

# 3D Computer Vision

Radim Šára    Martin Matoušek

Center for Machine Perception  
Department of Cybernetics  
Faculty of Electrical Engineering  
Czech Technical University in Prague

<https://cw.felk.cvut.cz/doku.php/courses/a4m33tdv/>

<http://cmp.felk.cvut.cz>

<mailto:sara@cmp.felk.cvut.cz>

phone ext. 7203

rev. January 2, 2018



Open Informatics Master's Course

# Perspective Camera

- 2.1 Basic Entities: Points, Lines
- 2.2 Homography: Mapping Acting on Points and Lines
- 2.3 Canonical Perspective Camera
- 2.4 Changing the Outer and Inner Reference Frames
- 2.5 Projection Matrix Decomposition
- 2.6 Anatomy of Linear Perspective Camera
- 2.7 Vanishing Points and Lines

**covered by**

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, Example: 2.19

## ► Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	$n$	$O$
plane		$\pi, \varphi$

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^T, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as  $\mathbf{m} = (u, v)$ ,  $\mathbf{X} = (x, y, z)$ , etc.

- vectors are always meant to be columns  $\mathbf{x} \in \mathbb{R}^{n,1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^T, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^T, \quad \underline{\mathbf{n}}$$

'in-line' forms:  $\underline{\mathbf{m}} = (m_1, m_2, m_3)$ ,  $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$ , etc.

- matrices are  $\mathbf{Q} \in \mathbb{R}^{m,n}$ , linear map of a  $\mathbb{R}^{n,1}$  vector is  $\mathbf{y} = \mathbf{Q}\mathbf{x}$
- $j$ -th element of vector  $\mathbf{m}_i$  is  $(\mathbf{m}_i)_j$ ; element  $i, j$  of matrix  $\mathbf{P}$  is  $\mathbf{P}_{ij}$

## ► Image Line (in 2D)

a finite line in the 2D  $(u, v)$  plane

$$a u + b v + c = 0$$

corresponds to a (homogeneous) vector

$$\underline{\mathbf{n}} \simeq (a, b, c)$$

and there is an equivalence class for  $\lambda \in \mathbb{R}, \lambda \neq 0$   $(\lambda a, \lambda b, \lambda c) \simeq (a, b, c)$

### 'Finite' lines

- standard representative for finite  $\underline{\mathbf{n}} = (n_1, n_2, n_3)$  is  $\lambda \underline{\mathbf{n}}$ , where  $\lambda = \frac{1}{\sqrt{n_1^2 + n_2^2}}$   
assuming  $n_1^2 + n_2^2 \neq 0$ ;  $\mathbf{1}$  is the unit, usually  $\mathbf{1} = 1$

### 'Infinite' line

- we augment the set of lines for a special entity called the **Ideal Line** (line at infinity)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1) \quad (\text{standard representative})$$

- the set of equivalence classes of vectors in  $\mathbb{R}^3 \setminus (0, 0, 0)$  forms the projective space  $\mathbb{P}^2$   
a set of rays  $\rightarrow 22$
- line at infinity is a proper member of  $\mathbb{P}^2$
- I may sometimes wrongly use  $=$  instead of  $\simeq$ , if you are in doubt, ask me

## ► Image Point

Finite point  $\mathbf{m} = (u, v)$  is incident on a finite line  $\mathbf{n} = (a, b, c)$  iff      iff = works either way!

$$a u + b v + c = 0$$

can be rewritten as (with scalar product):  $(u, v, \mathbf{1}) \cdot (a, b, c) = \mathbf{m}^\top \mathbf{n} = 0$

### 'Finite' points

- a finite point is also represented by a homogeneous vector  $\mathbf{m} \simeq (u, v, \mathbf{1})$
- the equivalence class for  $\lambda \in \mathbb{R}, \lambda \neq 0$  is  $(m_1, m_2, m_3) = \lambda \mathbf{m} \simeq \mathbf{m}$
- the standard representative for finite point  $\mathbf{m}$  is  $\lambda \mathbf{m}$ , where  $\lambda = \frac{1}{m_3}$  assuming  $m_3 \neq 0$
- when  $\mathbf{1} = 1$  then units are pixels and  $\lambda \mathbf{m} = (u, v, 1)$
- when  $\mathbf{1} = f$  then all components have a similar magnitude,  $f \sim$  image diagonal  
use  $\mathbf{1} = 1$  unless you know what you are doing;  
all entities participating in a formula must be expressed in the same units

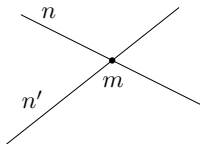
### 'Infinite' points

- we augment for **Ideal Points** (points at infinity)  $\mathbf{m}_\infty \simeq (m_1, m_2, 0)$   
proper members of  $\mathbb{P}^2$
- all such points lie on the ideal line (line at infinity)  $\mathbf{n}_\infty \simeq (0, 0, 1)$ , i.e.  $\mathbf{m}_\infty^\top \mathbf{n}_\infty = 0$

## ► Line Intersection and Point Join

The point of **intersection**  $m$  of image lines  $n$  and  $n'$ ,  $n \neq n'$  is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}'}$$



**proof:** If  $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}'}$  is the intersection point, it must be incident on both lines. Indeed, using known equivalences from vector algebra

$$\underline{\mathbf{n}}^T \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}'})}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}'^T} \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}'})}_{\underline{\mathbf{m}}} \equiv 0$$

The **join**  $n$  of two image points  $m$  and  $m'$ ,  $m \neq m'$  is

$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}'}$$

Parallel lines intersect (somewhere) on the line at infinity  $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$

$$a u + b v + c = 0,$$

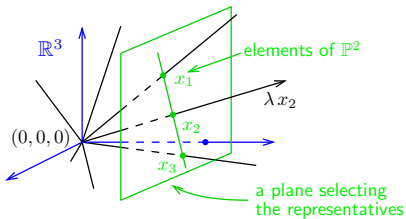
$$a u + b v + d = 0,$$

$$d \neq c$$

$$(a, b, c) \times (a, b, d) \simeq (b, -a, 0)$$

- all such intersections lie on  $\underline{\mathbf{n}}_\infty$
- line at infinity represents a set of directions in the plane
- Matlab: `m = cross(n, n_prime);`

## ► Homography in $\mathbb{P}^2$



Projective plane  $\mathbb{P}^2$ : Vector space of dimension 3 excluding the zero vector,  $\mathbb{R}^3 \setminus (0, 0, 0)$ , factorized to linear equivalence classes ('rays')

including 'points at infinity'

**Homography in  $\mathbb{P}^2$ :** Non-singular linear mapping in  $\mathbb{P}^2$

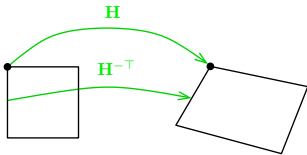
an analogic definition for  $\mathbb{P}^3$

$$\underline{\mathbf{x}}' \simeq \mathbf{H} \underline{\mathbf{x}}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

### Defining properties

- collinear image points are mapped to collinear image points  
lines of points are mapped to lines of points
- concurrent image lines are mapped to concurrent image lines  
concurrent = intersecting at a point
- and point-line incidence is preserved  
e.g. line intersection points mapped to line intersection points
- $\mathbf{H}$  is a  $3 \times 3$  non-singular matrix,  $\lambda \mathbf{H} \simeq \mathbf{H}$  equivalence class, 8 degrees of freedom
- homogeneous matrix representant:  $\det \mathbf{H} = 1$
- what we call homography here is often called 'projective collineation' in mathematics

## ► Mapping 2D Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H} \underline{\mathbf{m}} \quad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}} \quad \text{image line}$$

$$\mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$$

- incidence is preserved:  $(\underline{\mathbf{m}}')^{\top} \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top} \mathbf{H}^{\top} \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top} \underline{\mathbf{n}} = 0$

Mapping a finite 2D point  $\mathbf{m} = (u, v)$  to  $\underline{\mathbf{m}} = (u', v')$

- extend the Cartesian (pixel) coordinates to homogeneous coordinates,  $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
- map by homography,  $\underline{\mathbf{m}}' = \mathbf{H} \underline{\mathbf{m}}$
- if  $m'_3 \neq 0$  convert the result  $\underline{\mathbf{m}}' = (m'_1, m'_2, m'_3)$  back to Cartesian coordinates (pixels),

$$u' = \frac{m'_1}{m'_3} \mathbf{1}, \quad v' = \frac{m'_2}{m'_3} \mathbf{1}$$

- note that, typically,  $m'_3 \neq 1$
- an infinite point  $(u, v, 0)$  maps the same way

$m'_3 = 1$  when  $\mathbf{H}$  is affine



# Some Homographic Tasters

**Rectification of camera rotation:** →61 (geometry), →125 (homography estimation)



$$\mathbf{H} \simeq \mathbf{K} \mathbf{R}^T \mathbf{K}^{-1}$$

maps from image plane to facade plane

**Homographic Mouse for Visual Odometry:** [Mallis 2007]



illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

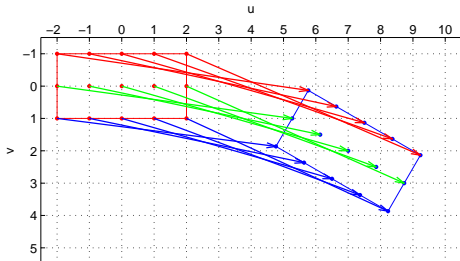
$$\mathbf{H} \simeq \mathbf{K} \left( \mathbf{R} - \frac{\mathbf{t} \mathbf{n}^T}{d} \right) \mathbf{K}^{-1} \quad [\text{H\&Z, p. 327}]$$

## ► Homography Subgroups: Euclidean Mapping (aka Rigid Motion)

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos \phi & -\sin \phi & t_x \\ \sin \phi & \cos \phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- eigenvalues  $(1, e^{-i\phi}, e^{i\phi})$



**EM = The most general homography preserving**

1. **areas:**  $\det \mathbf{H} = 1 \Rightarrow$  unit Jacobian

2. **lengths:** Let  $\underline{\mathbf{x}}'_i = \mathbf{H}\underline{\mathbf{x}}_i$  (check we can use = instead of  $\simeq$ ). Let  $(x_i)_3 = 1$ , Then

$$\|\underline{\mathbf{x}}'_2 - \underline{\mathbf{x}}'_1\| = \|\mathbf{H}\underline{\mathbf{x}}_2 - \mathbf{H}\underline{\mathbf{x}}_1\| = \|\mathbf{H}(\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1)\| = \dots = \|\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1\|$$

3. **angles** check the dot-product of normalized differences from a point  $(\mathbf{x} - \mathbf{z})^\top (\mathbf{y} - \mathbf{z})$  (Cartesian(!))

- eigenvectors when  $\phi \neq k\pi$ ,  $k = 0, 1, \dots$  (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot \frac{\phi}{2} \\ t_y - t_x \cot \frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

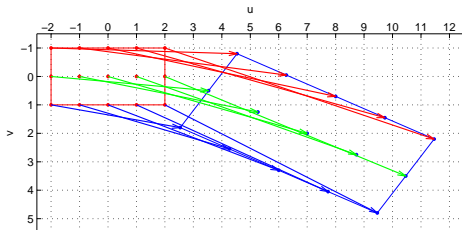
$\mathbf{e}_2, \mathbf{e}_3$  – circular points,  $i$  – imaginary unit

4. **circular points:** points at infinity  $(i, 1, 0)$ ,  $(-i, 1, 0)$  (preserved even by similarity)

- **similarity:** scaled Euclidean mapping (does not preserve lengths, areas)

## ► Homography Subgroups: Affine Mapping

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by  $30^\circ$   
then scaling by  $\text{diag}(1, 1.5, 1)$   
then translation by  $(7, 2)$

### AM = The most general homography preserving

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints)
- convex hull
- line at infinity  $\underline{n}_\infty$  (not pointwise)

### does not preserve

- lengths
- angles
- areas
- circular points

observe  $\mathbf{H}^T \underline{n}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{n}_\infty \Rightarrow \underline{n}_\infty \simeq \mathbf{H}^{-T} \underline{n}_\infty$

Euclidean mappings preserve all properties affine mappings preserve, of course

## ► Homography Subgroups: General Homography

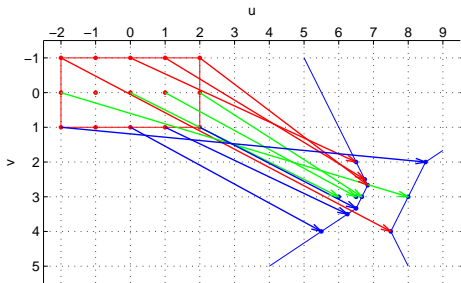
$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

### preserves only

- incidence and concurrency
- collinearity
- cross-ratio on the line  $\rightarrow 47$

### does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors (midpoints, etc.)
- convex hull
- line at infinity  $\underline{n}_\infty$



$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line  $\underline{n} = (1, 0, 1)$  is mapped to  $\underline{n}_\infty$ :  $\mathbf{H}^{-T} \underline{n} \simeq \underline{n}_\infty$

(where in the picture is the line  $\underline{n}$ ?)

# Elementary Decomposition of a Homography

**Unique decompositions:**  $\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P \quad (= \mathbf{H}'_P \mathbf{H}'_A \mathbf{H}'_S)$

$$\mathbf{H}_S = \begin{bmatrix} s \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \quad \text{similarity (scaled EM)}$$

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} \quad \text{special affine}$$

$$\mathbf{H}_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & w \end{bmatrix} \quad \text{special projective}$$

$\mathbf{K}$  – upper triangular matrix with positive diagonal entries

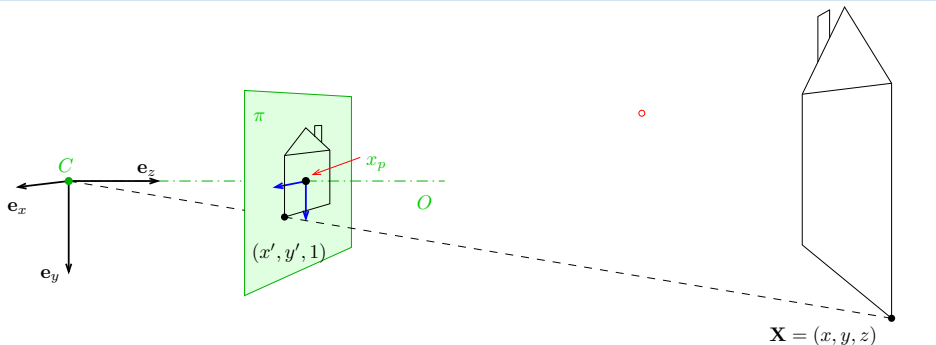
$\mathbf{R}$  – orthogonal,  $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ ,  $\det \mathbf{R} = 1$

$s, w \in \mathbb{R}$ ,  $s > 0$ ,  $w \neq 0$

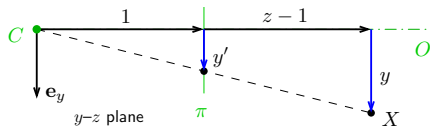
$$\mathbf{H} = \begin{bmatrix} s \mathbf{R} \mathbf{K} + \mathbf{t} \mathbf{v}^\top & w \mathbf{t} \\ \mathbf{v}^\top & w \end{bmatrix}$$

- must use 'thin' QR decomposition, which is unique [Golub & van Loan 2013, Sec. 5.2.6]
- $\mathbf{H}_S$ ,  $\mathbf{H}_A$ ,  $\mathbf{H}_P$  are homography subgroups (in the sense of group theory)  
(eg.  $\mathbf{K} = \mathbf{K}_1 \mathbf{K}_2$ ,  $\mathbf{K}^{-1}$ ,  $\mathbf{I}$  are all upper triangular with unit determinant, associativity holds)

## ► Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking 'down the street'
2. right-handed canonical coordinate system  $(x, y, z)$  with unit vectors  $e_x, e_y, e_z$
3. origin = center of projection  $C$
4. image plane  $\pi$  at unit distance from  $C$
5. optical axis  $O$  is perpendicular to  $\pi$
6. principal point  $x_p$ : intersection of  $O$  and  $\pi$
7. perspective camera is given by  $C$  and  $\pi$



projected point in the natural image coordinate system:

$$\frac{y'}{1} = y' = \frac{y}{1 + z - 1} = \frac{y}{z}, \quad x' = \frac{x}{z}$$

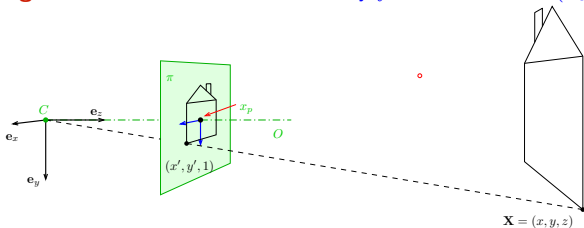
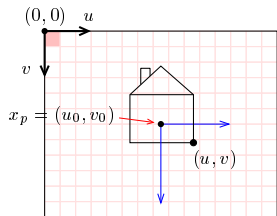
## ► Natural and Canonical Image Coordinate Systems

projected point **in canonical camera** ( $z \neq 0$ )

$$(x', y', 1) = \left( \frac{x}{z}, \frac{y}{z}, 1 \right) = \frac{1}{z}(x, y, z) \simeq \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \underline{\mathbf{X}}$$

projected point **in scanned image**

scale by  $f$  and translate to  $(u_0, v_0)$



$$\begin{aligned} u &= f \frac{x}{z} + u_0 \\ v &= f \frac{y}{z} + v_0 \end{aligned} \quad \frac{1}{z} \begin{bmatrix} f x + z u_0 \\ f y + z v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K} \mathbf{P}_0 \underline{\mathbf{X}} = \mathbf{P} \underline{\mathbf{X}}$$

- 'calibration' matrix  $\mathbf{K}$  transforms canonical  $\mathbf{P}_0$  to standard perspective camera  $\mathbf{P}$

## ► Computing with Perspective Camera Projection Matrix

$$\underline{\mathbf{m}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \simeq \begin{bmatrix} fx + u_0z \\ fy + v_0z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f}u_0 \\ y + \frac{z}{f}v_0 \\ \frac{z}{f} \end{bmatrix}}_{(a)}$$

$$\frac{m_1}{m_3} = \frac{fx}{z} + u_0 = u, \quad \frac{m_2}{m_3} = \frac{fy}{z} + v_0 = v \quad \text{when } m_3 \neq 0$$

