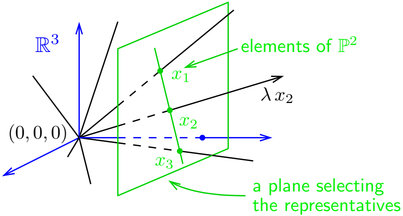
▶Homography in $\mathbb{P}^2$



Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$, factorized to linear equivalence classes ('rays')

including 'points at infinity'

Homography in $\mathbb{P}^2$: Non-singular linear mapping in $\mathbb{P}^2$

an analogic definition for $\mathbb{P}^3$

$$\underline{\mathbf{x}}' \simeq \mathbf{H}\,\underline{\mathbf{x}}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

Defining properties

- collinear image points are mapped to collinear image points

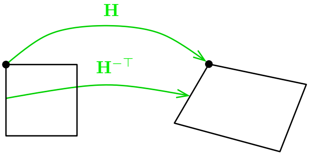
  lines of points are mapped to lines of points
- concurrent image lines are mapped to concurrent image lines

  concurrent = intersecting at a point
- and point-line incidence is preserved

  e.g. line intersection points mapped to line intersection points

- $\mathbf{H}$ is a 3×3 non-singular matrix, $\lambda\,\mathbf{H} \simeq \mathbf{H}$ equivalence class, 8 degrees of freedom
- homogeneous matrix representant: $\det \mathbf{H} = 1$
- what we call homography here is often called 'projective collineation' in mathematics

▶Mapping 2D Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{image line} \qquad \mathbf{H}^{-\top} = (\mathbf{H}^{-1})^\top = (\mathbf{H}^\top)^{-1}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^\top \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^\top \mathbf{H}^\top \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$

1. extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
2. map by homography, $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
3. if $m_3' \neq 0$ convert the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ back to Cartesian coordinates (pixels),

$$u' = \frac{m_1'}{m_3'}\,\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\,\mathbf{1}$$

- note that, typically, $m_3' \neq 1$ — $m_3' = 1$ when $\mathbf{H}$ is affine
- an infinite point $(u, v, 0)$ maps the same way

Some Homographic Tasters

Rectification of camera rotation: →59 (geometry), →122 (homography estimation)

$$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}$$ maps from image plane to facade plane

Homographic Mouse for Visual Odometry: [Mallis 2007]

illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

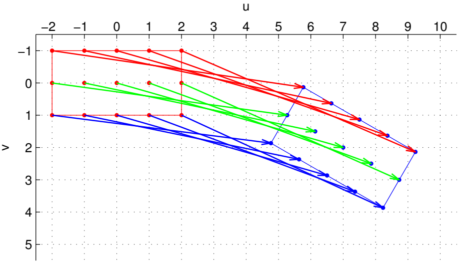
$$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\top}{d}\right)\mathbf{K}^{-1}$$ [H&Z, p. 327]

▶Homography Subgroups: Euclidean Mapping (aka Rigid Motion)

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- eigenvalues $\left(1,\, e^{-i\phi},\, e^{i\phi}\right)$



rotation by 30°, then translation by $(7,\ 2)$

EM = The most general homography preserving

1. **areas**: $\det\mathbf{H} = 1 \;\Rightarrow$ unit Jacobian
2. **lengths**: Let $\underline{\mathbf{x}}_i' = \mathbf{H}\underline{\mathbf{x}}_i$ (check we can use $=$ instead of $\simeq$). Let $(x_i)_3 = 1$, Then
$$\|\underline{\mathbf{x}}_2' - \underline{\mathbf{x}}_1'\| = \|\mathbf{H}\underline{\mathbf{x}}_2 - \mathbf{H}\underline{\mathbf{x}}_1\| = \|\mathbf{H}(\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1)\| = \cdots = \|\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1\|$$
3. **angles** check the dot-product of normalized differences from a point $(\mathbf{x}-\mathbf{z})^\top(\mathbf{y}-\mathbf{z})$ (Cartesian(!))

