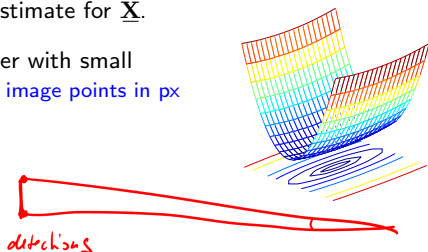


► Numerical Conditioning

- The equation $\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}$ in (15) may be ill-conditioned for numerical computation, which results in a poor estimate for $\underline{\mathbf{X}}$.

Why: on a row of \mathbf{D} there are big entries together with small entries, e.g. of orders projection centers in mm, image points in px

$$\begin{bmatrix} 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \\ 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \end{bmatrix}$$



Quick fix:

1. re-scale the problem by a regular diagonal conditioning matrix $\mathbf{S} \in \mathbb{R}^{4,4}$

$$\mathbf{0} = \mathbf{D}\underline{\mathbf{X}} = \mathbf{D}\mathbf{S}\mathbf{S}^{-1}\underline{\mathbf{X}} = \bar{\mathbf{D}}\bar{\underline{\mathbf{X}}}$$

choose \mathbf{S} to make the entries in $\hat{\mathbf{D}}$ all smaller than unity in absolute value:

$$\mathbf{S} = \text{diag}(10^{-3}, 10^{-3}, 10^{-3}, 10^{-6}) \quad \mathbf{S} = \text{diag}(1./\max(\text{abs}(\mathbf{D})), 1))$$

2. solve for $\bar{\underline{\mathbf{X}}}$ as before
3. get the final solution as $\underline{\mathbf{X}} = \mathbf{S}\bar{\underline{\mathbf{X}}}$

- when SVD is used in camera resection, conditioning is essential for success

→64

Algebraic Error vs Reprojection Error

- algebraic error (c – camera index, (u^c, v^c) – image coordinates)

from SVD → 91

$$\varepsilon^2 = \sigma_4^2 = \sum_{c=1}^2 \left[\left(u^c (\mathbf{p}_3^c)^T \mathbf{X} - (\mathbf{p}_1^c)^T \mathbf{X} \right)^2 + \left(v^c (\mathbf{p}_3^c)^T \mathbf{X} - (\mathbf{p}_2^c)^T \mathbf{X} \right)^2 \right]$$

- reprojection error

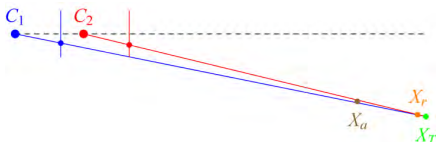
$$e^2 = \sum_{c=1}^2 \left[\left(u^c - \frac{(\mathbf{p}_1^c)^T \mathbf{X}}{(\mathbf{p}_3^c)^T \mathbf{X}} \right)^2 + \left(v^c - \frac{(\mathbf{p}_2^c)^T \mathbf{X}}{(\mathbf{p}_3^c)^T \mathbf{X}} \right)^2 \right]$$



- algebraic error zero \Rightarrow reprojection error zero
- epipolar constraint satisfied \Rightarrow equivalent results
- in general: minimizing algebraic error is cheap but it gives inferior results
- minimizing reprojection error is expensive but it gives good results
- the midpoint of the common perpendicular to both optical rays gives about 50% greater error in 3D
- the golden standard method – deferred to → 105

$\sigma_4 = 0 \Rightarrow$ non-trivial null space

Ex:



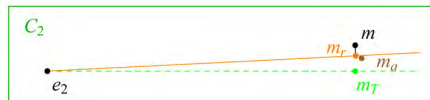
- forward camera motion
- error $f/50$ in image 2, orthogonal to epipolar plane

X_T – noiseless ground truth position

X_r – reprojection error minimizer

X_a – algebraic error minimizer

m – measurement (m_T with noise in v^2)



► We Have Added to The ZOO

continuation from →70

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	P	64
exterior orientation	K , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	R, t	68
fundamental matrix	7 img–img correspondences $\{(m_i, m'_i)\}_{i=1}^7$	F	84
relative orientation	K , 5 img–img correspondences $\{(m_i, m'_i)\}_{i=1}^5$	R, t	88
triangulation	P ₁ , P ₂ , 1 img–img correspondence (m_i, m'_i)	X	89

A bigger ZOO at <http://cmp.felk.cvut.cz/minimal/>

calibrated problems

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →117)

- algebraic error optimization (with SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on ‘optimizing optimally’

Optimization for 3D Vision

- 5.1 The Concept of Error for Epipolar Geometry
- 5.2 Levenberg-Marquardt's Iterative Optimization
- 5.3 The Correspondence Problem
- 5.4 Optimization by Random Sampling

covered by

- [1] [H&Z] Secs: 11.4, 11.6, 4.7
- [2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

additional references



P. D. Sampson. Fitting conic sections to 'very scattered' data: An iterative refinement of the Bookstein algorithm. *Computer Vision, Graphics, and Image Processing*, 18:97–108, 1982.