$f$  – ‘focal length’ – converts length ratios to pixels,  $[f] = \text{px}$ ,  $f > 0$

$(u_0, v_0)$  – principal point in pixels

### Perspective Camera:

1. dimension reduction since  $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change  $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$ , see (a)  
for convenience we use  $P_{11} = P_{22} = f$  rather than  $P_{33} = 1/f$  and the  $u_0, v_0$  in relative units
3.  $m_3 = 0$  represents points at infinity in image plane  $\pi$  i.e. points with  $z = 0$



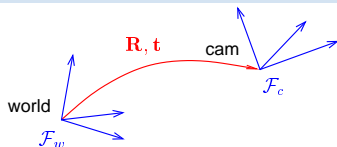
## ► Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:

$$\mathbf{X}_c = \mathbf{R} \mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$  – camera rotation matrix

$\mathbf{t}$  – camera translation vector



world origin in the camera coordinate frame  $\mathcal{F}_c$

world origin in the camera coordinate frame  $\mathcal{F}_c$

$$\mathbf{P} \underline{\mathbf{X}}_c = \mathbf{K} \mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K} \mathbf{P}_0 \begin{bmatrix} \mathbf{R} \mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K} \mathbf{P}_0 \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K} [\mathbf{R} \quad \mathbf{t}] \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$  (a  $3 \times 4$  mtx) selects the first 3 rows of  $\mathbf{T}$  and discards the last row

- $\mathbf{R}$  is rotation,  $\mathbf{R}^\top \mathbf{R} = \mathbf{I}$ ,  $\det \mathbf{R} = +1$   $\mathbf{I} \in \mathbb{R}^{3,3}$  identity matrix
- **6 extrinsic parameters:** 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K} [\mathbf{R} \quad \mathbf{t}] = \mathbf{K} \mathbf{R} [\mathbf{I} \quad -\mathbf{C}]$$

$\mathbf{C}$  – camera position in the world reference frame  $\mathcal{F}_w$   
 $\mathbf{r}_3^\top$  – optical axis in the world reference frame  $\mathcal{F}_w$

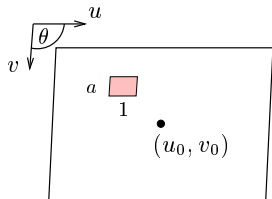
$\mathbf{t} = -\mathbf{R} \mathbf{C}$   
third row of  $\mathbf{R}$ :  $\mathbf{r}_3 = \mathbf{R}^{-1} [0, 0, 1]^\top$

- we can save some conversion and computation by noting that  $\mathbf{K} \mathbf{R} [\mathbf{I} \quad -\mathbf{C}] \underline{\mathbf{X}} = \mathbf{K} \mathbf{R} (\underline{\mathbf{X}} - \mathbf{C})$

## ► Changing the Inner (Image) Reference Frame

The general form of calibration matrix  $\mathbf{K}$  includes

- skew angle  $\theta$  of the digitization raster
- pixel aspect ratio  $a$



$$\mathbf{K} = \begin{bmatrix} f & -f \cot \theta & u_0 \\ 0 & f/(a \sin \theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units:  $[f] = \text{px}$ ,  $[u_0] = \text{px}$ ,  $[v_0] = \text{px}$ ,  $[a] = 1$

⊛ H1; 2pt: Verify this  $\mathbf{K}$ . Hints: (1) Map first by skew then by sampling scale then shift by  $u_0, v_0$ ; (2) Skew: express point  $\mathbf{x}$  as  $\mathbf{x} = u' \mathbf{e}_{u'} + v' \mathbf{e}_{v'} = u \mathbf{e}_u + v \mathbf{e}_v$ ,  $\mathbf{e}_u, \mathbf{e}_v$  etc. are unit basis vectors,  $\mathbf{K}$  maps from an orthogonal system to a skewed system  $[w' u', w' v', w']^T = \mathbf{K}[u, v, 1]^T$ ; deadline LD+2 wk

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters:  $f, u_0, v_0, a, \theta$
- 6 extrinsic parameters:  $\mathbf{t}, \mathbf{R}(\alpha, \beta, \gamma)$

finite camera:  $\det \mathbf{K} \neq 0$

$$\underline{\mathbf{m}} \simeq \mathbf{P} \underline{\mathbf{X}}, \quad \mathbf{P} = [\mathbf{Q} \quad \mathbf{q}] = \mathbf{K} [\mathbf{R} \quad \mathbf{t}] = \mathbf{K} \mathbf{R} [\mathbf{I} \quad -\mathbf{C}]$$

a recipe for filling  $\mathbf{P}$

Representation Theorem: The set of projection matrices  $\mathbf{P}$  of finite perspective cameras is isomorphic to the set of homogeneous  $3 \times 4$  matrices with the left  $3 \times 3$  submatrix  $\mathbf{Q}$  non-singular.



## RQ Decomposition Step

```
Q = Array [q_{#1,#2} &, {3, 3}];  
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q . R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c q_{1,2} + s q_{1,3} & -s q_{1,2} + c q_{1,3} \\ q_{2,1} & c q_{2,2} + s q_{2,3} & -s q_{2,2} + c q_{2,3} \\ q_{3,1} & c q_{3,2} + s q_{3,3} & -s q_{3,2} + c q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] = 0, c^2 + s^2 = 1}, {c, s}][[2]]
```

$$\left\{ c \rightarrow \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}}, s \rightarrow -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3} q_{3,2} + q_{1,2} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2} q_{3,2} + q_{1,3} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3} q_{3,2} + q_{2,2} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2} q_{3,2} + q_{2,3} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

## ► Center of Projection

**Observation:** finite  $\mathbf{P}$  has a non-trivial right null-space

rank 3 but 4 columns

### Theorem

Let there be  $\underline{\mathbf{B}} \neq \mathbf{0}$  s.t.  $\mathbf{P} \underline{\mathbf{B}} = \mathbf{0}$ . Then  $\underline{\mathbf{B}}$  is equal to the projection center  $\underline{\mathbf{C}}$  (homogeneous, in world coordinate frame).

### Proof.

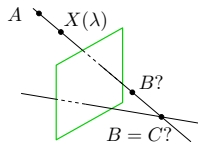
1. Consider spatial line  $AB$  ( $B$  is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda \underline{\mathbf{A}} + (1 - \lambda) \underline{\mathbf{B}}, \quad \lambda \in \mathbb{R}$$

2. it projects to

$$\mathbf{P} \underline{\mathbf{X}}(\lambda) \simeq \lambda \mathbf{P} \underline{\mathbf{A}} + (1 - \lambda) \mathbf{P} \underline{\mathbf{B}} \simeq \mathbf{P} \underline{\mathbf{A}}$$

- the entire line projects to a single point  $\Rightarrow$  it must pass through the optical center of  $\mathbf{P}$
- this holds for all choices of  $A \Rightarrow$  the only common point of the lines is the  $C$ , i.e.  $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$



Hence

$$\mathbf{0} = \mathbf{P} \underline{\mathbf{C}} = [\mathbf{Q} \quad \mathbf{q}] \begin{bmatrix} \underline{\mathbf{C}} \\ 1 \end{bmatrix} = \mathbf{Q} \underline{\mathbf{C}} + \mathbf{q} \Rightarrow \underline{\mathbf{C}} = -\mathbf{Q}^{-1} \mathbf{q}$$

$\underline{\mathbf{C}} = (c_j)$ , where  $c_j = (-1)^j \det \mathbf{P}^{(j)}$ , in which  $\mathbf{P}^{(j)}$  is  $\mathbf{P}$  with column  $j$  dropped

Matlab: `C_homo = null(P)`; or `C = -Q\q`;

## ► Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. consider line

$\mathbf{d}$  unit line direction vector,  $\|\mathbf{d}\| = 1$ ,  $\lambda \in \mathbb{R}$ , Cartesian representation

$$\mathbf{X} = \mathbf{C} + \lambda \mathbf{d}$$

2. the projection of the (finite) point  $X$  is

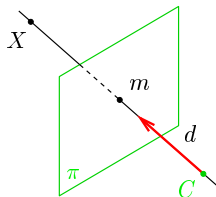
$$\begin{aligned} \underline{\mathbf{m}} &\simeq [\mathbf{Q} \quad \mathbf{q}] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda \mathbf{d}) + \mathbf{q} = \lambda \mathbf{Q} \mathbf{d} = \\ &= \lambda [\mathbf{Q} \quad \mathbf{q}] \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} \end{aligned}$$

... which is also the image of a point at infinity in  $\mathbb{P}^3$

- optical ray line corresponding to image point  $m$  is the set

$$\mathbf{X} = \mathbf{C} + (\lambda \mathbf{Q})^{-1} \underline{\mathbf{m}}, \quad \lambda \in \mathbb{R}$$

- optical ray direction may be represented by a point at infinity  $(\mathbf{d}, 0)$  in  $\mathbb{P}^3$



## ► Optical Axis

Optical axis: Optical ray that is perpendicular to image plane  $\pi$

1. points on a line parallel to  $\pi$  project to line at infinity in  $\pi$ :

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

2. therefore the set of points  $X$  is parallel to  $\pi$  iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

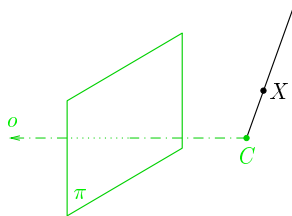
3. this is a plane with  $\pm \mathbf{q}_3$  as the normal vector
4. optical axis direction: substitution  $\mathbf{P} \mapsto \lambda \mathbf{P}$  must not change the direction
5. we select (assuming  $\det(\mathbf{R}) > 0$ )

$$\mathbf{o} = \det(\mathbf{Q}) \mathbf{q}_3$$

if  $\mathbf{P} \mapsto \lambda \mathbf{P}$  then  $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$  and  $\mathbf{q}_3 \mapsto \lambda \mathbf{q}_3$

[H&Z, p. 161]

- the axis is expressed in world coordinate frame



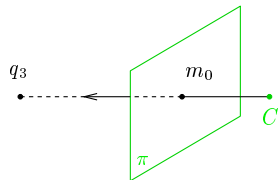
## ► Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw,  $\mathbf{q}_3$  is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to principal point  $m_0$

3. then

$$\underline{\mathbf{m}}_0 \simeq [\mathbf{Q} \quad \mathbf{q}] \begin{bmatrix} \mathbf{q}_3 \\ 0 \end{bmatrix} = \mathbf{Q} \mathbf{q}_3$$



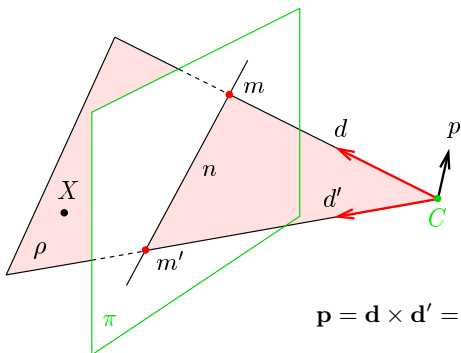
principal point:  $\underline{\mathbf{m}}_0 \simeq \mathbf{Q} \mathbf{q}_3$

- principal point is also the center of radial distortion



## ► Optical Plane

A spatial plane with normal  $p$  passing through optical center  $C$  and a given image line  $n$ .



optical ray given by  $m$      $\underline{d} = \mathbf{Q}^{-1} \underline{m}$

optical ray given by  $m'$      $\underline{d}' = \mathbf{Q}^{-1} \underline{m}'$

$$\underline{p} = \underline{d} \times \underline{d}' = (\mathbf{Q}^{-1} \underline{m}) \times (\mathbf{Q}^{-1} \underline{m}') = \mathbf{Q}^T (\underline{m} \times \underline{m}') = \mathbf{Q}^T \underline{n}$$

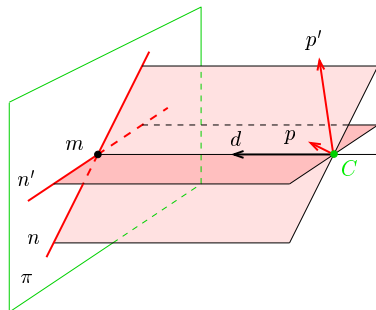
• note the way  $\mathbf{Q}$  factors out!

hence,  $0 = \underline{p}^T (\underline{X} - \underline{C}) = \underline{n}^T \underbrace{\mathbf{Q}(\underline{X} - \underline{C})}_{\rightarrow 32} = \underline{n}^T \mathbf{P} \underline{X} = (\mathbf{P}^T \underline{n})^T \underline{X}$  for every  $X$  in plane  $\rho$

optical plane is given by  $n$ :  $\rho \simeq \mathbf{P}^T \underline{n}$

$$\rho_1 x + \rho_2 y + \rho_3 z + \rho_4 = 0$$

## Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by  $\underline{n}$

$$\underline{p} = \mathbf{Q}^T \underline{n}$$

optical plane normal given by  $\underline{n}'$

$$\underline{p}' = \mathbf{Q}^T \underline{n}'$$

$$\underline{d} = \underline{p} \times \underline{p}' = (\mathbf{Q}^T \underline{n}) \times (\mathbf{Q}^T \underline{n}') = \mathbf{Q}^{-1}(\underline{n} \times \underline{n}') = \mathbf{Q}^{-1} \underline{m}$$

## ► Summary: Optical Center, Ray, Axis, Plane

General finite camera

$$\mathbf{P} = [\mathbf{Q} \quad \mathbf{q}] = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} = \mathbf{K} [\mathbf{R} \quad \mathbf{t}] = \mathbf{K} \mathbf{R} [\mathbf{I} \quad -\mathbf{C}]$$

$\underline{\mathbf{C}} \simeq \text{rnull}(\mathbf{P})$  optical center (world coords.)

$\mathbf{d} = \mathbf{Q}^{-1} \underline{\mathbf{m}}$  optical ray direction (world coords.)

$\det(\mathbf{Q}) \mathbf{q}_3$  outward optical axis (world coords.)

$\mathbf{Q} \mathbf{q}_3$  principal point (in image plane)

$\rho = \mathbf{P}^\top \underline{\mathbf{n}}$  optical plane (world coords.)

$\mathbf{K} = \begin{bmatrix} f & -f \cot \theta & u_0 \\ 0 & f/(a \sin \theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$  camera (calibration) matrix ( $f, u_0, v_0$  in pixels)

$\mathbf{R}$  camera rotation matrix (cam coords.)

$\mathbf{t}$  camera translation vector (cam coords.)

# What Can We Do with An 'Uncalibrated' Perspective Camera?



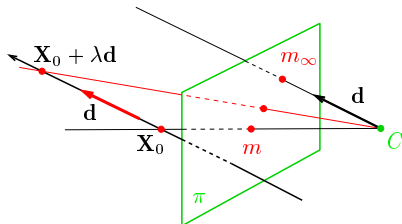
How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, image resolution is too low

We will review some life-saving theory...  
...and build a bit of geometric intuition...

## ► Vanishing Point

**Vanishing point:** the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



$$\underline{m}_\infty \simeq \lim_{\lambda \rightarrow \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda \mathbf{d} \\ 1 \end{bmatrix} = \dots \simeq \mathbf{Q} \mathbf{d}$$

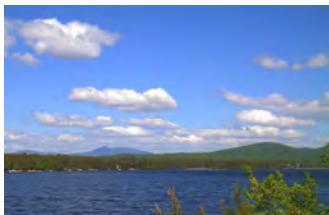
⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

- the V.P. of a spatial line with directional vector  $\mathbf{d}$  is  $\underline{m}_\infty \simeq \mathbf{Q} \mathbf{d}$
- V.P. is independent on line position  $\mathbf{X}_0$ , it depends on its directional vector only
- all parallel lines share the same V.P., including the optical ray defined by  $m_\infty$

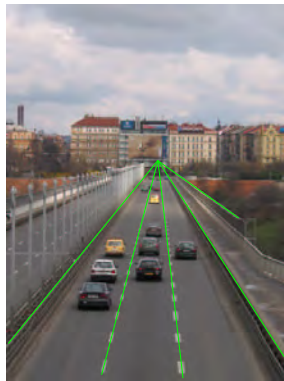
# Some Vanishing Point “Applications”



where is the sun?



what is the wind direction?  
(must have video)

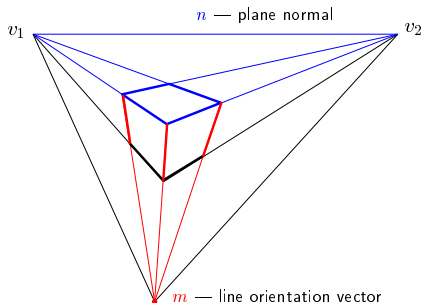


fly above the lane,  
at constant altitude!

## ► Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane  
and in all parallel planes

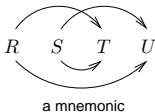


- V.L.  $n$  corresponds to spatial plane of normal vector  $\mathbf{p} = \mathbf{Q}^T \underline{n}$   
because this is the normal vector of a parallel optical plane (!) →40
- a spatial plane of normal vector  $\mathbf{p}$  has a V.L. represented by  $\underline{n} = \mathbf{Q}^{-T} \mathbf{p}$ .

