5}, & y_3 &= -\frac{3}{5} \\
x_3 &= \frac{4}{5}, & y_3 &= \frac{3}{5}
\end{aligned}
$$

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