O. Chum, J. Matas, and J. Kittler. Locally optimized RANSAC. In *Proc DAGM, LNCS 2781:236–243*. Springer-Verlag, 2003.

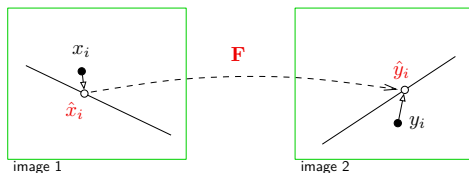


O. Chum, T. Werner, and J. Matas. Epipolar geometry estimation via RANSAC benefits from the oriented epipolar constraint. In *Proc ICPR, vol 1:112–115*, 2004.

► The Concept of Error for Epipolar Geometry

Problem: Given at least 8 matched points $x_i \leftrightarrow y_j$ in a general position, estimate the most likely (or most probable) fundamental matrix \mathbf{F} .

$$\mathbf{x}_i = (u_i^1, v_i^1), \quad \mathbf{y}_i = (u_i^2, v_i^2), \quad i = 1, 2, \dots, k, \quad k \geq 8$$



- detected points (measurements) x_i, y_i
- we introduce matches $\mathbf{Z}_i = (u_i^1, v_i^1, u_i^2, v_i^2) \in \mathbb{R}^4$; $S = \{\mathbf{Z}_i\}_{i=1}^k$
- corrected points \hat{x}_i, \hat{y}_i ; $\hat{\mathbf{Z}}_i = (\hat{u}_i^1, \hat{v}_i^1, \hat{u}_i^2, \hat{v}_i^2)$; $\hat{S} = \{\hat{\mathbf{Z}}_i\}_{i=1}^k$ are correspondences
- correspondences satisfy the epipolar geometry exactly $\hat{\mathbf{y}}_i^\top \mathbf{F} \hat{\mathbf{x}}_i = 0, i = 1, \dots, k$
- small correction is more probable
- let $\mathbf{e}_i(\cdot)$ be the 'reprojection error' (vector) per match i ,

$$\mathbf{e}_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \begin{bmatrix} \mathbf{x}_i - \hat{\mathbf{x}}_i \\ \mathbf{y}_i - \hat{\mathbf{y}}_i \end{bmatrix} = \mathbf{e}_i(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F}) = \mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F}) \quad (16)$$

$$\|\mathbf{e}_i(\cdot)\|^2 \stackrel{\text{def}}{=} \mathbf{e}_i^2(\cdot) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F})\|^2$$

- the total reprojection error (of all data) then is

$$L(S \mid \hat{S}, \mathbf{F}) = \sum_{i=1}^k \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \sum_{i=1}^k \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F})$$

- and the optimization problem is \mathbf{F}

$$(\hat{S}^*, \mathbf{F}^*) = \arg \min_{\substack{\mathbf{F} \\ \text{rank } \mathbf{F} = 2}} \min_{\substack{\hat{S} \\ \hat{\mathbf{y}}_i^\top \mathbf{F} \hat{\mathbf{x}}_i = 0}} \sum_{i=1}^k \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \quad (17)$$

Three possible approaches

- they differ in how the correspondences \hat{x}_i, \hat{y}_i are obtained:
 - direct optimization of reprojection error over all variables \hat{S}, \mathbf{F} →98
 - Sampson optimal correction = partial correction of \mathbf{Z}_i towards $\hat{\mathbf{Z}}_i$ used in an iterative minimization over \mathbf{F} →99
 - removing \hat{x}_i, \hat{y}_i altogether = marginalization of $L(S, \hat{S} \mid \mathbf{F})$ over \hat{S} followed by minimization over \mathbf{F} not covered, the marginalization is difficult