## ► Cross Ratio

Four distinct collinear spatial points  $R, S, T, U$  define cross-ratio

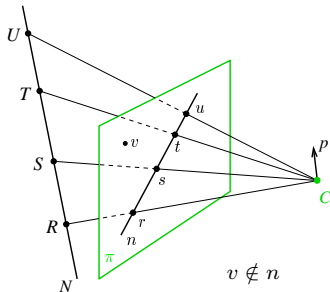
$$[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{RU}|} \frac{|\overrightarrow{SU}|}{|\overrightarrow{ST}|}$$



$|\overrightarrow{RT}|$  – distance from  $R$  to  $T$  in the arrow direction

6 cross-ratios from four points:

$$[SRUT] = [RSTU], [RSUT] = \frac{1}{[RSTU]}, [RTSU] = 1 - [RSTU], \dots$$



**Obs:**  $[RSTU] = \frac{|\underline{\mathbf{r}} \ \underline{\mathbf{t}} \ \underline{\mathbf{v}}|}{|\underline{\mathbf{r}} \ \underline{\mathbf{u}} \ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{s}} \ \underline{\mathbf{u}} \ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}} \ \underline{\mathbf{t}} \ \underline{\mathbf{v}}|}, \quad |\underline{\mathbf{r}} \ \underline{\mathbf{t}} \ \underline{\mathbf{v}}| = \det [\underline{\mathbf{r}} \ \underline{\mathbf{t}} \ \underline{\mathbf{v}}] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \quad (1)$

### Corollaries:

- cross ratio is invariant under homographies  $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$  plug  $\mathbf{H}\underline{\mathbf{x}}$  in (1):  $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection:  $[RSTU] = [rstu]$
- 4 collinear points: any perspective camera will “see” the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points  $R, S, T, U$  may be at infinity (we take the limit, in effect  $\frac{\infty}{\infty} = 1$ )



## ► 1D Projective Coordinates

The 1-D projective coordinate of a point  $P$  is defined by the following cross-ratio:

$$[P] = [P_\infty P_0 P_I P] = [p_\infty p_0 p_I p] = \frac{|\overrightarrow{p_\infty p_I}|}{|\overrightarrow{p_0 p_I}|} \frac{|\overrightarrow{p_0 p}|}{|\overrightarrow{p_\infty p}|} = [p]$$

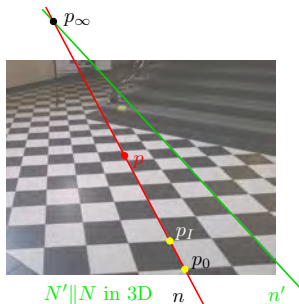
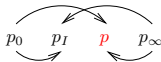
naming convention:

$P_0$ – the origin	$[P_0] = 0$
$P_I$ – the unit point	$[P_I] = 1$
$P_\infty$ – the supporting point	$[P_\infty] = \pm\infty$

$[P]$  is equal to Euclidean coordinate along  $N$

$[p]$  is its measurement in the image plane

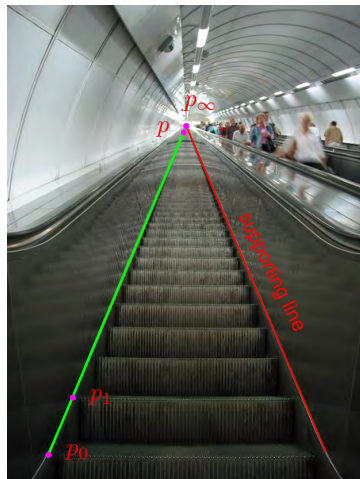
moving  $p_\infty$  to the opposite side



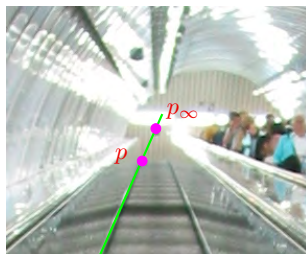
### Applications

- Given the image of a 3D line  $N$ , the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point  $P \in N$  can be determined →49
- Finding v.p. of a line through a regular object →50

# Application: Counting Steps



- Namesti Miru underground station in Prague

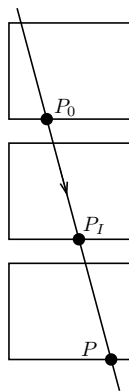
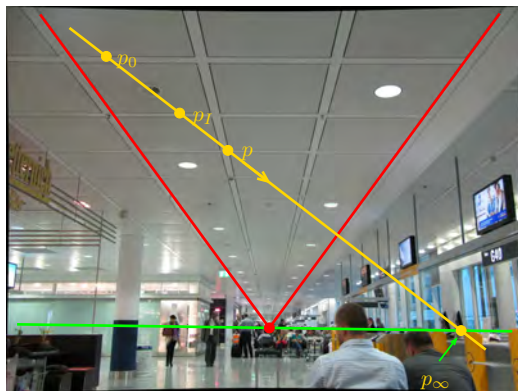


detail around the vanishing point

**Result:**  $[P] = 214$  steps (correct answer is 216 steps)

4Mpx camera

# Application: Finding the Horizon from Repetitions



[H&Z, p. 218]

in 3D:  $|P_0P| = 2|P_0P_I|$  then

$$[P_\infty P_0 P_I P] = \frac{|P_0P|}{|P_0P_I|} = 2 \quad \Rightarrow \quad x_\infty = \frac{x_0(2x - x_1) - xx_1}{x + x_0 - 2x_1}$$

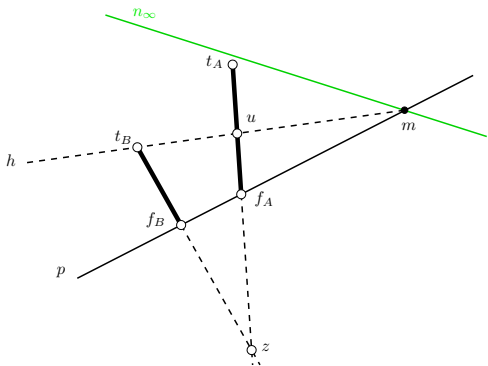
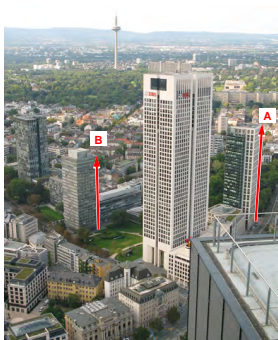
- $x$  - 1D coordinate along the yellow line, positive in the arrow direction
- could be applied to counting steps ( $\rightarrow 49$ ) if there was no supporting line

⊛ P1; 1pt: How high is the camera above the floor?

# Homework Problem

⊛ H2; 3pt: What is the ratio of heights of Building A to Building B?

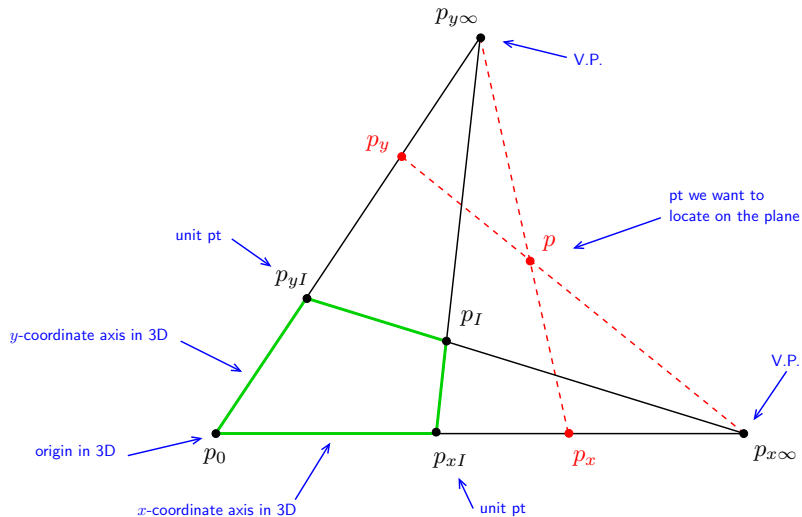
- expected: conceptual solution; use notation from this figure
- deadline: LD+2 weeks



## Hints

1. What are the interesting properties of line  $h$  connecting the top  $t_B$  of Building B with the point  $m$  at which the horizon intersects the line  $p$  joining the feet  $f_A, f_B$  of both buildings? [1 point]
2. How do we actually get the horizon  $n_\infty$ ? (we do not see it directly, there are some hills there...) [1 point]
3. Give the formula for measuring the length ratio. [formula = 1 point]

# 2D Projective Coordinates



$$[P_x] = [P_{x\infty} \ P_0 \ P_{xI} \ P_x]$$

$$[P_y] = [P_{y\infty} \ P_0 \ P_{yI} \ P_y]$$

## Application: Measuring on the Floor (Wall, etc)



San Giovanni in Laterano, Rome

- measuring distances on the floor in terms of tile units
- what are the dimensions of the seal? Is it circular (assuming square tiles)?
- needs no explicit camera calibration

because we can see the calibrating object (vanishing points)

## Part III

# Computing with a Single Camera

- 3.1 Calibration: Internal Camera Parameters from Vanishing Points and Lines
- 3.2 Camera Resection: Projection Matrix from 6 Known Points
- 3.3 Exterior Orientation: Camera Rotation and Translation from 3 Known Points

### covered by

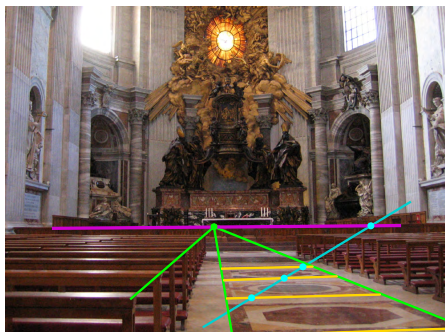
- [1] [H&Z] Secs: 8.6, 7.1, 22.1
- [2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981
- [3] [Golub & van Loan 2013, Sec. 2.5]

# Obtaining Vanishing Points and Lines

- orthogonal direction pairs can be collected from more images by camera rotation



- vanishing line can be obtained without vanishing points ( $\rightarrow$ 50)





## ► Camera Calibration from Vanishing Points and Lines

**Problem:** Given finite vanishing points and/or vanishing lines, compute  $\mathbf{K}$

$$\mathbf{d}_i \simeq \mathbf{Q}^{-1} \mathbf{v}_i, \quad i = 1, 2, 3 \quad \rightarrow 44 \quad (2)$$

$$\mathbf{p}_{ij} \simeq \mathbf{Q}^\top \mathbf{n}_{ij}, \quad i, j = 1, 2, 3, i \neq j \quad \rightarrow 40$$

- simple method: solve (2) after eliminating nuisance pars.

### Special Configurations

1. orthogonal rays  $\mathbf{d}_1 \perp \mathbf{d}_2$  in space then

$$0 = \mathbf{d}_1^\top \mathbf{d}_2 = \mathbf{v}_1^\top \mathbf{Q}^{-\top} \mathbf{Q}^{-1} \mathbf{v}_2 = \mathbf{v}_1^\top \underbrace{(\mathbf{K}\mathbf{K}^\top)^{-1}}_{\omega \text{ (IAC)}} \mathbf{v}_2$$

2. orthogonal planes  $\mathbf{p}_{ij} \perp \mathbf{p}_{ik}$  in space

$$0 = \mathbf{p}_{ij}^\top \mathbf{p}_{ik} = \mathbf{n}_{ij}^\top \mathbf{Q}\mathbf{Q}^\top \mathbf{n}_{ik} = \mathbf{n}_{ij}^\top \omega^{-1} \mathbf{n}_{ik}$$

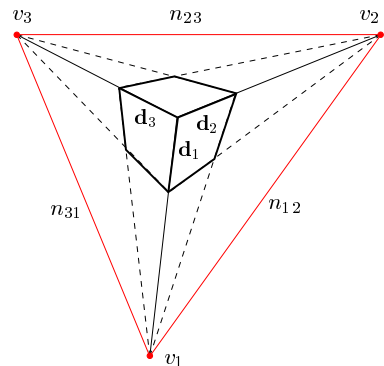
3. orthogonal ray and plane  $\mathbf{d}_k \parallel \mathbf{p}_{ij}, k \neq i, j$  normal parallel to optical ray

$$\mathbf{p}_{ij} \simeq \mathbf{d}_k \Rightarrow \mathbf{Q}^\top \mathbf{n}_{ij} = \lambda \mathbf{Q}^{-1} \mathbf{v}_k \Rightarrow \mathbf{n}_{ij} = \lambda \mathbf{Q}^{-\top} \mathbf{Q}^{-1} \mathbf{v}_k = \lambda \omega \mathbf{v}_k, \quad \lambda \neq 0$$

- $n_{ij}$  may be constructed from non-orthogonal  $v_i$  and  $v_j$ , e.g. using the cross-ratio

- $\omega$  is a symmetric, positive definite  $3 \times 3$  matrix

IAC = Image of Absolute Conic



	configuration	equation	# constraints
(3)	orthogonal v.p.	$\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j = 0$	1
(4)	orthogonal v.l.	$\underline{\mathbf{n}}_{ij}^\top \boldsymbol{\omega}^{-1} \underline{\mathbf{n}}_{ik} = 0$	1
(5)	v.p. orthogonal to v.l.	$\underline{\mathbf{n}}_{ij} = \lambda \boldsymbol{\omega} \underline{\mathbf{v}}_k$	2
(6)	orthogonal raster $\theta = \pi/2$	$\omega_{12} = \omega_{21} = 0$	1
(7)	unit aspect $a = 1$ when $\theta = \pi/2$	$\omega_{11} - \omega_{22} = 0$	1
(8)	known principal point $u_0 = v_0 = 0$	$\omega_{13} = \omega_{31} = \omega_{23} = \omega_{32} = 0$	2

- these are homogeneous linear equations for the 5 parameters in  $\boldsymbol{\omega}$  in the form  $\mathbf{D}\boldsymbol{\omega} = \mathbf{0}$   
 $\lambda$  can be eliminated from (5)
- we need at least 5 constraints for full  $\boldsymbol{\omega}$  symmetric  $3 \times 3$
- we get  $\mathbf{K}$  from  $\boldsymbol{\omega}^{-1} = \mathbf{K}\mathbf{K}^\top$  by Choleski decomposition  
the decomposition returns a positive definite upper triangular matrix  
one avoids solving an explicit set of quadratic equations for the parameters in  $\mathbf{K}$
- unlike in the naive method (2), we can introduce constraints on  $\mathbf{K}$ , e.g. (6)–(8)

# Examples

Assuming orthogonal raster, unit aspect (ORUA):  $\theta = \pi/2$ ,  $a = 1$

$$\boldsymbol{\omega} \simeq \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ -u_0 & -v_0 & f^2 + u_0^2 + v_0^2 \end{bmatrix}$$

## Ex 1:

Assuming ORUA and known  $m_0 = (u_0, v_0)$ , two finite orthogonal vanishing points give  $f$

$$\underline{\mathbf{v}}_1^\top \boldsymbol{\omega} \underline{\mathbf{v}}_2 = 0 \quad \Rightarrow \quad f^2 = |(\mathbf{v}_1 - \mathbf{m}_0)^\top (\mathbf{v}_2 - \mathbf{m}_0)|$$

in this formula,  $\mathbf{v}_i$ ,  $\mathbf{m}_0$  are not homogeneous!

## Ex 2:

Non-orthogonal vanishing points  $\mathbf{v}_i$ ,  $\mathbf{v}_j$ , known angle  $\phi$ :  $\cos \phi = \frac{\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j}{\sqrt{\underline{\mathbf{v}}_i^\top \boldsymbol{\omega} \underline{\mathbf{v}}_i} \sqrt{\underline{\mathbf{v}}_j^\top \boldsymbol{\omega} \underline{\mathbf{v}}_j}}$

- leads to polynomial equations
- e.g. ORUA and  $u_0 = v_0 = 0$  gives

$$(f^2 + \mathbf{v}_i^\top \mathbf{v}_j)^2 = (f^2 + \|\mathbf{v}_i\|^2) \cdot (f^2 + \|\mathbf{v}_j\|^2) \cdot \cos^2 \phi$$

# Image of Absolute Conic

This is the  $\mathbf{K}$  matrix:

$$\mathbf{K} = \{\{f, s, u_0\}, \{0, a \cdot f, v_0\}, \{0, 0, 1\}\}$$

$$\begin{pmatrix} f & s & u_0 \\ 0 & a f & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

The  $\omega$  matrix:

$$\omega = \text{Inverse}[\mathbf{K}.\text{Transpose}[\mathbf{K}]] * \text{Det}[\mathbf{K}]^2 // \text{Simplify}$$

$$\begin{pmatrix} a^2 f^2 & -a f s & a f (s v_0 - a f u_0) \\ -a f s & f^2 + s^2 & a f s u_0 - (f^2 + s^2) v_0 \\ a f (s v_0 - a f u_0) & a f s u_0 - (f^2 + s^2) v_0 & a^2 f^4 + a^2 u_0^2 f^2 - 2 a s u_0 v_0 f + (f^2 + s^2) v_0^2 \end{pmatrix}$$

The  $\omega$  matrix with no skew:

$$\omega / f^2 /. s \rightarrow 0 // \text{Simplify} // \text{MatrixForm}$$

$$\begin{pmatrix} a^2 & 0 & -a^2 u_0 \\ 0 & 1 & -v_0 \\ -a^2 u_0 & -v_0 & a^2 f^2 + a^2 u_0^2 + v_0^2 \end{pmatrix}$$

ORUA

$$\omega / f^2 /. \{a \rightarrow 1, s \rightarrow 0\} // \text{Simplify}$$

$$\begin{pmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ -u_0 & -v_0 & f^2 + u_0^2 + v_0^2 \end{pmatrix}$$

## ► Camera Orientation from Two Finite Vanishing Points

**Problem:** Given  $\mathbf{K}$  and two vanishing points corresponding to two known orthogonal directions  $\mathbf{d}_1, \mathbf{d}_2$ , compute camera orientation  $\mathbf{R}$  with respect to the plane.