- eigenvectors when $\phi \neq k\pi,\ k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

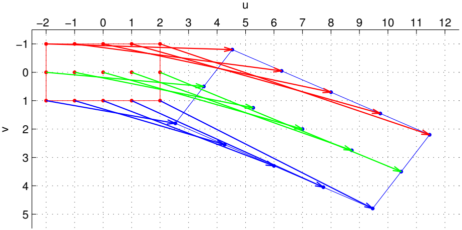
$\mathbf{e}_2$, $\mathbf{e}_3$ – circular points, i – imaginary unit

4. **circular points**: points at infinity $(i, 1, 0)$, $(-i, 1, 0)$ (preserved even by similarity)

- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

▶Homography Subgroups: Affine Mapping

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by 30°
then scaling by $\mathrm{diag}(1,\ 1.5,\ 1)$
then translation by $(7,\ 2)$

AM = The most general homography preserving

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

does not preserve

- lengths
- angles
- areas
- circular points

observe $\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\mathbf{n}}_\infty \quad \Rightarrow \quad \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$

Euclidean mappings preserve all properties affine mappings preserve, of course

▶Homography Subgroups: General Homography

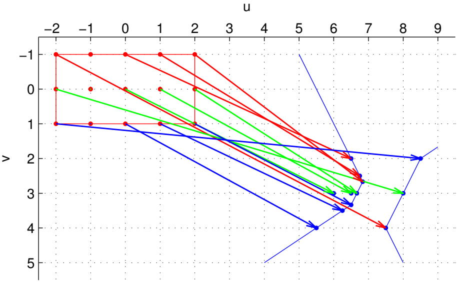
$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

preserves only

- incidence and concurrency
- collinearity
- cross-ratio on the line →47

does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors (midpoints, etc.)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$



$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

(where in the picture is the line $\mathbf{n}$?)

Elementary Decomposition of a Homography

Unique decompositions: $\mathbf{H} = \mathbf{H}_S\,\mathbf{H}_A\,\mathbf{H}_P \quad (= \mathbf{H}'_P\,\mathbf{H}'_A\,\mathbf{H}'_S)$

$$\mathbf{H}_S = \begin{bmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$ similarity (scaled EM)

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$ special affine

$$\mathbf{H}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix}$$ special projective

$\mathbf{K}$ – upper triangular matrix with positive diagonal entries

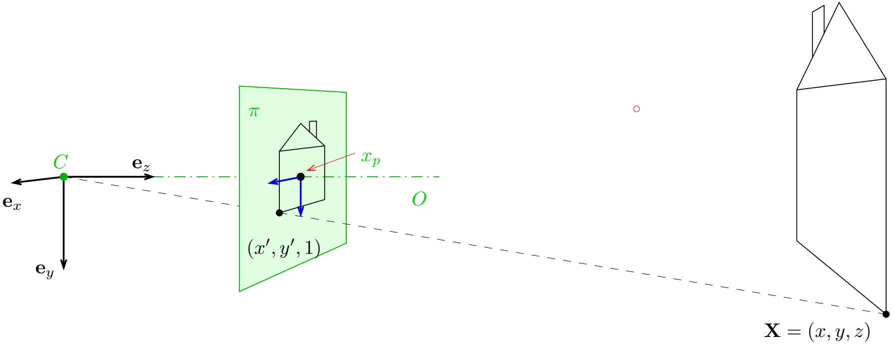
$\mathbf{R}$ – orthogonal, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = 1$

$s, w \in \mathbb{R}$, $s > 0$, $w \neq 0$

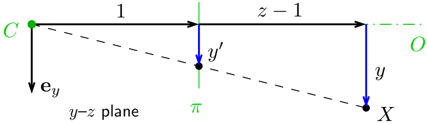
$$\mathbf{H} = \begin{bmatrix} s\mathbf{R}\mathbf{K} + \mathbf{t}\,\mathbf{v}^\top & w\,\mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