Method 1: Geometric Error Optimization

- we need to encode the constraints $\hat{\mathbf{y}}_i^T \mathbf{F} \hat{\mathbf{x}}_i = 0$, $\text{rank } \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z, Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = [\mathbf{I} \quad \mathbf{0}], \quad \mathbf{P}_2 = \begin{bmatrix} [\mathbf{e}_2]_{\times} \mathbf{F} + \mathbf{e}_2 \mathbf{e}_1^T & \mathbf{e}_2 \end{bmatrix} \quad (18)$$

⊛ H3; 2pt: Verify that \mathbf{F} is a f.m. of $\mathbf{P}_1, \mathbf{P}_2$. Hint: \mathbf{A} is skew symmetric iff $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$ for all \mathbf{x} .

1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm $\rightarrow 84$; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$ using (18)
2. triangulate 3D points $\hat{\mathbf{X}}_i^{(0)}$ from matches (x_i, y_i) for all $i = 1, \dots, k$ $\rightarrow 89$
3. starting from $\mathbf{P}_2^{(0)}, \hat{\mathbf{X}}^{(0)}$ minimize the reprojection error (16)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2, \hat{\mathbf{X}}} \sum_{i=1}^k e_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

where

$$\hat{\mathbf{Z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \text{ (Cartesian)}, \quad \hat{\mathbf{x}}_i \simeq \mathbf{P}_1 \hat{\mathbf{X}}_i, \quad \hat{\mathbf{y}}_i \simeq \mathbf{P}_2 \hat{\mathbf{X}}_i \text{ (homogeneous)}$$

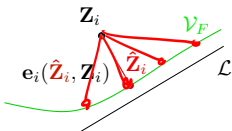
Non-linear, non-convex problem

4. compute \mathbf{F} from $\mathbf{P}_1, \mathbf{P}_2^*$
 - $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all i (correspondences!), non-latent: \mathbf{P}_2
 - minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ $\rightarrow 140$
 - there are pitfalls; this is essentially bundle adjustment; we will return to this later $\rightarrow 132$

► Method 2: First-Order Error Approximation

An elegant method for solving problems like (17):

- we will get rid of the latent parameters \hat{X} needed for obtaining the correction [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error $\epsilon = \underline{y}^\top \mathbf{F} \underline{x}$ from $\rightarrow 84$
- consider matches \mathbf{Z}_i , correspondences $\hat{\mathbf{Z}}_i$, and reprojection error $\mathbf{e}_i = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$
- correspondences satisfy $\hat{\underline{y}}_i^\top \mathbf{F} \hat{\underline{x}}_i = 0$, $\hat{\underline{x}}_i = (\hat{u}^1, \hat{v}^1, 1)$, $\hat{\underline{y}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold $\mathcal{V}_F \in \mathbb{R}^4$: a set of points $\hat{\mathbf{Z}} = (\hat{u}^1, \hat{v}^1, \hat{u}^2, \hat{v}^2)$ consistent with \mathbf{F}
- algebraic error vanishes for $\hat{\mathbf{Z}}_i$: $\mathbf{0} = \epsilon_i(\hat{\mathbf{Z}}_i) = \hat{\underline{y}}_i^\top \mathbf{F} \hat{\underline{x}}_i$



Sampson's idea: Linearize the algebraic error $\epsilon(\mathbf{Z})$ at \mathbf{Z}_i (where it is non-zero) and evaluate the resulting linear function at $\hat{\mathbf{Z}}_i$ (where it is zero). The zero-crossing replaces \mathcal{V}_F by a linear manifold \mathcal{L} . The point on \mathcal{V}_F closest to \mathbf{Z}_i is replaced by the closest point on \mathcal{L} .

$$\epsilon_i(\hat{\mathbf{Z}}_i) \approx \epsilon_i(\mathbf{Z}_i) + \frac{\partial \epsilon_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i} (\hat{\mathbf{Z}}_i - \mathbf{Z}_i) = \phi$$

► Sampson's Approximation of Reprojection Error

- linearize $\varepsilon(\mathbf{Z})$ at match \mathbf{Z}_i , evaluate it at correspondence $\hat{\mathbf{Z}}_i$

$$0 = \varepsilon_i(\hat{\mathbf{Z}}_i) \approx \varepsilon_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \varepsilon_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \stackrel{\text{def}}{=} \varepsilon_i(\mathbf{Z}_i) + \mathbf{J}_i(\mathbf{Z}_i) \mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$$