- 3D coordinate system choice, e.g.:

$$\mathbf{d}_1 = (1, 0, 0), \quad \mathbf{d}_2 = (0, 1, 0)$$

- we know that

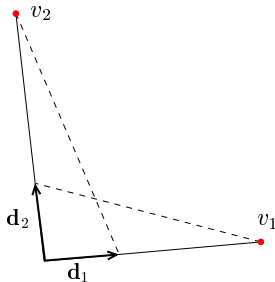
$$\mathbf{d}_i \simeq \mathbf{Q}^{-1} \mathbf{v}_i = (\mathbf{K}\mathbf{R})^{-1} \mathbf{v}_i = \mathbf{R}^{-1} \underbrace{\mathbf{K}^{-1} \mathbf{v}_i}_{\mathbf{w}_i}$$

$$\mathbf{R}\mathbf{d}_i \simeq \mathbf{w}_i$$

- knowing  $\mathbf{d}_{1,2}$  we conclude that  $\mathbf{w}_i / \|\mathbf{w}_i\|$  is the  $i$ -th column  $\mathbf{r}_i$  of  $\mathbf{R}$
- the third column is orthogonal:

$$\mathbf{r}_3 \simeq \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{R} = \begin{bmatrix} \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} & \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} & \frac{\mathbf{w}_1 \times \mathbf{w}_2}{\|\mathbf{w}_1 \times \mathbf{w}_2\|} \end{bmatrix}$$

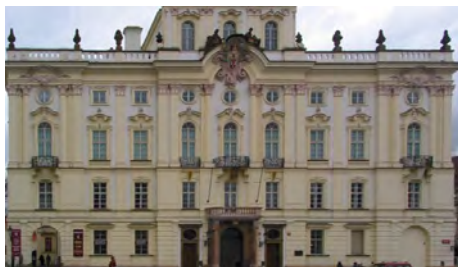


some suitable scenes



# Application: Planar Rectification

**Principle:** Rotate camera parallel to the plane of interest.



$$\underline{\mathbf{m}} \simeq \mathbf{K}\mathbf{R} [\mathbf{I} \quad -\mathbf{C}] \underline{\mathbf{X}}$$

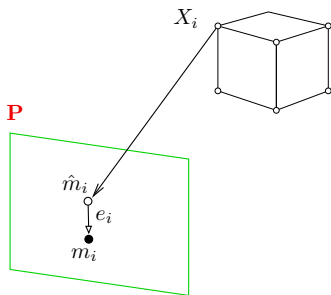
$$\underline{\mathbf{m}}' \simeq \mathbf{K} [\mathbf{I} \quad -\mathbf{C}] \underline{\mathbf{X}}$$

$$\underline{\mathbf{m}}' \simeq \mathbf{K}(\mathbf{K}\mathbf{R})^{-1} \underline{\mathbf{m}} = \mathbf{K}\mathbf{R}^{\top} \mathbf{K}^{-1} \underline{\mathbf{m}} = \mathbf{H} \underline{\mathbf{m}}$$

- $\mathbf{H}$  is the rectifying homography
- both  $\mathbf{K}$  and  $\mathbf{R}$  can be calibrated from two finite vanishing points [assuming ORUA](#) →58
- not possible when one (or both) of them are infinite
- without ORUA we would need 4 additional views to calibrate  $\mathbf{K}$  as on →55

## ► Camera Resection

Camera calibration and orientation from a known set of  $k \geq 6$  reference points and their images  $\{(X_i, m_i)\}_{i=1}^6$ .

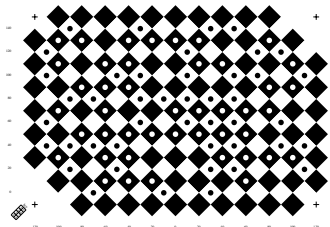


- $X_i$  are considered exact
- $m_i$  is a measurement subject to detection error

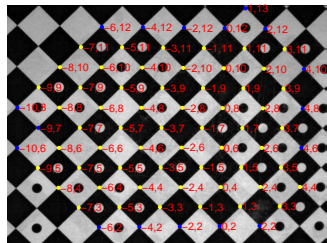
$$\mathbf{m}_i = \hat{\mathbf{m}}_i + \mathbf{e}_i \quad \text{Cartesian}$$

- where  $\underline{\hat{\mathbf{m}}}_i \simeq \mathbf{P}\underline{\mathbf{X}}_i$

# Resection Targets



calibration chart



automatic calibration point detection



resection target with translation stage

- target translated at least once
- by a calibrated (known) translation
- $X_i$  point locations looked up in a table based on their code



## ► The Minimal Problem for Camera Resection

**Problem:** Given  $k = 6$  corresponding pairs  $\{(X_i, m_i)\}_{i=1}^k$ , find  $\mathbf{P}$

$$\lambda_i \underline{m}_i = \mathbf{P} \underline{X}_i, \quad \mathbf{P} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \quad \begin{array}{l} \underline{X}_i = (x_i, y_i, z_i, 1), \quad i = 1, 2, \dots, k, \quad k = 6 \\ \underline{m}_i = (u_i, v_i, 1), \quad \lambda_i \in \mathbb{R}, \quad \lambda_i \neq 0 \end{array}$$

easy to modify for infinite points  $X_i$  but be aware of  $\rightarrow 66$

expanded:  $\lambda_i u_i = \mathbf{q}_1^\top \mathbf{X}_i + q_{14}, \quad \lambda_i v_i = \mathbf{q}_2^\top \mathbf{X}_i + q_{24}, \quad \lambda_i = \mathbf{q}_3^\top \mathbf{X}_i + q_{34}$

after elimination of  $\lambda_i$ :  $(\mathbf{q}_3^\top \mathbf{X}_i + q_{34})u_i = \mathbf{q}_1^\top \mathbf{X}_i + q_{14}, \quad (\mathbf{q}_3^\top \mathbf{X}_i + q_{34})v_i = \mathbf{q}_2^\top \mathbf{X}_i + q_{24}$

Then

$$\mathbf{A} \mathbf{q} = \begin{bmatrix} \mathbf{X}_1^\top & 1 & \mathbf{0}^\top & 0 & -u_1 \mathbf{X}_1^\top & -u_1 \\ \mathbf{0}^\top & 0 & \mathbf{X}_1^\top & 1 & -v_1 \mathbf{X}_1^\top & -v_1 \\ \vdots & & & & & \\ \mathbf{X}_k^\top & 1 & \mathbf{0}^\top & 0 & -u_k \mathbf{X}_k^\top & -u_k \\ \mathbf{0}^\top & 0 & \mathbf{X}_k^\top & 1 & -v_k \mathbf{X}_k^\top & -v_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ q_{14} \\ \mathbf{q}_2 \\ q_{24} \\ \mathbf{q}_3 \\ q_{34} \end{bmatrix} = \mathbf{0} \quad (9)$$

- we need 11 independent parameters for  $\mathbf{P}$
- $\mathbf{A} \in \mathbb{R}^{2k, 12}$ ,  $\mathbf{q} \in \mathbb{R}^{12}$
- 6 points in a general position give  $\text{rank } \mathbf{A} = 12$  and there is no non-trivial null space
- drop one row to get rank 11 matrix, then the basis vector of the null space of  $\mathbf{A}$  gives  $\mathbf{q}$

## ► The Jack-Knife Solution for $k = 6$

- given the 6 correspondences, we have 12 equations for the 11 parameters
- can we use all the information present in the 6 points?

### Jack-knife estimation

1.  $n := 0$
2. for  $i = 1, 2, \dots, 2k$  do
  - a) delete  $i$ -th row from  $\mathbf{A}$ , this gives  $\mathbf{A}_i$
  - b) if  $\dim \text{null } \mathbf{A}_i > 1$  continue with the next  $i$
  - c)  $n := n + 1$
  - d) compute the right null-space  $\mathbf{q}_i$  of  $\mathbf{A}_i$
  - e)  $\hat{\mathbf{q}}_i := \mathbf{q}_i$  normalized to  $q_{34} = 1$  and dimension-reduced
3. from all  $n$  vectors  $\hat{\mathbf{q}}_i$  collected in Step 1d compute

$$\mathbf{q} = \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{q}}_i, \quad \text{var}[\mathbf{q}] = \frac{n-1}{n} \text{diag} \sum_{i=1}^n (\hat{\mathbf{q}}_i - \mathbf{q})(\hat{\mathbf{q}}_i - \mathbf{q})^\top \quad \text{regular for } n \geq 11$$

- have a solution + an error estimate, per individual elements of  $\mathbf{P}$  (except  $P_{34}$ )
- at least 5 points must be in a general position ( $\rightarrow 66$ )
- large error indicates near degeneracy
- computation not efficient with  $k > 6$  points, needs  $\binom{2k}{11}$  draws, e.g.  $k = 7 \Rightarrow 364$  draws
- better error estimation method: decompose  $\mathbf{P}_i$  to  $\mathbf{K}_i, \mathbf{R}_i, \mathbf{t}_i$  ( $\rightarrow 34$ ), represent  $\mathbf{R}_i$  with 3 parameters (e.g. Euler angles, or in Cayley representation  $\rightarrow 140$ ) and compute the errors for the parameters



e.g. by 'economy-size' SVD  
assuming finite cam. with  $P_{3,4} = 1$

## ► Degenerate (Critical) Configurations for Camera Resection

Let  $\mathcal{X} = \{X_i; i = 1, \dots\}$  be a set of points and  $\mathbf{P}_1 \neq \mathbf{P}_j$  be two regular (rank-3) cameras. Then two configurations  $(\mathbf{P}_1, \mathcal{X})$  and  $(\mathbf{P}_j, \mathcal{X})$  are image-equivalent if

$$\mathbf{P}_1 \underline{\mathbf{X}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i \quad \text{for all } X_i \in \mathcal{X}$$

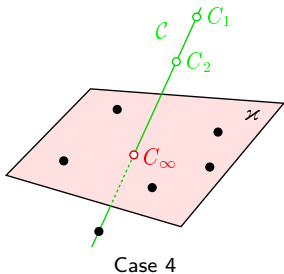
there is a non-trivial set of other cameras that see the same image

- **importantly:** If all calibration points  $X_i \in \mathcal{X}$  lie on a plane  $\varkappa$  then camera resection is non-unique and all image-equivalent camera centers lie on a spatial line  $\mathcal{C}$  with the  $C_\infty = \varkappa \cap \mathcal{C}$  excluded

this also means we cannot resect if all  $X_i$  are infinite

- by adding points  $X_i \in \mathcal{X}$  to  $\mathcal{C}$  we gain nothing
- there are additional image-equivalent configurations, see next

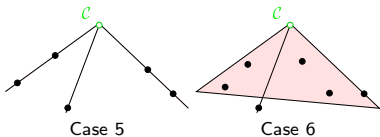
proof sketch in [H&Z, Sec. 22.1.2]



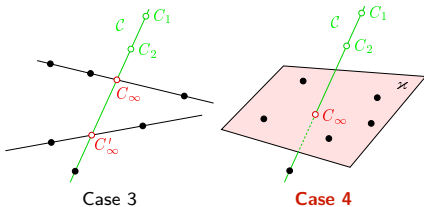
Note that if  $\mathbf{Q}, \mathbf{T}$  are suitable homographies then  $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$ , where  $\mathbf{P}_0$  is canonical and the analysis can be made with  $\hat{\mathbf{P}}_j \simeq \mathbf{Q}^{-1}\mathbf{P}_j$

$$\mathbf{P}_0 \underbrace{\underline{\mathbf{T}}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \simeq \hat{\mathbf{P}}_j \underbrace{\underline{\mathbf{T}}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \quad \text{for all } Y_i \in \mathcal{Y}$$

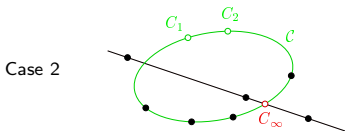
# cont'd (all cases)



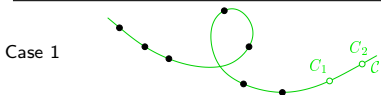
- cameras  $C_1, C_2$  co-located at point  $C$
- points on three optical rays or one optical ray and one optical plane
- Case 5: camera sees 3 isolated point images
- Case 6: cam. sees a line of points and an isolated point



- cameras lie on a line  $C \setminus \{C_\infty, C'_\infty\}$
- points lie on  $C$  and
  1. on two lines meeting  $C$  at  $C_\infty, C'_\infty$
  2. or on a plane meeting  $C$  at  $C_\infty$
- Case 3: camera sees 2 lines of points



- cameras lie on a planar conic  $C \setminus \{C_\infty\}$   
not necessarily an ellipse
- points lie on  $C$  and an additional line meeting the conic at  $C_\infty$
- Case 2: camera sees 2 lines of points



- cameras and points all lie on a twisted cubic  $C$
- Case 1: camera sees a conic

## ► Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

**Problem:** Given  $\mathbf{K}$  and three corresponding pairs  $\{(m_i, X_i)\}_{i=1}^3$ , find  $\mathbf{R}$ ,  $\mathbf{C}$  by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}(\mathbf{X}_i - \mathbf{C}), \quad i = 1, 2, 3$$

1. Transform  $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$ . Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}(\mathbf{X}_i - \mathbf{C}). \quad (10)$$

2. Eliminate  $\mathbf{R}$  by taking rotation preserves length:  $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \quad (11)$$

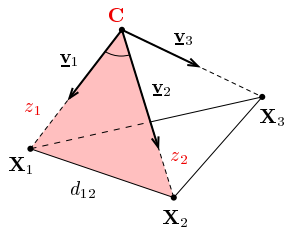
3. Consider only angles among  $\underline{\mathbf{v}}_i$  and apply Cosine Law per triangle  $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$   $i, j = 1, 2, 3, i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2 z_i z_j c_{ij},$$

$$z_i = \|\mathbf{X}_i - \mathbf{C}\|, \quad d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|, \quad c_{ij} = \cos(\angle \underline{\mathbf{v}}_i \underline{\mathbf{v}}_j)$$

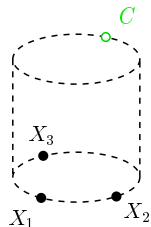
4. Solve system of 3 quadratic eqs in 3 unknowns  $z_i$  [Fischler & Bolles, 1981]  
there may be no real root; there are up to 4 solutions that cannot be ignored (verify on additional points)
5. Compute  $\mathbf{C}$  by trilateration (3-sphere intersection) from  $\mathbf{X}_i$  and  $z_i$ ; then  $\lambda_i$  from (11) and  $\mathbf{R}$  from (10)

configuration w/o rotation in (11)



Similar problems (P4P with unknown  $f$ ) at <http://cmp.felk.cvut.cz/minimal/> (with code)

# Degenerate (Critical) Configurations for Exterior Orientation



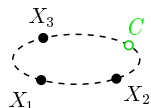
## unstable solution

- center of projection  $C$  located on the orthogonal circular cylinder with base circumscribing the three points  $X_i$

unstable: a small change of  $X_i$  results in a large change of  $C$   
can be detected by error propagation

## degenerate

- camera  $C$  is coplanar with points  $(X_1, X_2, X_3)$  but is not on the circumscribed circle of  $(X_1, X_2, X_3)$  camera sees a line



## no solution

- $C$  cocyclic with  $(X_1, X_2, X_3)$

camera sees a line

- additional critical configurations depend on the method to solve the quadratic equations

## ► Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	<b>P</b>	64
exterior orientation	<b>K</b> , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	<b>R, C</b>	68

- camera resection and exterior orientation are similar problems in a sense:
  - we do resectioning when our camera is uncalibrated
  - we do orientation when our camera is calibrated
- more problems to come

## Computing with a Camera Pair

- 4.1 Camera Motions Inducing Epipolar Geometry
- 4.2 Estimating Fundamental Matrix from 7 Correspondences
- 4.3 Estimating Essential Matrix from 5 Correspondences
- 4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

### covered by

- [1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1
- [2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR 2006*, pp. 630–633

### additional references



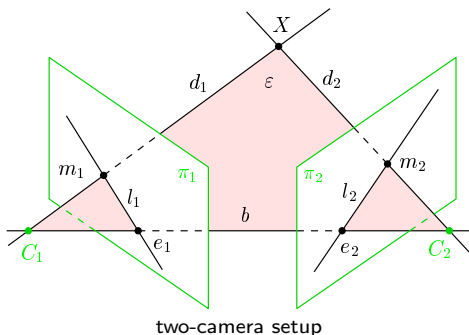
H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.



## ► Geometric Model of a Camera Pair

### Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



### Description

- baseline  $b$  joins projection centers  $C_1, C_2$   
$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$
- epipole  $e_i \in \pi_i$  is the image of  $C_j$ :  
$$\mathbf{e}_1 \simeq \mathbf{P}_1 \mathbf{C}_2, \quad \mathbf{e}_2 \simeq \mathbf{P}_2 \mathbf{C}_1$$
- $l_i \in \pi_i$  is the image of epipolar plane  
$$\varepsilon = (C_2, X, C_1)$$
- $l_j$  is the epipolar line in image  $\pi_j$  induced by  $m_i$  in image  $\pi_i$

**Epipolar constraint:** corresponding  $d_2, b, d_1$  are coplanar    a necessary condition, see →85

# Epipolar Geometry Example: Forward Motion

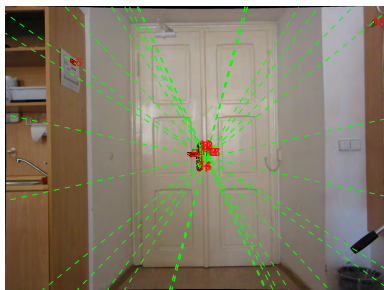


image 1

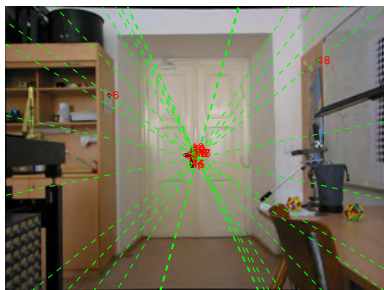
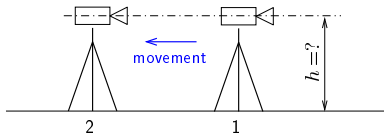


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs  
same ID in both images

How high was the camera above the floor?