- must use 'thin' QR decomposition, which is unique [Golub & van Loan 2013, Sec. 5.2.6]
- $\mathbf{H}_S$, $\mathbf{H}_A$, $\mathbf{H}_P$ are homography subgroups (in the sense of group theory)
  (eg. $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$, $\mathbf{K}^{-1}$, $\mathbf{I}$ are all upper triangular with unit determinant, associativity holds)

►Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking 'down the street'
2. right-handed canonical coordinate system (x, y, z) with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$
3. origin $=$ center of projection C
4. image plane π at <u>unit</u> distance from C
5. optical axis O is perpendicular to π
6. principal point x_p: intersection of O and π
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

$$\frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}\,, \qquad x' = \frac{x}{z}$$

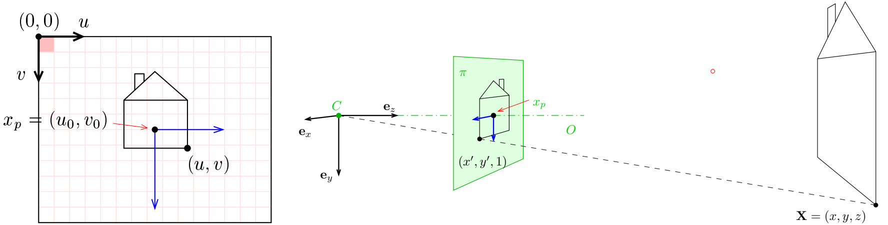
►Natural and Canonical Image Coordinate Systems

projected point **in canonical camera** $(z \neq 0)$

$$(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) = \frac{1}{z}(x, y, z) \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \, \underline{\mathbf{X}}$$

projected point **in scanned image**

scale by f and translate to (u_0, v_0)



$$u = f\,\frac{x}{z} + u_0 \qquad v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0\,\underline{\mathbf{X}} = \mathbf{P}\,\underline{\mathbf{X}}$$

- 'calibration' matrix $\mathbf{K}$ transforms <u>canonical</u> $\mathbf{P}_0$ to <u>standard</u> perspective camera $\mathbf{P}$

►Computing with Perspective Camera Projection Matrix

$$\underline{\mathbf{m}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \simeq \begin{bmatrix} f x + u_0 z \\ f y + v_0 z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}}_{(a)}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

(u_0, v_0) – principal point in pixels

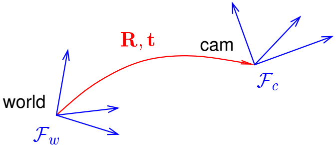
Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)

   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $m_3 = 0$ represents points at infinity in image plane π — i.e. points with $z = 0$

►Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:



$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame $\mathcal{F}_c$

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame $\mathcal{F}_c$

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ (a 3×4 mtx) selects the first 3 rows of $\mathbf{T}$ and discards the last row

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

$\mathbf{C}$ – camera position in the world reference frame $\mathcal{F}_w$ — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

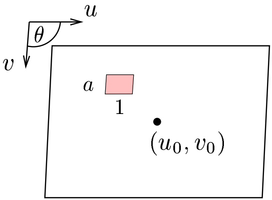
$\mathbf{r}_3^\top$ – optical axis in the world reference frame $\mathcal{F}_w$ — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

►Changing the Inner (Image) Reference Frame

The general form of calibration matrix $\mathbf{K}$ includes

- skew angle θ of the digitization raster
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} f & -f\,\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Verify this $\mathbf{K}$. Hints: (1) Map first by skew then by sampling scale then shift by u_0, v_0; (2) Skew: express point $\mathbf{x}$ as $\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u\mathbf{e}_u + v\mathbf{e}_v$, $\mathbf{e}_u$, $\mathbf{e}_v$ etc. are unit basis vectors, $\mathbf{K}$ maps from an orthogonal system to a skewed system $[w'u', w'v', w']^\top = \mathbf{K}[u, v, 1]^\top$; deadline LD+2 wk

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite perspective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left 3×3 submatrix $\mathbf{Q}$ non-singular.