- goal: compute $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$ from $\varepsilon_i(\mathbf{Z}_i)$, where $\mathbf{e}_i(\cdot)$ is the distance of $\hat{\mathbf{Z}}_i$ from \mathbf{Z}_i
- we have a linear underconstrained equation for $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$
- we look for a minimal $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i) \stackrel{\text{def}}{=} \mathbf{e}_i$ per match i

$$\mathbf{e}_i^* = \arg \min_{\mathbf{e}_i} \|\mathbf{e}_i\|^2 \quad \text{subject to} \quad \varepsilon_i + \mathbf{J}_i \mathbf{e}_i = 0$$

- which has a closed-form solution **note that \mathbf{J}_i is not invertible!**

⊗ P1; 1pt: derive \mathbf{e}_i^*

$$\begin{aligned} \mathbf{e}_i^* &= -\mathbf{J}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \varepsilon_i \\ \|\mathbf{e}_i^*\|^2 &= \varepsilon_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \varepsilon_i \end{aligned} \quad \begin{matrix} \varepsilon(\mathbf{F}) \\ \mathbf{J}(\mathbf{F}) \end{matrix} \quad (19)$$

- this maps $\varepsilon_i(\cdot)$ to an estimate of $\mathbf{e}_i(\cdot)$ per correspondence
- we often do not need \mathbf{e}_i , just $\|\mathbf{e}_i\|^2$ exception: triangulation → 105
- the unknown parameters \mathbf{F} are inside: $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $\varepsilon_i = \varepsilon_i(\mathbf{F})$, $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

► Example: Fitting A Circle To Scattered Points

Problem: Fit a zero-centered circle \mathcal{C} to a set of 2D points $\{x_i\}_{i=1}^k$, $\mathcal{C}: \|\mathbf{x}\|^2 - r^2 = 0$.

1. consider radial error as the 'algebraic error' $\epsilon(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2$
2. linearize it at $\hat{\mathbf{x}}$ we are dropping i in ϵ_i , \mathbf{e}_i etc for clarity

$$\epsilon(\hat{\mathbf{x}}) \approx \epsilon(\mathbf{x}) + \underbrace{\frac{\partial \epsilon(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x})=2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}}, \mathbf{x})} = \dots = 2\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \stackrel{\text{def}}{=} \epsilon_L(\hat{\mathbf{x}})$$

$\epsilon_L(\hat{\mathbf{x}}) = 0$ is a line with normal $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and intercept $\frac{r^2 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$

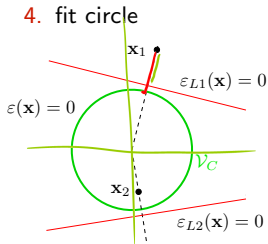
not tangent to \mathcal{C} , outside!

3. using (19), express error approximation \mathbf{e}^* as

$$\|\mathbf{e}^*\|^2 = \boldsymbol{\epsilon}^\top (\mathbf{J}\mathbf{J}^\top)^{-1} \boldsymbol{\epsilon} = \frac{(\|\mathbf{x}\|^2 - r^2)^2}{4\|\mathbf{x}\|^2}$$



4. fit circle



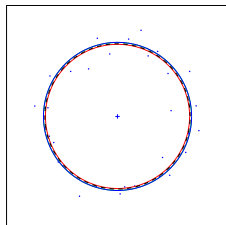
$$r^* = \arg \min_r \sum_{i=1}^k \frac{(\|\mathbf{x}_i\|^2 - r^2)^2}{4\|\mathbf{x}_i\|^2} = \dots = \left(\frac{1}{k} \sum_{i=1}^k \frac{1}{\|\mathbf{x}_i\|^2} \right)^{-\frac{1}{2}}$$

- this example results in a convex quadratic optimization problem
- note that

$$r^* = \arg \min_r \sum_{i=1}^k (\|\mathbf{x}_i\|^2 - r^2)^2 = \left(\frac{1}{k} \sum_{i=1}^k \|\mathbf{x}_i\|^2 \right)^{\frac{1}{2}}$$