## ► Cross Products and Maps by Skew-Symmetric $3 \times 3$ Matrices

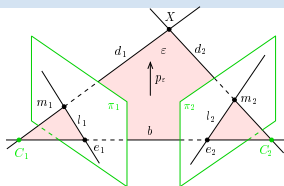
- There is an equivalence  $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_{\times} \mathbf{m}$ , where  $[\mathbf{b}]_{\times}$  is a  $3 \times 3$  skew-symmetric matrix

$$[\mathbf{b}]_{\times} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \quad \text{assuming } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

### Some properties

- $[\mathbf{b}]_{\times}^{\top} = -[\mathbf{b}]_{\times}$  the general antisymmetry property
- $\mathbf{A}$  is skew-symmetric iff  $\mathbf{x}^{\top} \mathbf{A} \mathbf{x} = 0$  for all  $\mathbf{x}$  skew-sym mtx generalizes cross products
- $[\mathbf{b}]_{\times}^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_{\times}$
- $\|[\mathbf{b}]_{\times}\|_F = \sqrt{2} \|\mathbf{b}\|$  Frobenius norm ( $\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^{\top} \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$ )
- $[\mathbf{b}]_{\times} \mathbf{b} = \mathbf{0}$
- $\text{rank} [\mathbf{b}]_{\times} = 2$  iff  $\|\mathbf{b}\| > 0$  check minors of  $[\mathbf{b}]_{\times}$
- eigenvalues of  $[\mathbf{b}]_{\times}$  are  $(0, \lambda, -\lambda)$
- for any regular  $\mathbf{B}$ :  $[\mathbf{Bz}]_{\times} \mathbf{B} = \det \mathbf{B} \cdot \mathbf{B}^{-\top} [\mathbf{z}]_{\times}$  follows from the factoring on  $\rightarrow 40$
- special case: if  $\mathbf{R} \mathbf{R}^{\top} = \mathbf{I}$  then  $[\mathbf{Rb}]_{\times} = \mathbf{R} [\mathbf{b}]_{\times} \mathbf{R}^{\top}$ 
  - note that if  $\mathbf{R}_b$  is rotation about  $\mathbf{b}$  then  $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
  - note  $[\mathbf{b}]_{\times}$  is not a homography; it is not a rotation matrix it is a logarithm of a rotation mtx

## ► Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = [\mathbf{Q}_i \quad \mathbf{q}_i] = \mathbf{K}_i [\mathbf{R}_i \quad \mathbf{t}_i], \quad i = 1, 2$$

$\mathbf{R}_{21}$  – relative camera rotation,  $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^\top$

$\mathbf{t}_{21}$  – relative camera translation,  $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21} \mathbf{t}_1 = -\mathbf{R}_2 \mathbf{b}$

$\mathbf{b}$  – baseline (world coordinate system)

remember:  $\mathbf{C} = -\mathbf{Q}^{-1} \mathbf{q} = -\mathbf{R}^\top \mathbf{t}$

→34 and 36

$$0 = \mathbf{d}_2^\top \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1} \mathbf{m}_2)^\top}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^\top \mathbf{l}_1}_{\text{optical plane}} = \mathbf{m}_2^\top \underbrace{\mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top (\mathbf{e}_1 \times \mathbf{m}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \mathbf{m}_2^\top \underbrace{(\mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{e}_1]_\times)}_{\text{fundamental matrix } \mathbf{F}} \mathbf{m}_1$$

**Epipolar constraint**  $\mathbf{m}_2^\top \mathbf{F} \mathbf{m}_1 = 0$  is a point-line incidence constraint

- point  $\mathbf{m}_2$  is incident on epipolar line  $\mathbf{l}_2 \simeq \mathbf{F} \mathbf{m}_1$
- point  $\mathbf{m}_1$  is incident on epipolar line  $\mathbf{l}_1 \simeq \mathbf{F}^\top \mathbf{m}_2$
- $\mathbf{F} \mathbf{e}_1 = \mathbf{F}^\top \mathbf{e}_2 = \mathbf{0}$  (non-trivially)
- all epipolars meet at the epipole

$$\mathbf{e}_1 \simeq \mathbf{Q}_1 \mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1 \mathbf{C}_2 - \mathbf{Q}_1 \mathbf{C}_1 = \mathbf{K}_1 \mathbf{R}_1 \mathbf{b} = -\mathbf{K}_1 \mathbf{R}_1 \mathbf{R}_2^\top \mathbf{t}_{21} = -\mathbf{K}_1 \mathbf{R}_{21}^\top \mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{e}_1]_\times = \mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{K}_1 \mathbf{R}_1 \mathbf{b}]_\times = \dots \simeq \mathbf{K}_2^{-\top} [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} \mathbf{K}_1^{-1} \quad \text{fundamental}$$

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2 \mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1 \mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21} [-\mathbf{R}_{21} \mathbf{t}_{21}]_\times \quad \text{essential}$$

## ► The Structure and the Key Properties of the Fundamental Matrix

$$\mathbf{F} = \underbrace{(\mathbf{Q}_2 \mathbf{Q}_1^{-1})^{-\top}}_{\text{epipolar homography } \mathbf{H}} [\mathbf{e}_1]_{\times} = \underbrace{\mathbf{K}_2^{-\top} \mathbf{R}_{21} \mathbf{K}_1^{\top}}_{\mathbf{H}^{-\top}} \overbrace{[\mathbf{e}_1]_{\times}}^{\text{left epipole}} \simeq \overbrace{[\mathbf{H} \mathbf{e}_1]_{\times}}^{\text{right epipole}} \mathbf{H} = \mathbf{K}_2^{-\top} \underbrace{[-\mathbf{t}_{21}]_{\times} \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}} \mathbf{K}_1^{-1}$$

1.  $\mathbf{E}$  captures relative camera pose only (the change of the world coordinate system does not change  $\mathbf{E}$ ) [Longuet-Higgins 1981]

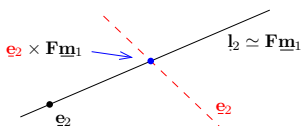
$$[\mathbf{R}'_i \quad \mathbf{t}'_i] = [\mathbf{R}_i \quad \mathbf{t}_i] \cdot \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} = [\mathbf{R}_i \mathbf{R} \quad \mathbf{R}_i \mathbf{t} + \mathbf{t}_i],$$

then

$$\mathbf{R}'_{21} = \mathbf{R}'_2 \mathbf{R}'_1{}^{\top} = \dots = \mathbf{R}_{21}$$

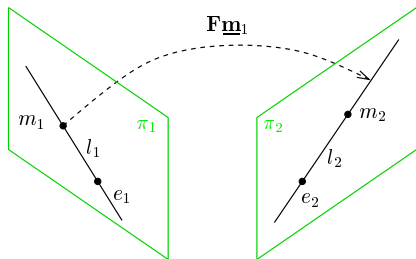
$$\mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21} \mathbf{t}'_1 = \dots = \mathbf{t}_{21}$$

2. the translation length  $\mathbf{t}_{21}$  is lost since  $\mathbf{E}$  is homogeneous
3.  $\mathbf{F}$  maps points to lines and it is not a homography
4.  $\mathbf{e}_2 \times (\mathbf{e}_2 \times \mathbf{F} \underline{\mathbf{m}}_1) \simeq \mathbf{F} \underline{\mathbf{m}}_1$ , in general  $\mathbf{F} \simeq [\mathbf{e}_2]_{\times}^{2a} \mathbf{F} [\mathbf{e}_1]_{\times}^{2b}$  for any  $a, b \in \mathbb{N}$



- by point/line 'transmutation' (left)
- point  $\mathbf{e}_2$  does not lie on line  $\mathbf{e}_2$  (dashed):  $\mathbf{e}_2^{\top} \mathbf{e}_2 \neq 0$
- application:  $\mathbf{F}^{\top} (\mathbf{e}_2 \times \mathbf{l}_2) \simeq \mathbf{F}^{\top} (\mathbf{e}_2 \times \mathbf{F} \underline{\mathbf{m}}_1) \simeq \mathbf{F}^{\top} \underline{\mathbf{m}}_2 \simeq \mathbf{l}_1$
- $\mathbf{F}^{\top} [\mathbf{e}_2]_{\times}$  maps epipolar lines to epi. lines but it is not a homography

## ► Some Mappings by the Fundamental Matrix

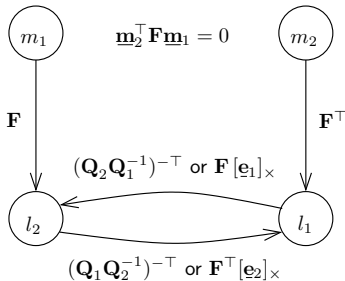


$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F} \underline{\mathbf{m}}_1$$

$$\mathbf{e}_1 \simeq \text{null}(\mathbf{F}), \quad \mathbf{e}_2 \simeq \text{null}(\mathbf{F}^\top)$$

$$\mathbf{l}_2 = \mathbf{F} \underline{\mathbf{m}}_1 \quad \mathbf{l}_1 = \mathbf{F}^\top \underline{\mathbf{m}}_2$$

$$\mathbf{l}_2 = \mathbf{F}[\mathbf{e}_1]_\times \mathbf{l}_1 \quad \mathbf{l}_1 = \mathbf{F}^\top[\mathbf{e}_2]_\times \mathbf{l}_2$$



- $\mathbf{l}_2 \simeq \mathbf{F}[\mathbf{e}_1]_\times \mathbf{l}_1$ : by 'transmutation' →76
- $\mathbf{F}[\mathbf{e}_1]_\times$  maps lines to lines but it is not a homography
- $\mathbf{H} = \mathbf{Q}_2 \mathbf{Q}_1^{-1}$  is the epipolar homography →76 mapping epipolar lines to epipolar lines, hence

$$\mathbf{H} = \mathbf{Q}_2 \mathbf{Q}_1^{-1} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$$

you have seen this →61

## ► Representation Theorem for Fundamental Matrices

### Theorem

Every  $3 \times 3$  matrix of rank 2 is a fundamental matrix.

### Proof.

Converse: By the definition  $\mathbf{F} = \mathbf{H}^{-\top} [\mathbf{e}_1]_{\times}$  is a  $3 \times 3$  matrix of rank 2.

Direct:

1. let  $\mathbf{A} = \mathbf{UDV}^{\top}$  be the SVD of a  $3 \times 3$  matrix  $\mathbf{A}$  of rank 2; then  $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, 0)$
2. we can write  $\mathbf{D} = \mathbf{BC}$ , where  $\mathbf{B} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ ,  $\mathbf{C} = \text{diag}(1, 1, 0)$ ,  $\lambda_3 \neq 0$
3. then  $\mathbf{A} = \mathbf{UBCV}^{\top} = \mathbf{UBC} \underbrace{\mathbf{WW}^{\top}}_{\mathbf{I}} \mathbf{V}^{\top}$  with  $\mathbf{W}$  rotation
4. we look for a rotation  $\mathbf{W}$  that maps  $\mathbf{C}$  to a skew-symmetric  $\mathbf{S}$ , i.e.  $\mathbf{S} = \mathbf{CW}$

5. then  $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $|\alpha| = 1$ , and  $\mathbf{S} = [\mathbf{s}]_{\times}$ ,  $\mathbf{s} = (0, 0, 1)$

6. we can write

$$\mathbf{A} = \mathbf{UB}[\mathbf{s}]_{\times} \mathbf{W}^{\top} \mathbf{V}^{\top} = \dots \stackrel{\textcircled{*} 1}{=} \underbrace{\mathbf{UB}(\mathbf{VW})^{\top}}_{\mathbf{H}^{-\top}} [\mathbf{v}_3]_{\times}, \quad \mathbf{v}_3 - \text{3rd column of } \mathbf{V} \quad (12)$$

7.  $\mathbf{H}$  regular  $\Rightarrow \mathbf{A}$  does the job of a fundamental matrix, with epipole  $\mathbf{v}_3$  and epipolar homography  $\mathbf{H}$  □

- we also got a (non-unique:  $\lambda_3, \alpha = \pm 1$ ) decomposition formula for fundamental matrices
- it follows there is no constraint on  $\mathbf{F}$  except the rank

## ► Representation Theorem for Essential Matrices

### Theorem

Let  $\mathbf{E}$  be a  $3 \times 3$  matrix with SVD  $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ . Then  $\mathbf{E}$  is essential iff  $\mathbf{D} \simeq \text{diag}(1, 1, 0)$ .

### Proof.

Direct:

If  $\mathbf{E}$  is an essential matrix, then the epipolar homography is a rotation ( $\rightarrow$ 76) and  $\mathbf{U}\mathbf{B}(\mathbf{V}\mathbf{W})^\top$  in (12) must be orthogonal, therefore  $\mathbf{B} = \lambda\mathbf{I}$ .

Converse:

$\mathbf{E}$  is fundamental with  $\mathbf{D} = \lambda \text{diag}(1, 1, 0)$  then we do not need  $\mathbf{B}$  (as if  $\mathbf{B} = \lambda\mathbf{I}$ ) in (12) and  $\mathbf{U}(\mathbf{V}\mathbf{W})^\top$  is orthogonal, as required.

□



## ► Essential Matrix Decomposition

We are decomposing  $\mathbf{E}$  to  $\mathbf{E} = [-\mathbf{t}_{21}]_{\times} \mathbf{R}_{21} = \mathbf{R}_{21} [-\mathbf{R}_{21}^{\top} \mathbf{t}_{21}]_{\times}$  [H&Z, sec. 9.6]

1. compute SVD of  $\mathbf{E} = \mathbf{U} \mathbf{D} \mathbf{V}^{\top}$  and verify  $\mathbf{D} = \lambda \text{diag}(1, 1, 0)$
2. if  $\det \mathbf{U} < 0$  change signs  $\mathbf{U} \mapsto -\mathbf{U}$ ,  $\mathbf{V} \mapsto -\mathbf{V}$  the overall sign is dropped
3. compute

$$\mathbf{R}_{21} = \mathbf{U} \underbrace{\begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{W}} \mathbf{V}^{\top}, \quad \mathbf{t}_{21} = -\beta \mathbf{u}_3, \quad |\alpha| = 1, \quad \beta \neq 0 \quad (13)$$

### Notes

- $\mathbf{v}_3 \simeq \mathbf{R}_{21}^{\top} \mathbf{t}_{21}$ , hence  $\mathbf{R}_{21} \mathbf{v}_3 \simeq \mathbf{t}_{21} \simeq \mathbf{u}_3$  since it must fall in left null space by  $\mathbf{E} \simeq [\mathbf{u}_3]_{\times} \mathbf{R}$
- $\mathbf{t}_{21}$  is recoverable up to scale  $\beta$  and direction  $\text{sign } \beta$
- the result for  $\mathbf{R}_{21}$  is unique up to  $\alpha = \pm 1$  despite non-uniqueness of SVD
- change of sign in  $\alpha$  rotates the solution by  $180^\circ$  about  $\mathbf{t}_{21}$  since  $-\mathbf{W} = \mathbf{W}^{\top}$

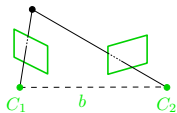
$\mathbf{R}(\alpha) = \mathbf{U} \mathbf{W} \mathbf{V}^{\top}$ ,  $\mathbf{R}(-\alpha) = \mathbf{U} \mathbf{W}^{\top} \mathbf{V}^{\top} \Rightarrow \mathbf{T} = \mathbf{R}(-\alpha) \mathbf{R}^{\top}(\alpha) = \dots = \mathbf{U} \text{diag}(-1, -1, 1) \mathbf{U}^{\top}$   
which is a rotation by  $180^\circ$  about  $\mathbf{u}_3 = \mathbf{t}_{21}$ :

$$\mathbf{U} \text{diag}(-1, -1, 1) \mathbf{U}^{\top} \mathbf{u}_3 = \mathbf{U} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_3$$

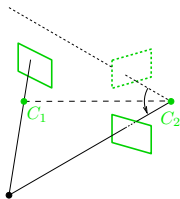
- 4 solution sets for 4 sign combinations of  $\alpha$ ,  $\beta$  see next for geometric interpretation

## ► Four Solutions to Essential Matrix Decomposition

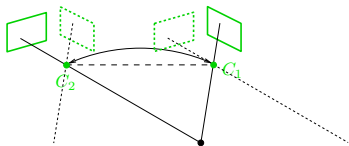
Transform the world coordinate system so that the origin is in Camera 2. Then  $t_{21} = -\mathbf{b}$  and  $\mathbf{W}$  rotates about the baseline  $\mathbf{b}$ . →75



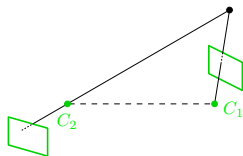
$\alpha, \beta$



$-\alpha, \beta$  (twisted by  $\mathbf{W}$ )



$\alpha, -\beta$  (baseline reversal)



$-\alpha, -\beta$  (combination of both)

- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case

[H&Z, Sec. 9.6.3]

## ►7-Point Algorithm for Estimating Fundamental Matrix

**Problem:** Given a set  $\{(x_i, y_i)\}_{i=1}^k$  of  $k = 7$  correspondences, estimate f. m.  $\mathbf{F}$ .

$$\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i = 0, \quad i = 1, \dots, k, \quad \text{known: } \underline{\mathbf{x}}_i = (u_i^1, v_i^1, 1), \quad \underline{\mathbf{y}}_i = (u_i^2, v_i^2, 1)$$

terminology: correspondence = truth, later: match = algorithm's result; hypothesized corresp.