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \quad \longrightarrow \quad \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$	full rank	(if finite perspective camera)
$\mathbf{K} \in \mathbb{R}^{3,3}$	upper triangular with positive diagonal entries	
$\mathbf{R} \in \mathbb{R}^{3,3}$	rotation:	$\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{I} & \mathbf{Q}^{-1}\mathbf{q}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & -\mathbf{R}\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$ also →36
2. RQ decomposition of $\mathbf{Q} = \mathbf{K}\mathbf{R}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q}\,\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g. (see next slide for derivation details)

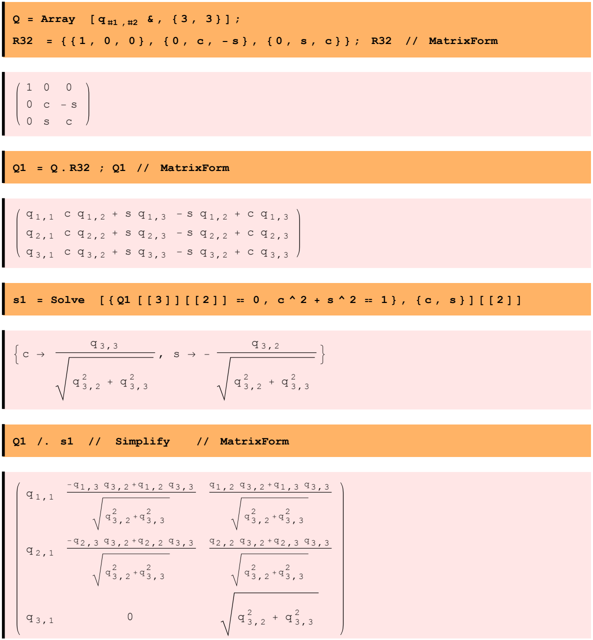
$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix} \text{ gives } \begin{matrix} c^2 + s^2 = 1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33}\end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{K}\mathbf{R} = \mathbf{K}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}$, where $\mathbf{T} = \mathrm{diag}(-1, -1, 1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive

  'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step



```
Q = Array [q_{#1,#2} &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q . R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1 [[3]] [[2]] == 0, c ^ 2 + s ^ 2 == 1}, {c, s}] [[2]]
```

$$\left\{ c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\, q_{3,2} + q_{1,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2}\, q_{3,2} + q_{1,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\, q_{3,2} + q_{2,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2}\, q_{3,2} + q_{2,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

▶Center of Projection

Observation: finite $\mathbf{P}$ has a non-trivial right null-space rank 3 but 4 columns

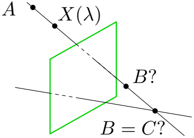
Theorem

Let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ *s.t.* $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. *Then* $\underline{\mathbf{B}}$ *is equal to the projection center* $\underline{\mathbf{C}}$ *(in world coordinate frame).*

Proof.

1. Consider spatial line $A\,B$ (B is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. it projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + (1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}} \simeq \mathbf{P}\,\underline{\mathbf{A}}$$

- the entire line projects to a single point ⇒ it must pass through the optical center of $\mathbf{P}$
- this holds for all choices of A ⇒ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \Rightarrow \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. consider line

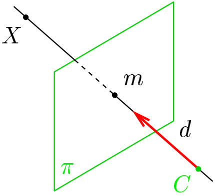
   $\mathbf{d}$ unit line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X} = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the projection of the (finite) point X is

$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X} \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$



…which is also the image of a point at infinity in $\mathbb{P}^3$

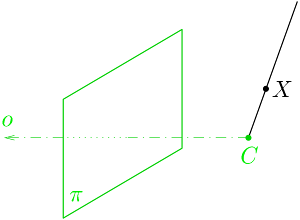
- optical ray line corresponding to image point m is

$$\boxed{\mathbf{X} = \mathbf{C} + (\lambda\mathbf{Q})^{-1}\underline{\mathbf{m}}, \qquad \lambda \in \mathbb{R}}$$

- optical ray may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π



1. a line parallel to π projects to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