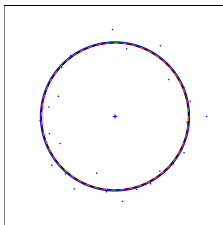
Circle Fitting: Some Results

medium isotropic noise



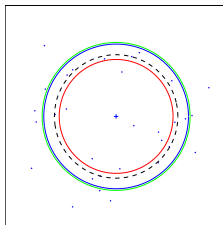
opt=1.8, dir=2.0, Smp=2.2

medium radial noise



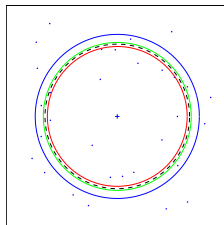
1.8, 1.9, 2.3

big isotropic noise



1.6, 2.0, 2.4

big radial noise



1.6, 1.8, 2.6

mean ranks over 10 000 random trials with $k = 32$ samples

green – ground truth

red – Sampson error minimizer

blue – direct radial error minimizer

black – optimal estimator for isotropic error

optimal estimator for isotropic error (black, dashed):

$$r \approx \frac{3}{4k} \sum_{i=1}^k \|\mathbf{x}_i\| + \sqrt{\left(\frac{3}{4k} \sum_{i=1}^k \|\mathbf{x}_i\| \right)^2 - \frac{1}{2k} \sum_{i=1}^k \|\mathbf{x}_i\|^2}$$

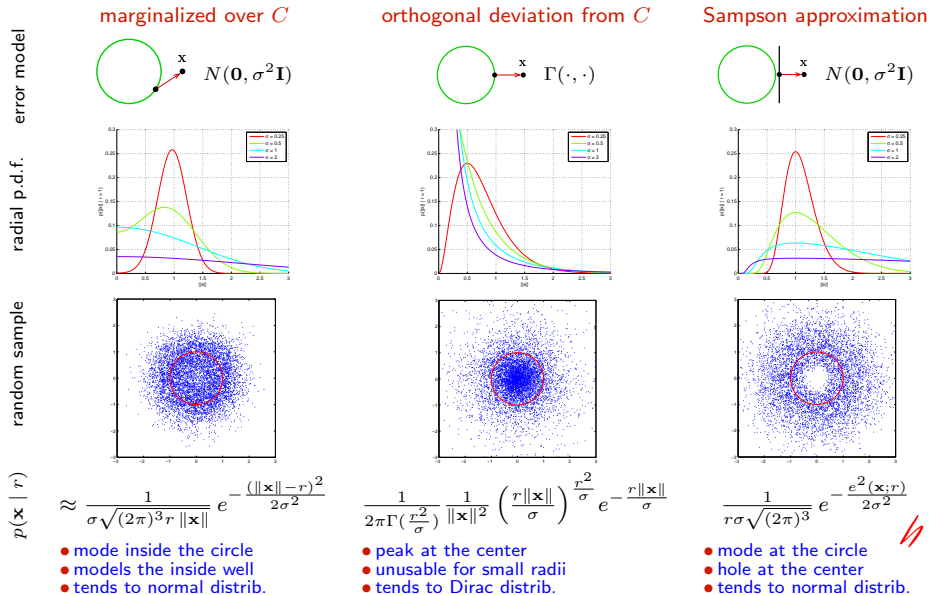
*Q: what error gave the ranks?
what radius*

which method is better?

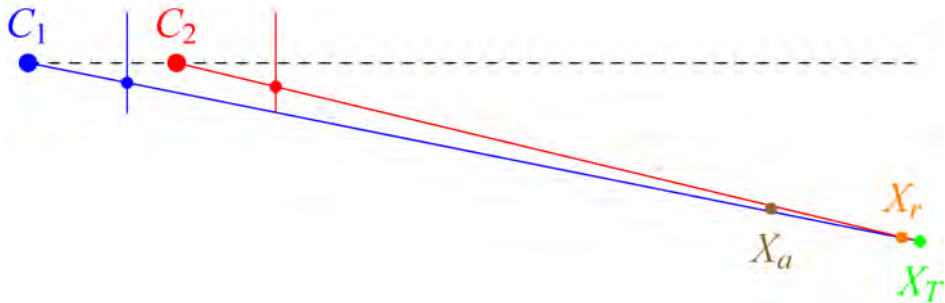
- error should model noise, radial noise and isotropic noise behave differently
- ground truth: Normally distributed isotropic error, Gamma-distributed radial error
- Sampson: better for the radial distribution model; Direct: better for the isotropic model
- no matter how corrected, the algebraic error minimizer is not an unbiased parameter estimator
Cramér-Rao bound tells us how close one can get with unbiased estimator and given k

Discussion: On The Art of Probabilistic Model Design...

- a few models for fitting zero-centered circle C of radius r to points in \mathbb{R}^2



Thank You



C_1 