**Solution:**

$$\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i = (\text{vec}(\mathbf{y}_i \mathbf{x}_i^\top))^\top \text{vec}(\mathbf{F}),$$

$$\text{vec}(\mathbf{F}) = [f_{11} \quad f_{21} \quad f_{31} \quad \dots \quad f_{33}]^\top \in \mathbb{R}^9 \quad \text{column vector from matrix}$$

$$\mathbf{D} = \begin{bmatrix} (\text{vec}(\mathbf{y}_1 \mathbf{x}_1^\top))^\top \\ (\text{vec}(\mathbf{y}_2 \mathbf{x}_2^\top))^\top \\ (\text{vec}(\mathbf{y}_3 \mathbf{x}_3^\top))^\top \\ \vdots \\ (\text{vec}(\mathbf{y}_k \mathbf{x}_k^\top))^\top \end{bmatrix} = \begin{bmatrix} u_1^1 u_1^2 & u_1^1 v_1^2 & u_1^1 & u_1^2 v_1^1 & v_1^1 v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ u_2^1 u_2^2 & u_2^1 v_2^2 & u_2^1 & u_2^2 v_2^1 & v_2^1 v_2^2 & v_2^1 & u_2^2 & v_2^2 & 1 \\ u_3^1 u_3^2 & u_3^1 v_3^2 & u_3^1 & u_3^2 v_3^1 & v_3^1 v_3^2 & v_3^1 & u_3^2 & v_3^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_k^1 u_k^2 & u_k^1 v_k^2 & u_k^1 & u_k^2 v_k^1 & v_k^1 v_k^2 & v_k^1 & u_k^2 & v_k^2 & 1 \end{bmatrix} \in \mathbb{R}^{k,9}$$

$$\mathbf{D} \text{vec}(\mathbf{F}) = \mathbf{0}$$

## ►7-Point Algorithm Continued

$$\mathbf{D} \operatorname{vec}(\mathbf{F}) = \mathbf{0}, \quad \mathbf{D} \in \mathbb{R}^{k,9}$$

- for  $k = 7$  we have a rank-deficient system, the null-space of  $\mathbf{D}$  is 2-dimensional
- but we know that  $\det \mathbf{F} = 0$ , hence

1. find a basis of the null space of  $\mathbf{D}$ :  $\mathbf{F}_1, \mathbf{F}_2$  by SVD or QR factorization
2. get up to 3 real solutions for  $\alpha$  from

$$\det(\alpha \mathbf{F}_1 + (1 - \alpha) \mathbf{F}_2) = 0 \quad \text{cubic equation in } \alpha$$

3. get up to 3 fundamental matrices  $\mathbf{F} = \alpha_i \mathbf{F}_1 + (1 - \alpha_i) \mathbf{F}_2$  (check rank  $\mathbf{F} = 2$ )

- the result may depend on image transformations
- normalization improves conditioning →90
- this gives a good starting point for the full algorithm →108
- dealing with mismatches need not be a part of the 7-point algorithm →109

## ► Degenerate Configurations for Fundamental Matrix Estimation

When is  $\mathbf{F}$  not uniquely determined from any number of correspondences? [H&Z, Sec. 11.9]

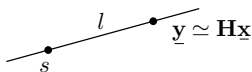
1. when images are related by homography

a) camera centers coincide  $\mathbf{t}_{21} = 0$ :  $\mathbf{H} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$

b) camera moves but all 3D points lie in a plane  $(\mathbf{n}, d)$ :  $\mathbf{H} = \mathbf{K}_2 (\mathbf{R}_{21} - \mathbf{t}_{21} \mathbf{n}^\top / d) \mathbf{K}_1^{-1}$

• in both cases: epipolar geometry is not defined

• we do get a solution from the 7-point algorithm but it has the form of  $\mathbf{F} = [\underline{\mathbf{s}}]_{\times} \mathbf{H}$   
note that  $[\underline{\mathbf{s}}]_{\times} \mathbf{H} \simeq \mathbf{H}' [\underline{\mathbf{s}}']_{\times} \rightarrow 74$



• given (arbitrary)  $\underline{\mathbf{s}}$

• and correspondence  $x \leftrightarrow y$

•  $y$  is the image of  $x$ :  $\underline{\mathbf{y}} \simeq \mathbf{H}\underline{\mathbf{x}}$

• a necessary condition:  $y \in l$ ,  $\underline{\mathbf{1}} \simeq \underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}$

$$0 = \underline{\mathbf{y}}^\top (\underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}) = \underline{\mathbf{y}}^\top [\underline{\mathbf{s}}]_{\times} \mathbf{H}\underline{\mathbf{x}} \quad \text{for any } \underline{\mathbf{x}}, \underline{\mathbf{s}} (!)$$

2. both camera centers and all 3D points lie on a ruled quadric

hyperboloid of one sheet, cones, cylinders, two planes

• there are 3 solutions for  $\mathbf{F}$

### notes

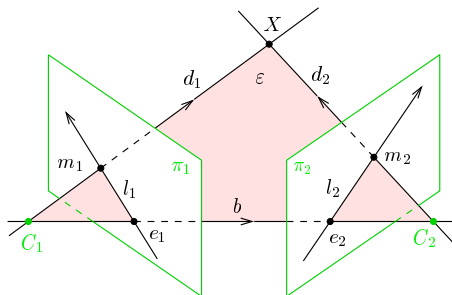
• estimation of  $\mathbf{E}$  can deal with planes:  $[\underline{\mathbf{s}}]_{\times} \mathbf{H}$  is essential matrix iff  $\underline{\mathbf{s}} = \lambda \mathbf{t}_{21}$

• a complete treatment with additional degenerate configurations in [H&Z, sec. 22.2]

• a stronger epipolar constraint could reject some configurations

# A Note on Oriented Epipolar Constraint

- a tighter epipolar constraint preserves orientations
- requires all points and cameras be on the same side of the plane at infinity



$$\mathbf{e}_2 \times \mathbf{m}_2 \stackrel{+}{\sim} \mathbf{F} \mathbf{m}_1$$

notation:  $\mathbf{m} \stackrel{+}{\sim} \mathbf{n}$  means  $\mathbf{m} = \lambda \mathbf{n}$ ,  $\lambda > 0$

- note that the constraint is not invariant to the change of either sign of  $\mathbf{m}_i$
- all 7 correspondence in 7-point alg. must have the same sign
- this may help reject some wrong matches, see  $\rightarrow 109$
- an even more tight constraint: scene points in front of both cameras

see later

[Chum et al. 2004]

expensive

this is called chirality constraint

## ►5-Point Algorithm for Relative Camera Orientation

**Problem:** Given  $\{m_i, m'_i\}_{i=1}^5$  corresponding image points and calibration matrix  $\mathbf{K}$ , recover the camera motion  $\mathbf{R}, \mathbf{t}$ .

**Obs:**

1.  $\mathbf{E}$  – 8 numbers
2.  $\mathbf{R}$  – 3DOF,  $\mathbf{t}$  – 2DOF only, in total 5 DOF  $\rightarrow$  we need  $8 - 5 = 3$  constraints on  $\mathbf{E}$
3.  $\mathbf{E}$  essential iff it has two equal singular values and the third is zero  $\rightarrow$ 79

**This gives an equation system:**

$$\underline{\mathbf{v}}_i^\top \mathbf{E} \underline{\mathbf{v}}'_i = 0 \quad 5 \text{ linear constraints } (\underline{\mathbf{v}} \simeq \mathbf{K}^{-1} \underline{\mathbf{m}})$$

$$\det \mathbf{E} = 0 \quad 1 \text{ cubic constraint}$$

$$\mathbf{E} \mathbf{E}^\top \mathbf{E} - \frac{1}{2} \text{tr}(\mathbf{E} \mathbf{E}^\top) \mathbf{E} = 0 \quad 9 \text{ cubic constraints, 2 independent}$$

⊛ P1; 1pt: verify this equation from  $\mathbf{E} = \mathbf{U} \mathbf{D} \mathbf{V}^\top$ ,  $\mathbf{D} = \lambda \text{diag}(1, 1, 0)$

1. estimate  $\mathbf{E}$  by SVD from  $\underline{\mathbf{v}}_i^\top \mathbf{E} \underline{\mathbf{v}}'_i = 0$  by the null-space method 4D null space
2. this gives  $\mathbf{E} = x \mathbf{E}_1 + y \mathbf{E}_2 + z \mathbf{E}_3 + \mathbf{E}_4$
3. at most 10 (complex) solutions for  $x, y, z$  from the cubic constraints

- when all 3D points lie on a plane: at most 2 real solutions (twisted-pair) can be disambiguated in 3 views  
or by chirality constraint ( $\rightarrow$ 81) unless all 3D points are closer to one camera
- 6-point problem for unknown  $f$  [Kukelova et al. BMVC 2008]
- resources at [http://cmp.felk.cvut.cz/minimal/5\\_pt\\_relative.php](http://cmp.felk.cvut.cz/minimal/5_pt_relative.php)

## ► The Triangulation Problem

**Problem:** Given cameras  $\mathbf{P}_1, \mathbf{P}_2$  and a correspondence  $x \leftrightarrow y$  compute a 3D point  $\mathbf{X}$  projecting to  $x$  and  $y$

$$\lambda_1 \underline{\mathbf{x}} = \mathbf{P}_1 \underline{\mathbf{X}}, \quad \lambda_2 \underline{\mathbf{y}} = \mathbf{P}_2 \underline{\mathbf{X}}, \quad \underline{\mathbf{x}} = \begin{bmatrix} u^1 \\ v^1 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{y}} = \begin{bmatrix} u^2 \\ v^2 \\ 1 \end{bmatrix}, \quad \mathbf{P}_i = \begin{bmatrix} (\mathbf{p}_1^i)^\top \\ (\mathbf{p}_2^i)^\top \\ (\mathbf{p}_3^i)^\top \end{bmatrix}$$

### Linear triangulation method

$$\begin{aligned} u^1 (\mathbf{p}_3^1)^\top \underline{\mathbf{X}} &= (\mathbf{p}_1^1)^\top \underline{\mathbf{X}}, & u^2 (\mathbf{p}_3^2)^\top \underline{\mathbf{X}} &= (\mathbf{p}_1^2)^\top \underline{\mathbf{X}}, \\ v^1 (\mathbf{p}_3^1)^\top \underline{\mathbf{X}} &= (\mathbf{p}_2^1)^\top \underline{\mathbf{X}}, & v^2 (\mathbf{p}_3^2)^\top \underline{\mathbf{X}} &= (\mathbf{p}_2^2)^\top \underline{\mathbf{X}}, \end{aligned}$$

Gives

$$\mathbf{D} \underline{\mathbf{X}} = \mathbf{0}, \quad \mathbf{D} = \begin{bmatrix} u^1 (\mathbf{p}_3^1)^\top - (\mathbf{p}_1^1)^\top \\ v^1 (\mathbf{p}_3^1)^\top - (\mathbf{p}_2^1)^\top \\ u^2 (\mathbf{p}_3^2)^\top - (\mathbf{p}_1^2)^\top \\ v^2 (\mathbf{p}_3^2)^\top - (\mathbf{p}_2^2)^\top \end{bmatrix}, \quad \mathbf{D} \in \mathbb{R}^{4,4}, \quad \underline{\mathbf{X}} \in \mathbb{R}^4 \quad (14)$$

- back-projected rays will generally not intersect due to image error, see next
- using Jack-knife ( $\rightarrow 65$ ) not recommended sensitive to small error
- we will use SVD ( $\rightarrow 88$ )
- but the result will not be invariant to projective frame  
replacing  $\mathbf{P}_1 \mapsto \mathbf{P}_1 \mathbf{H}, \mathbf{P}_2 \mapsto \mathbf{P}_2 \mathbf{H}$  does not always result in  $\underline{\mathbf{X}} \mapsto \mathbf{H}^{-1} \underline{\mathbf{X}}$
- note the homogeneous form in (14) can represent points at infinity



## ► The Least-Squares Triangulation by SVD

- if  $\mathbf{D}$  is full-rank we may minimize the algebraic least-squares error

$$\epsilon^2(\underline{\mathbf{X}}) = \|\mathbf{D}\underline{\mathbf{X}}\|^2 \quad \text{s.t.} \quad \|\underline{\mathbf{X}}\| = 1, \quad \underline{\mathbf{X}} \in \mathbb{R}^4$$

- let  $\mathbf{D}_i$  be the  $i$ -th row of  $\mathbf{D}$ , then

$$\|\mathbf{D}\underline{\mathbf{X}}\|^2 = \sum_{i=1}^4 (\mathbf{D}_i \underline{\mathbf{X}})^2 = \sum_{i=1}^4 \underline{\mathbf{X}}^\top \mathbf{D}_i^\top \mathbf{D}_i \underline{\mathbf{X}} = \underline{\mathbf{X}}^\top \mathbf{Q} \underline{\mathbf{X}}, \quad \text{where } \mathbf{Q} = \sum_{i=1}^4 \mathbf{D}_i^\top \mathbf{D}_i = \mathbf{D}^\top \mathbf{D} \in \mathbb{R}^{4,4}$$

- we write the SVD of  $\mathbf{Q}$  as  $\mathbf{Q} = \sum_{j=1}^4 \sigma_j^2 \mathbf{u}_j \mathbf{u}_j^\top$ , in which [Golub & van Loan 2013, Sec. 2.5]

$$\sigma_1^2 \geq \dots \geq \sigma_4^2 \geq 0 \quad \text{and} \quad \mathbf{u}_l^\top \mathbf{u}_m = \begin{cases} 0 & \text{if } l \neq m \\ 1 & \text{otherwise} \end{cases}$$

- then  $\underline{\mathbf{X}} = \arg \min_{\mathbf{q}, \|\mathbf{q}\|=1} \mathbf{q}^\top \mathbf{Q} \mathbf{q} = \mathbf{u}_4$

**Proof (by contradiction).**

$$\mathbf{q}^\top \mathbf{Q} \mathbf{q} = \sum_{j=1}^4 \sigma_j^2 \mathbf{q}^\top \mathbf{u}_j \mathbf{u}_j^\top \mathbf{q} = \sum_{j=1}^4 \sigma_j^2 (\mathbf{u}_j^\top \mathbf{q})^2 \text{ is a sum of non-negative terms } 0 \leq (\mathbf{u}_j^\top \mathbf{q})^2 \leq 1$$

Let  $\mathbf{q} = \mathbf{u}_4 \cos \alpha + \bar{\mathbf{q}} \sin \alpha$  s.t.  $\bar{\mathbf{q}} \perp \mathbf{u}_4$  and  $\|\bar{\mathbf{q}}\| = 1$ , then  $\|\mathbf{q}\| = 1$ ,  $\bar{\mathbf{q}} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3$  with  $a_1^2 + a_2^2 + a_3^2 = 1$  and

$$\mathbf{q}^\top \mathbf{Q} \mathbf{q} = \dots = \sigma_4^2 \cos^2 \alpha + \sin^2 \alpha \underbrace{\sum_{j=1}^3 \sigma_j^2 (\mathbf{u}_j^\top \bar{\mathbf{q}})^2}_{\geq \sigma_4^2} \geq \sigma_4^2$$

- if  $\sigma_4 \ll \sigma_3$ , there is a unique solution  $\underline{\mathbf{X}} = \mathbf{u}_4$  with residual error  $(\mathbf{D} \underline{\mathbf{X}})^2 = \sigma_4^2$   
the quality (conditioning) of the solution may be expressed as  $q = \sigma_3/\sigma_4$  (greater is better)

Matlab code for the least-squares solver:

```
[U,0,V] = svd(D);  
X = V(:,end);  
q = 0(3,3)/0(4,4);
```

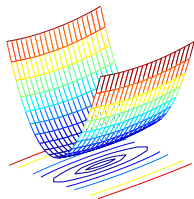
⊗ P1; 1pt: Why did we decompose  $\mathbf{D}$  and not  $\mathbf{Q} = \mathbf{D}^\top \mathbf{D}$ ?

## ► Numerical Conditioning

- The equation  $\mathbf{D}\underline{\mathbf{X}} = \mathbf{0}$  in (14) may be ill-conditioned for numerical computation, which results in a poor estimate for  $\underline{\mathbf{X}}$ .

**Why:** on a row of  $\mathbf{D}$  there are big entries together with small entries, e.g. of orders projection centers in mm, image points in px

$$\begin{bmatrix} 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \\ 10^3 & 0 & 10^3 & 10^6 \\ 0 & 10^3 & 10^3 & 10^6 \end{bmatrix}$$



### Quick fix:

1. re-scale the problem by a regular diagonal conditioning matrix  $\mathbf{S} \in \mathbb{R}^{4,4}$

$$\mathbf{0} = \mathbf{D}\underline{\mathbf{X}} = \mathbf{D}\mathbf{S}\mathbf{S}^{-1}\underline{\mathbf{X}} = \bar{\mathbf{D}}\bar{\underline{\mathbf{X}}}$$

choose  $\mathbf{S}$  to make the entries in  $\hat{\mathbf{D}}$  all smaller than unity in absolute value:

$$\mathbf{S} = \text{diag}(10^{-3}, 10^{-3}, 10^{-3}, 10^{-6}) \quad \mathbf{S} = \text{diag}(1./\max(\text{abs}(\mathbf{D}), 1))$$

2. solve for  $\bar{\underline{\mathbf{X}}}$  as before
3. get the final solution as  $\underline{\mathbf{X}} = \mathbf{S}\bar{\underline{\mathbf{X}}}$

- when SVD is used in camera resection, conditioning is essential for success

→64

# Algebraic Error vs Reprojection Error

- algebraic error ( $c$  – camera index,  $(u^c, v^c)$  – image coordinates)

from SVD → 89

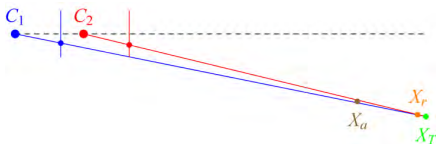
$$\varepsilon^2 = \sigma_4^2 = \sum_{c=1}^2 \left[ \left( u^c (\mathbf{p}_3^c)^\top \mathbf{X} - (\mathbf{p}_1^c)^\top \mathbf{X} \right)^2 + \left( v^c (\mathbf{p}_3^c)^\top \mathbf{X} - (\mathbf{p}_2^c)^\top \mathbf{X} \right)^2 \right]$$

- reprojection error

$$e^2 = \sum_{c=1}^2 \left[ \left( u^c - \frac{(\mathbf{p}_1^c)^\top \mathbf{X}}{(\mathbf{p}_3^c)^\top \mathbf{X}} \right)^2 + \left( v^c - \frac{(\mathbf{p}_2^c)^\top \mathbf{X}}{(\mathbf{p}_3^c)^\top \mathbf{X}} \right)^2 \right]$$

- algebraic error zero  $\Rightarrow$  reprojection error zero  $\sigma_4 = 0 \Rightarrow$  non-trivial null space
- epipolar constraint satisfied  $\Rightarrow$  equivalent results
- in general: minimizing algebraic error is cheap but it gives inferior results
- minimizing reprojection error is expensive but it gives good results
- the midpoint of the common perpendicular to both optical rays gives about 50% greater error in 3D
- the golden standard method – deferred to → 103

Ex:



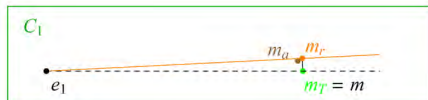
- forward camera motion
- error  $f/50$  in image 2, orthogonal to epipolar plane

$X_T$  – noiseless ground truth position

$X_r$  – reprojection error minimizer

$X_a$  – algebraic error minimizer

$m$  – measurement ( $m_T$  with noise in  $v^2$ )



## ► We Have Added to The ZOO

continuation from →70

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	<b>P</b>	64
exterior orientation	<b>K</b> , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	<b>R, t</b>	68
fundamental matrix	7 img–img correspondences $\{(m_i, m'_i)\}_{i=1}^7$	<b>F</b>	82
relative orientation	<b>K</b> , 5 img–img correspondences $\{(m_i, m'_i)\}_{i=1}^5$	<b>R, t</b>	86
triangulation	<b>P</b> <sub>1</sub> , <b>P</b> <sub>2</sub> , 1 img–img correspondence $(m_i, m'_i)$	<b>X</b>	87

A bigger ZOO at <http://cmp.felk.cvut.cz/minimal/>

### calibrated problems

- have fewer degenerate configurations
- can do with fewer points (good for geometry proposal generators →116)
- algebraic error optimization (with SVD) makes sense in camera resection and triangulation only
- but it is not the best method; we will now focus on ‘optimizing optimally’

## Optimization for 3D Vision

- 5.1 The Concept of Error for Epipolar Geometry
- 5.2 Levenberg-Marquardt's Iterative Optimization
- 5.3 The Correspondence Problem
- 5.4 Optimization by Random Sampling

### covered by

- [1] [H&Z] Secs: 11.4, 11.6, 4.7
- [2] Fischler, M.A. and Bolles, R.C . Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM* 24(6):381–395, 1981

### additional references



P. D. Sampson. Fitting conic sections to 'very scattered' data: An iterative refinement of the Bookstein algorithm. *Computer Vision, Graphics, and Image Processing*, 18:97–108, 1982.