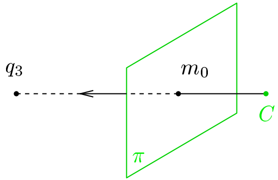
$$\boxed{\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3}$$

if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to principal point m_0



3. then

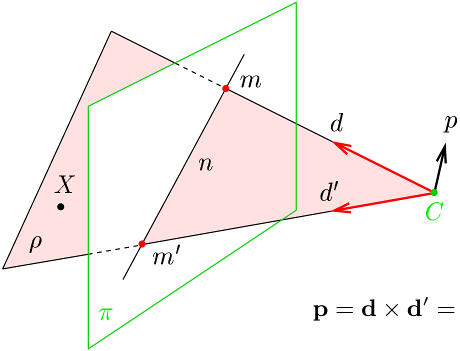
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\boxed{\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3}$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p passing through optical center C and a given image line n.



optical ray given by m $\quad \mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' = \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} = \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

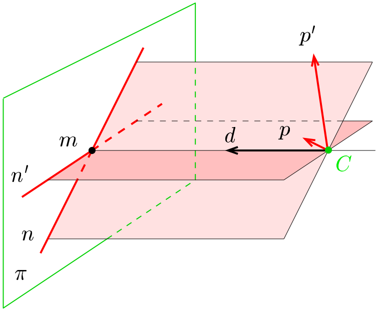
- note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X}-\mathbf{C}) = \underline{\mathbf{n}}^{\top}\underbrace{\mathbf{Q}(\mathbf{X}-\mathbf{C})}_{\to 32} = \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \boldsymbol{\rho} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}$

$\rho_1\,x + \rho_2\,y + \rho_3\,z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\quad\quad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\quad\quad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1} (\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1} \underline{\underline{\mathbf{m}}}$$

▶Summary: Optical Center, Ray, Axis, Plane

General finite camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \mathrm{rnull}(\mathbf{P})$ optical center (world coords.)

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ optical ray direction (world coords.)

$\det(\mathbf{Q})\,\mathbf{q}_3$ outward optical axis (world coords.)

$\mathbf{Q}\,\mathbf{q}_3$ principal point (in image plane)

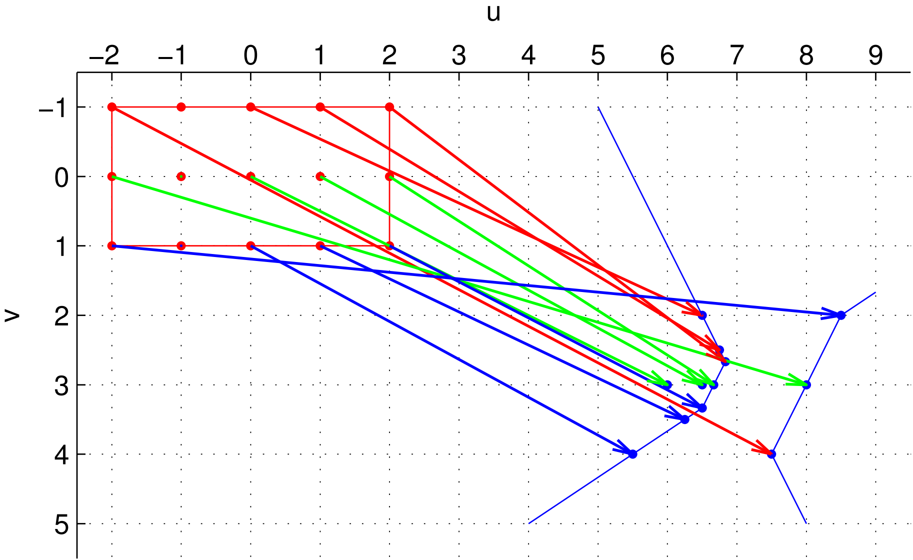
$\boldsymbol{\rho} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ optical plane (world coords.)

$$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$ camera (calibration) matrix (f, u_0, v_0 in pixels)

$\mathbf{R}$ camera rotation matrix (cam coords.)

$\mathbf{t}$ camera translation vector (cam coords.)

Thank You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