O. Chum, J. Matas, and J. Kittler. Locally optimized RANSAC. In *Proc DAGM, LNCS 2781:236–243*. Springer-Verlag, 2003.

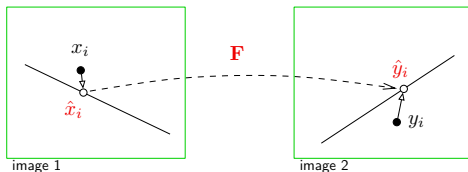


O. Chum, T. Werner, and J. Matas. Epipolar geometry estimation via RANSAC benefits from the oriented epipolar constraint. In *Proc ICPR, vol 1:112–115*, 2004.

## ► The Concept of Error for Epipolar Geometry

**Problem:** Given at least 8 matched points  $x_i \leftrightarrow y_j$  in a general position, estimate the most likely (or most probable) fundamental matrix  $\mathbf{F}$ .

$$\mathbf{x}_i = (u_i^1, v_i^1), \quad \mathbf{y}_i = (u_i^2, v_i^2), \quad i = 1, 2, \dots, k, \quad k \geq 8$$



- detected points (measurements)  $x_i, y_i$
- we introduce matches  $\mathbf{Z}_i = (u_i^1, v_i^1, u_i^2, v_i^2) \in \mathbb{R}^4$ ;  $S = \{\mathbf{Z}_i\}_{i=1}^k$
- corrected points  $\hat{x}_i, \hat{y}_i$ ;  $\hat{\mathbf{Z}}_i = (\hat{u}_i^1, \hat{v}_i^1, \hat{u}_i^2, \hat{v}_i^2)$ ;  $\hat{S} = \{\hat{\mathbf{Z}}_i\}_{i=1}^k$  are correspondences
- correspondences satisfy the epipolar geometry exactly  $\hat{\mathbf{y}}_i^\top \mathbf{F} \hat{\mathbf{x}}_i = 0, i = 1, \dots, k$
- small correction is more probable
- let  $\mathbf{e}_i(\cdot)$  be the 'reprojection error' (vector) per match  $i$ ,

$$\mathbf{e}_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \begin{bmatrix} \mathbf{x}_i - \hat{\mathbf{x}}_i \\ \mathbf{y}_i - \hat{\mathbf{y}}_i \end{bmatrix} = \mathbf{e}_i(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F}) = \mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F}) \quad (15)$$

$$\|\mathbf{e}_i(\cdot)\|^2 \stackrel{\text{def}}{=} \mathbf{e}_i^2(\cdot) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i(\mathbf{F})\|^2$$

- the total reprojection error (of all data) then is

$$L(S \mid \hat{S}, \mathbf{F}) = \sum_{i=1}^k \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) = \sum_{i=1}^k \mathbf{e}_i^2(\mathbf{Z}_i \mid \hat{\mathbf{Z}}_i, \mathbf{F})$$

- and the optimization problem is

$$(\hat{S}^*, \mathbf{F}^*) = \arg \min_{\substack{\mathbf{F} \\ \text{rank } \mathbf{F} = 2}} \min_{\substack{\hat{S} \\ \hat{\mathbf{y}}_i^\top \mathbf{F} \hat{\mathbf{x}}_i = 0}} \sum_{i=1}^k \mathbf{e}_i^2(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \quad (16)$$

### Three possible approaches

- they differ in how the correspondences  $\hat{x}_i, \hat{y}_i$  are obtained:
  - direct optimization of reprojection error over all variables  $\hat{S}, \mathbf{F}$  →96
  - Sampson optimal correction = partial correction of  $\mathbf{Z}_i$  towards  $\hat{\mathbf{Z}}_i$  used in an iterative minimization over  $\mathbf{F}$  →97
  - removing  $\hat{x}_i, \hat{y}_i$  altogether = marginalization of  $L(S, \hat{S} \mid \mathbf{F})$  over  $\hat{S}$  followed by minimization over  $\mathbf{F}$  not covered, the marginalization is difficult



# Method 1: Geometric Error Optimization

- we need to encode the constraints  $\hat{\mathbf{y}}_i^T \mathbf{F} \hat{\mathbf{x}}_i = 0$ ,  $\text{rank } \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z, Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = [\mathbf{I} \quad \mathbf{0}], \quad \mathbf{P}_2 = \left[ [\mathbf{e}_2]_{\times} \mathbf{F} + \mathbf{e}_2 \mathbf{e}_1^T \quad \mathbf{e}_2 \right] \quad (17)$$

⊗ H3; 2pt: Assuming  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  are epipoles of  $\mathbf{F}$ , verify that  $\mathbf{F}$  is a fundamental matrix of  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ .

Hint:  $\mathbf{A}$  is skew symmetric iff  $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$  for all vectors  $\mathbf{x}$ .

- 
1. compute  $\mathbf{F}^{(0)}$  by the 7-point algorithm  $\rightarrow 82$ ; construct camera  $\mathbf{P}_2^{(0)}$  from  $\mathbf{F}^{(0)}$  using (17)
  2. triangulate 3D points  $\hat{\mathbf{X}}_i^{(0)}$  from matches  $(x_i, y_i)$  for all  $i = 1, \dots, k$   $\rightarrow 87$
  3. starting from  $\mathbf{P}_2^{(0)}$ ,  $\hat{\mathbf{X}}^{(0)}$  minimize the reprojection error (15)

$$(\hat{\mathbf{X}}^*, \mathbf{P}_2^*) = \arg \min_{\mathbf{P}_2, \hat{\mathbf{X}}} \sum_{i=1}^k \mathbf{e}_i^2(\mathbf{z}_i \mid \hat{\mathbf{z}}_i(\hat{\mathbf{X}}_i, \mathbf{P}_2))$$

where

$$\hat{\mathbf{z}}_i = (\hat{\mathbf{x}}_i, \hat{\mathbf{y}}_i) \text{ (Cartesian)}, \quad \hat{\mathbf{x}}_i \simeq \mathbf{P}_1 \hat{\mathbf{X}}_i, \quad \hat{\mathbf{y}}_i \simeq \mathbf{P}_2 \hat{\mathbf{X}}_i \text{ (homogeneous)}$$

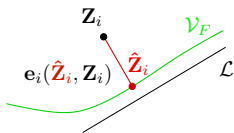
Non-linear, non-convex problem

4. compute  $\mathbf{F}$  from  $\mathbf{P}_1$ ,  $\mathbf{P}_2^*$ 
  - $3k + 12$  parameters to be found: latent:  $\hat{\mathbf{X}}_i$ , for all  $i$  (correspondences!), non-latent:  $\mathbf{P}_2$
  - minimal representation:  $3k + 7$  parameters,  $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$   $\rightarrow 144$
  - there are pitfalls: this is essentially bundle adjustment: we will return to this later  $\rightarrow 135$

## ► Method 2: First-Order Error Approximation

An elegant method for solving problems like (16):

- we will get rid of the latent parameters  $\hat{X}$  needed for obtaining the correction [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error  $\epsilon = \underline{y}^\top \mathbf{F} \underline{x}$  from  $\rightarrow 82$
- consider matches  $\mathbf{Z}_i$ , correspondences  $\hat{\mathbf{Z}}_i$ , and reprojection error  $\mathbf{e}_i = \|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2$
- correspondences satisfy  $\hat{\underline{y}}_i^\top \mathbf{F} \hat{\underline{x}}_i = 0$ ,  $\hat{\underline{x}}_i = (\hat{u}^1, \hat{v}^1, 1)$ ,  $\hat{\underline{y}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold  $\mathcal{V}_F \in \mathbb{R}^4$ : a set of points  $\hat{\mathbf{Z}} = (\hat{u}^1, \hat{v}^1, \hat{u}^2, \hat{v}^2)$  consistent with  $\mathbf{F}$
- algebraic error vanishes for  $\hat{\mathbf{Z}}_i$ :  $\mathbf{0} = \epsilon_i(\hat{\mathbf{Z}}_i) = \hat{\underline{y}}_i^\top \mathbf{F} \hat{\underline{x}}_i$



**Sampson's idea:** Linearize the algebraic error  $\epsilon(\mathbf{Z})$  at  $\mathbf{Z}_i$  (where it is non-zero) and evaluate the resulting linear function at  $\hat{\mathbf{Z}}_i$  (where it is zero). The zero-crossing replaces  $\mathcal{V}_F$  by a linear manifold  $\mathcal{L}$ . The point on  $\mathcal{V}_F$  closest to  $\mathbf{Z}_i$  is replaced by the closest point on  $\mathcal{L}$ .

$$\epsilon_i(\hat{\mathbf{Z}}_i) \approx \epsilon_i(\mathbf{Z}_i) + \frac{\partial \epsilon_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i} (\hat{\mathbf{Z}}_i - \mathbf{Z}_i)$$

## ► Sampson's Approximation of Reprojection Error

- linearize  $\varepsilon(\mathbf{Z})$  at match  $\mathbf{Z}_i$ , evaluate it at correspondence  $\hat{\mathbf{Z}}_i$

$$0 = \varepsilon_i(\hat{\mathbf{Z}}_i) \approx \varepsilon_i(\mathbf{Z}_i) + \underbrace{\frac{\partial \varepsilon_i(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \stackrel{\text{def}}{=} \varepsilon_i(\mathbf{Z}_i) + \mathbf{J}_i(\mathbf{Z}_i) \mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$$

- goal: compute  $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$  from  $\varepsilon_i(\mathbf{Z}_i)$ , where  $\mathbf{e}_i(\cdot)$  is the distance of  $\hat{\mathbf{Z}}_i$  from  $\mathbf{Z}_i$
- we have a linear underconstrained equation for  $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$
- we look for a minimal  $\mathbf{e}_i(\hat{\mathbf{Z}}_i, \mathbf{Z}_i) \stackrel{\text{def}}{=} \mathbf{e}_i$  per match  $i$

$$\mathbf{e}_i^* = \arg \min_{\mathbf{e}_i} \|\mathbf{e}_i\|^2 \quad \text{subject to} \quad \varepsilon_i + \mathbf{J}_i \mathbf{e}_i = 0$$

- which has a closed-form solution **note that  $\mathbf{J}_i$  is not invertible!**      ⊗ P1; 1pt: derive  $\mathbf{e}_i^*$

$$\begin{aligned} \mathbf{e}_i^* &= -\mathbf{J}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \varepsilon_i \\ \|\mathbf{e}_i^*\|^2 &= \varepsilon_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \varepsilon_i \end{aligned} \tag{18}$$

- this maps  $\varepsilon_i(\cdot)$  to an estimate of  $\mathbf{e}_i(\cdot)$  per correspondence
- we often do not need  $\mathbf{e}_i$ , just  $\|\mathbf{e}_i\|^2$       **exception: triangulation** →103
- the unknown parameters  $\mathbf{F}$  are inside:  $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$ ,  $\varepsilon_i = \varepsilon_i(\mathbf{F})$ ,  $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

## ► Example: Fitting A Circle To Scattered Points

**Problem:** Fit a zero-centered circle  $\mathcal{C}$  to a set of 2D points  $\{x_i\}_{i=1}^k$ ,  $\mathcal{C}: \|\mathbf{x}\|^2 - r^2 = 0$ .

1. consider radial error as the 'algebraic error'  $\varepsilon(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2$
2. linearize it at  $\hat{\mathbf{x}}$  we are dropping  $i$  in  $\varepsilon_i$ ,  $\mathbf{e}_i$  etc for clarity

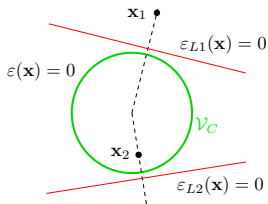
$$\varepsilon(\hat{\mathbf{x}}) \approx \varepsilon(\mathbf{x}) + \underbrace{\frac{\partial \varepsilon(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x})=2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}}, \mathbf{x})} = \dots = 2\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \stackrel{\text{def}}{=} \varepsilon_L(\hat{\mathbf{x}})$$

$\varepsilon_L(\hat{\mathbf{x}}) = 0$  is a line with normal  $\frac{\mathbf{x}}{\|\mathbf{x}\|}$  and intercept  $\frac{r^2 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$  not tangent to  $\mathcal{C}$ , outside!

3. using (18), express error approximation  $\mathbf{e}^*$  as

$$\|\mathbf{e}^*\|^2 = \varepsilon^\top (\mathbf{J}\mathbf{J}^\top)^{-1} \varepsilon = \frac{(\|\mathbf{x}\|^2 - r^2)^2}{4\|\mathbf{x}\|^2}$$

4. fit circle



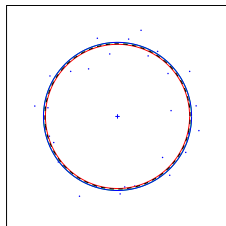
$$r^* = \arg \min_r \sum_{i=1}^k \frac{(\|\mathbf{x}_i\|^2 - r^2)^2}{4\|\mathbf{x}_i\|^2} = \dots = \left( \frac{1}{k} \sum_{i=1}^k \frac{1}{\|\mathbf{x}_i\|^2} \right)^{-\frac{1}{2}}$$

- this example results in a convex quadratic optimization problem
- note that

$$\arg \min_r \sum_{i=1}^k (\|\mathbf{x}_i\|^2 - r^2)^2 = \left( \frac{1}{k} \sum_{i=1}^k \|\mathbf{x}_i\|^2 \right)^{\frac{1}{2}}$$

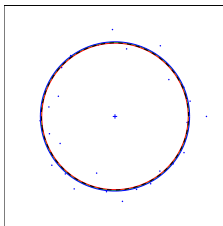
# Circle Fitting: Some Results

medium isotropic noise



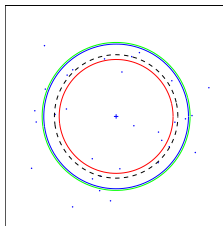
opt=1.8, dir=2.0, Smp=2.2

medium radial noise



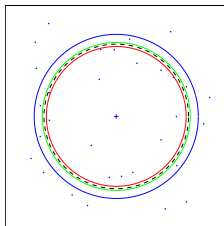
1.8, 1.9, 2.3

big isotropic noise



1.6, 2.0, 2.4

big radial noise



1.6, 1.8, 2.6

mean ranks over 10 000 random trials with  $k = 32$  samples

green – ground truth

red – Sampson error minimizer

blue – direct radial error minimizer

black – optimal estimator for isotropic error

optimal estimator for isotropic error (black, dashed):

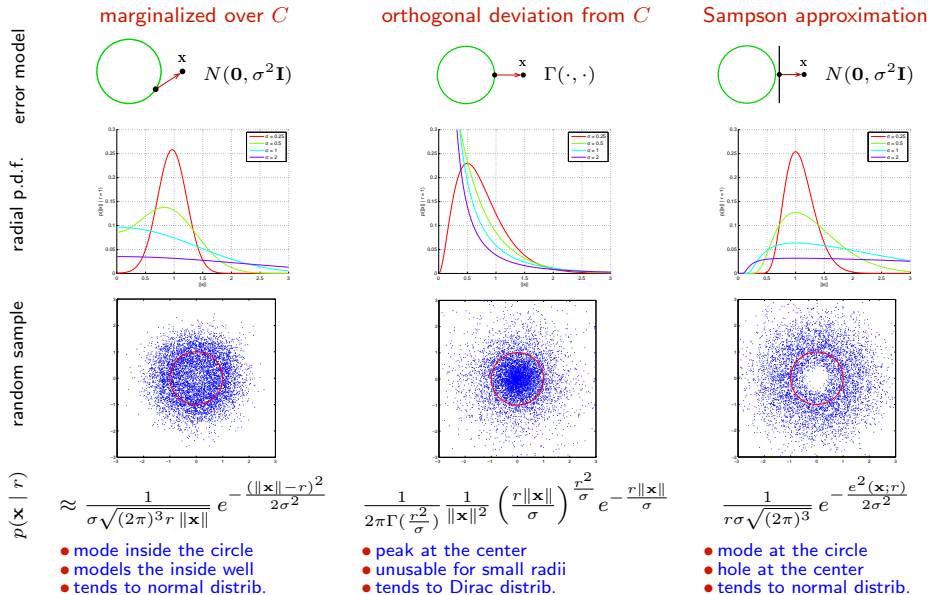
$$r \approx \frac{3}{4k} \sum_{i=1}^k \|\mathbf{x}_i\| + \sqrt{\left( \frac{3}{4k} \sum_{i=1}^k \|\mathbf{x}_i\| \right)^2 - \frac{1}{2k} \sum_{i=1}^k \|\mathbf{x}_i\|^2}$$

## which method is better?

- error should model noise, radial noise and isotropic noise behave differently
- ground truth: Normally distributed isotropic error, Gamma-distributed radial error
- Sampson: better for the radial distribution model; Direct: better for the isotropic model
- no matter how corrected, the algebraic error minimizer is not an unbiased parameter estimator  
Cramér-Rao bound tells us how close one can get with unbiased estimator and given  $k$

# Discussion: On The Art of Probabilistic Model Design...

- a few models for fitting zero-centered circle  $C$  of radius  $r$  to points in  $\mathbb{R}^2$



## ► Sampson Error for Fundamental Matrix Manifold

The epipolar algebraic error is

$$\varepsilon_i(\mathbf{F}) = \underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i, \quad \mathbf{x}_i = (u_i^1, v_i^1), \quad \mathbf{y}_i = (u_i^2, v_i^2), \quad \varepsilon_i \in \mathbb{R}$$

Let  $\mathbf{F} = [\mathbf{F}_1 \quad \mathbf{F}_2 \quad \mathbf{F}_3]$  (per columns) =  $\begin{bmatrix} (\mathbf{F}^1)^\top \\ (\mathbf{F}^2)^\top \\ (\mathbf{F}^3)^\top \end{bmatrix}$  (per rows),  $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , then

### Sampson

$$\mathbf{J}_i(\mathbf{F}) = \left[ \frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^1}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial u_i^2}, \frac{\partial \varepsilon_i(\mathbf{F})}{\partial v_i^2} \right] \quad \mathbf{J}_i \in \mathbb{R}^{1,4} \quad \text{derivatives over point coords.}$$

$$= [(\mathbf{F}_1)^\top \underline{\mathbf{y}}_i, (\mathbf{F}_2)^\top \underline{\mathbf{y}}_i, (\mathbf{F}^1)^\top \underline{\mathbf{x}}_i, (\mathbf{F}^2)^\top \underline{\mathbf{x}}_i]$$

$$\mathbf{e}_i(\mathbf{F}) = -\frac{\mathbf{J}_i(\mathbf{F}) \varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|^2} \quad \mathbf{e}_i(\mathbf{F}) \in \mathbb{R}^4 \quad \text{Sampson error vector}$$

$$e_i(\mathbf{F}) = \|\mathbf{e}_i(\mathbf{F})\| = \frac{\varepsilon_i(\mathbf{F})}{\|\mathbf{J}_i(\mathbf{F})\|} = \frac{\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i}{\sqrt{\|\mathbf{S} \mathbf{F} \underline{\mathbf{x}}_i\|^2 + \|\mathbf{S} \mathbf{F}^\top \underline{\mathbf{y}}_i\|^2}} \quad e_i(\mathbf{F}) \in \mathbb{R} \quad \text{scalar Sampson error}$$

- Sampson error 'normalizes' the algebraic error
- automatically copes with multiplicative factors  $\mathbf{F} \mapsto \lambda \mathbf{F}$
- actual optimization not yet covered →107

## ► Back to Triangulation: The Golden Standard Method

Given  $\mathbf{P}_1, \mathbf{P}_2$  and a correspondence  $x \leftrightarrow y$ , look for 3D point  $\mathbf{X}$  projecting to  $x$  and  $y$ . →87

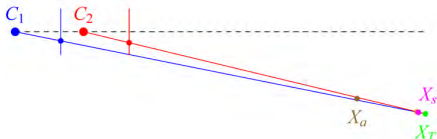
Idea:

1. if not given, compute  $\mathbf{F}$  from  $\mathbf{P}_1, \mathbf{P}_2$ , e.g.  $\mathbf{F} = (\mathbf{Q}_1 \mathbf{Q}_2^{-1})^\top [\mathbf{q}_1 - (\mathbf{Q}_1 \mathbf{Q}_2^{-1}) \mathbf{q}_2]_\times$
2. correct measurement by the linear estimate of the correction vector →98

$$\begin{bmatrix} \hat{u}^1 \\ \hat{v}^1 \\ \hat{u}^2 \\ \hat{v}^2 \end{bmatrix} \approx \begin{bmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \end{bmatrix} - \frac{\varepsilon}{\|\mathbf{J}\|^2} \mathbf{J}^\top = \begin{bmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \end{bmatrix} - \frac{\underline{\mathbf{y}}^\top \mathbf{F} \underline{\mathbf{x}}}{\|\mathbf{S} \mathbf{F} \underline{\mathbf{x}}\|^2 + \|\mathbf{S} \mathbf{F}^\top \underline{\mathbf{y}}\|^2} \begin{bmatrix} (\mathbf{F}_1)^\top \underline{\mathbf{y}} \\ (\mathbf{F}_2)^\top \underline{\mathbf{y}} \\ (\mathbf{F}^1)^\top \underline{\mathbf{x}} \\ (\mathbf{F}^2)^\top \underline{\mathbf{x}} \end{bmatrix}$$

3. use the SVD triangulation algorithm with numerical conditioning →88; iteration possible

Ex (cont'd from →91):



- $X_T$  – noiseless ground truth position
- – reprojection error minimizer
- $X_s$  – Sampson-corrected algebraic error minimizer
- $X_a$  – algebraic error minimizer
- $m$  – measurement ( $m_T$  with noise in  $v^2$ )





## ► Back to Fundamental Matrix Estimation

**Goal:** Given a set  $X = \{(x_i, y_i)\}_{i=1}^k$  of  $k \gg 7$  inlier correspondences, compute a statistically efficient estimate for fundamental matrix  $\mathbf{F}$ .

### What we have so far

- 7-point algorithm for  $\mathbf{F}$  (5-point algorithm for  $\mathbf{E}$ )  $\rightarrow$ 82
- definition of Sampson error per correspondence  $e_i(\mathbf{F} \mid x_i, y_i) \rightarrow$ 102

### What we need

- an optimization algorithm for

$$\mathbf{F}^* = \arg \min_{\mathbf{F}} \sum_{i=1}^k e_i^2(\mathbf{F} \mid X)$$

- the 7-point estimate is a good starting point  $\mathbf{F}_0$

# Levenberg-Marquardt (LM) Iterative Estimation in a Nutshell

Consider error function  $\mathbf{e}_i(\boldsymbol{\theta}) = f(\mathbf{x}_i, \mathbf{y}_i, \boldsymbol{\theta}) \in \mathbb{R}^m$ , with  $\mathbf{x}_i, \mathbf{y}_i$  given,  $\boldsymbol{\theta} \in \mathbb{R}^q$  unknown  
 $\theta = \mathbf{F}$ ,  $q = 9$ ,  $m = 1$  for f.m. estimation

**Our goal:**  $\boldsymbol{\theta}^* = \arg \min_{\boldsymbol{\theta}} \sum_{i=1}^k \|\mathbf{e}_i(\boldsymbol{\theta})\|^2$

**Idea 1** (Gauss-Newton approximation): proceed iteratively for  $s = 0, 1, 2, \dots$

$$\boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s, \quad \text{where } \mathbf{d}_s = \arg \min_{\mathbf{d}} \sum_{i=1}^k \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d})\|^2 \quad (19)$$

$$\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}) \approx \mathbf{e}_i(\boldsymbol{\theta}^s) + \mathbf{L}_i \mathbf{d},$$

$$(\mathbf{L}_i)_{jl} = \frac{\partial (\mathbf{e}_i(\boldsymbol{\theta}))_j}{\partial (\boldsymbol{\theta})_l}, \quad \mathbf{L}_i \in \mathbb{R}^{m,q} \quad \text{typically a long matrix}$$

Then the solution to Problem (19) is a set of normal eqs

$$-\underbrace{\sum_{i=1}^k \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s)}_{\mathbf{e} \in \mathbb{R}^{q,1}} = \underbrace{\left( \sum_{i=1}^k \mathbf{L}_i^\top \mathbf{L}_i \right)}_{\mathbf{L} \in \mathbb{R}^{q,q}} \mathbf{d}_s, \quad (20)$$

- $\mathbf{d}_s$  can be solved for by Gaussian elimination using Choleski decomposition of  $\mathbf{L}$   
 $\mathbf{L}$  symmetric  $\Rightarrow$  use Choleski, almost  $2\times$  faster than Gauss-Seidel, see bundle adjustment  $\rightarrow 138$
- such updates do not lead to stable convergence  $\rightarrow$  ideas of Levenberg and Marquardt

**Idea 2** (Levenberg): replace  $\sum_i \mathbf{L}_i^\top \mathbf{L}_i$  with  $\sum_i \mathbf{L}_i^\top \mathbf{L}_i + \lambda \mathbf{I}$  for some damping factor  $\lambda \geq 0$

**Idea 3** (Marquardt): replace  $\lambda \mathbf{I}$  with  $\lambda \sum_i \text{diag}(\mathbf{L}_i^\top \mathbf{L}_i)$  to adapt to local curvature:

$$-\sum_{i=1}^k \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s) = \left( \sum_{i=1}^k (\mathbf{L}_i^\top \mathbf{L}_i + \lambda \text{diag}(\mathbf{L}_i^\top \mathbf{L}_i)) \right) \mathbf{d}_s$$

**Idea 4** (Marquardt): adaptive  $\lambda$       small  $\lambda \rightarrow$  Gauss-Newton, large  $\lambda \rightarrow$  gradient descend

1. choose  $\lambda \approx 10^{-3}$  and compute  $\mathbf{d}_s$
2. if  $\sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}_s)\|^2 < \sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s)\|^2$  then accept  $\mathbf{d}_s$  and set  $\lambda := \lambda/10$ ,  $s := s + 1$
3. otherwise set  $\lambda := 10\lambda$  and recompute  $\mathbf{d}_s$

- sometimes different constants are needed for the 10 and  $10^{-3}$
- note that  $\mathbf{L}_i \in \mathbb{R}^{m,q}$  (long matrix) but each contribution  $\mathbf{L}_i^\top \mathbf{L}_i$  is a square singular  $q \times q$  matrix (always singular for  $k < q$ )
- error can be made robust to outliers, see the trick  $\rightarrow 110$
- we have approximated the least squares Hessian by ignoring second derivatives of the error function (Gauss-Newton approximation) See [Triggs et al. 1999, Sec. 4.3]
- $\lambda$  helps avoid the consequences of gauge freedom  $\rightarrow 140$

**Sampson** (derived by linearization over point coordinates  $u^1, v^1, u^2, v^2$ )

$$e_i(\mathbf{F}) = \frac{\varepsilon_i}{\|\mathbf{J}_i\|} = \frac{\mathbf{y}_i^\top \mathbf{F} \mathbf{x}_i}{\sqrt{\|\mathbf{S} \mathbf{F} \mathbf{x}_i\|^2 + \|\mathbf{S} \mathbf{F}^\top \mathbf{y}_i\|^2}} \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**LM** (by linearization over parameters  $\mathbf{F}$ )

$$\mathbf{L}_i = \frac{\partial e_i(\mathbf{F})}{\partial \mathbf{F}} = \dots = \frac{1}{2\|\mathbf{J}_i\|} \left[ \left( \mathbf{y}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S} \mathbf{F} \mathbf{x}_i \right) \mathbf{x}_i^\top + \mathbf{y}_i \left( \mathbf{x}_i - \frac{2e_i}{\|\mathbf{J}_i\|} \mathbf{S} \mathbf{F}^\top \mathbf{y}_i \right)^\top \right] \quad (21)$$

- $\mathbf{L}_i$  from (21) is a  $3 \times 3$  matrix, must be reshaped to dimension-9 vector  $\text{vec}(\mathbf{L}_i)$
- $\mathbf{x}_i$  and  $\mathbf{y}_i$  in Sampson error are normalized to unit homogeneous coordinate (21) relies on this
- reinforce rank  $\mathbf{F} = 2$  after each LM update to stay in the fundamental matrix manifold and  $\|\mathbf{F}\| = 1$  to avoid gauge freedom by SVD  $\rightarrow 108$
- LM linearization could be done by numerical differentiation (small dimension)

## ► Local Optimization for Fundamental Matrix Estimation

Given a set  $X = \{(x_i, y_i)\}_{i=1}^k$  of  $k \gg 7$  inlier correspondences, compute a statistically efficient estimate for fundamental matrix  $\mathbf{F}$ .

### Summary so far

1. Find the conditioned ( $\rightarrow 90$ ) 7-point  $\mathbf{F}_0$  ( $\rightarrow 82$ ) from a suitable 7-tuple
2. Improve the  $\mathbf{F}_0^*$  using the LM optimization ( $\rightarrow 105-106$ ) and the Sampson error ( $\rightarrow 107$ ) on all inliers, reinforce rank-2, unit-norm  $\mathbf{F}_k^*$  after each LM iteration using SVD

### We are not yet done

- if there are no wrong correspondences (mismatches, outliers), this gives a local optimum given the 7-point initial estimate
- the algorithm breaks under contamination of (inlier) correspondences by outliers
- the full problem involves finding the inliers!
- in addition, we need a mechanism for jumping out of local minima (and exploring the space of all fundamental matrices)

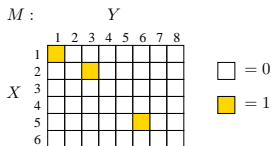
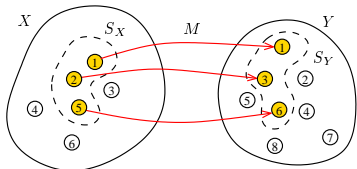
## ► The Full Problem of Matching and Fundamental Matrix Estimation

**Problem:** Given image point sets  $X = \{x_i\}_{i=1}^m$  and  $Y = \{y_j\}_{j=1}^n$  and their descriptors  $D$ , find the most probable

1. inliers  $S_X \subseteq X$ ,  $S_Y \subseteq Y$
2. one-to-one perfect matching  $M: S_X \rightarrow S_Y$
3. fundamental matrix  $\mathbf{F}$  such that  $\text{rank } \mathbf{F} = 2$
4. such that for each  $x_i \in S_X$  and  $y_j = M(x_i)$  it is probable that
  - a) the image descriptor  $D(x_i)$  is similar to  $D(y_j)$ , and
  - b) the total geometric error  $E = \sum_{ij} e_{ij}^2(\mathbf{F})$  is small
5. inlier-outlier and outlier-outlier matches are improbable

perfect matching: 1-factor of the bipartite graph

note a slight change in notation:  $e_{ij}$



$$(M^*, \mathbf{F}^*) = \arg \max_{M, \mathbf{F}} p(E, D, \mathbf{F} | M) P(M) \quad (22)$$

- probabilistic model: an efficient language for problem formulation it also unifies 4.a and 4.b
- the (22) is a Bayesian probabilistic model there is a constant number of random variables!
- binary matching table  $M_{ij} \in \{0, 1\}$  of fixed size  $m \times n$ 
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point  $x_i / y_j$

# Deriving A Robust Matching Model by Approximate Marginalization

For algorithmic efficiency, instead of  $(M^*, \mathbf{F}^*) = \arg \max_{M, \mathbf{F}} p(E, D, \mathbf{F} | M) P(M)$  solve

$$\mathbf{F}^* = \arg \max_{\mathbf{F}} p(E, D, \mathbf{F}) \quad (23)$$

by marginalization of  $p(E, D, \mathbf{F} | M) P(M)$  over  $M$  this changes the problem!

ignoring that  $M$  are 1:1 matchings and assuming correspondence-wise independence:

$$p(E, D, \mathbf{F} | M) P(M) = \prod_{i=1}^m \prod_{j=1}^n p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij})$$

- $e_{ij}$  represents geometric error for match  $x_i \leftrightarrow y_j$ :  $e_{ij}(x_i, y_j, \mathbf{F})$
- $d_{ij}$  represents descriptor similarity for match  $x_i \leftrightarrow y_j$ :  $d_{ij} = \|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|$

Marginalization:

$$\begin{aligned} p(E, D, \mathbf{F}) &\approx \sum_{m_{11} \in \{0,1\}} \sum_{m_{12}} \cdots \sum_{m_{mn}} p(E, D, \mathbf{F} | M) P(M) = \\ &= \sum_{m_{11}} \cdots \sum_{m_{mn}} \prod_{i=1}^m \prod_{j=1}^n p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij}) = \overset{*}{\dots} \overset{!}{=} \\ &= \prod_{i=1}^m \prod_{j=1}^n \underbrace{\sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} | m_{ij}) P(m_{ij})}_{\text{we will continue with this term}} \end{aligned}$$

## Robust Matching Model (cont'd)

$$\begin{aligned}
 & \sum_{m_{ij} \in \{0,1\}} p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij}) P(m_{ij}) = \\
 & = \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 1)}_{p_1(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 1)}_{1 - P_0} + \underbrace{p_e(e_{ij}, d_{ij}, \mathbf{F} \mid m_{ij} = 0)}_{p_0(e_{ij}, d_{ij}, \mathbf{F})} \underbrace{P(m_{ij} = 0)}_{P_0} = \\
 & = (1 - P_0) p_1(e_{ij}, d_{ij}, \mathbf{F}) + P_0 p_0(e_{ij}, d_{ij}, \mathbf{F}) \quad (24)
 \end{aligned}$$

- the  $p_0(e_{ij}, d_{ij}, \mathbf{F})$  is a penalty for 'missing a correspondence' but it should be a p.d.f. (cannot be a constant)  $(\rightarrow 112 \text{ for a simplification})$

choose  $P_0 \rightarrow 1$ ,  $p_0(\cdot) \rightarrow 0$  so that  $\frac{P_0}{1 - P_0} p_0(\cdot) \approx \text{const}$

- the  $p_1(e_{ij}, d_{ij}, \mathbf{F})$  is typically an easy-to-design term: assuming independence of geometric error and descriptor similarity:

$$p_1(e_{ij}, d_{ij}, \mathbf{F}) = p_1(e_{ij} \mid \mathbf{F}) p_F(\mathbf{F}) p_1(d_{ij})$$

- we choose, eg.

$$p_1(e_{ij} \mid \mathbf{F}) = \frac{1}{T_e(\sigma_1)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}}, \quad p_1(d_{ij}) = \frac{1}{T_d(\sigma_d, \text{dim } \mathbf{d})} e^{-\frac{\|\mathbf{d}(x_i) - \mathbf{d}(y_j)\|^2}{2\sigma_d^2}} \quad (25)$$

- $\mathbf{F}$  is a random variable and  $\sigma_1, \sigma_d, P_0$  are parameters
- the form of  $T(\sigma_1)$  depends on error definition, it may depend on  $x_i, y_j$  but not on  $\mathbf{F}$
- we will continue with the result from (24)



## ► Simplified Robust Energy (Error) Function

- assuming the choice of  $p_1$  as in (25), we are simplifying

$$\begin{aligned} p(E, D, \mathbf{F}) &= p(E, D \mid \mathbf{F}) p_F(\mathbf{F}) = \\ &= p_F(\mathbf{F}) \prod_{i=1}^m \prod_{j=1}^n \left[ (1 - P_0) p_1(e_{ij}, d_{ij} \mid \mathbf{F}) + P_0 p_0(e_{ij}, d_{ij} \mid \mathbf{F}) \right] \end{aligned}$$

- we choose  $\sigma_0 \gg \sigma_1$  and omit  $d_{ij}$  for simplicity; then the square-bracket term is

$$\frac{1 - P_0}{T_e(\sigma_1)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \frac{P_0}{T_e(\sigma_0)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}$$

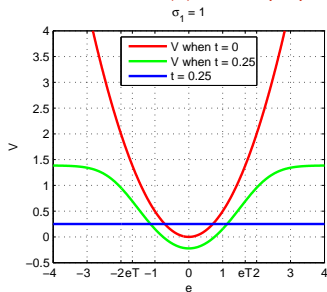
- we define the 'potential function' as:  $V(x) = -\log p(x)$ , then

$$\begin{aligned} V(E, D \mid \mathbf{F}) &= \sum_{i=1}^m \sum_{j=1}^n \left[ \underbrace{-\log \frac{1 - P_0}{T_e(\sigma_1)}}_{\Delta = \text{const}} - \log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + \underbrace{\frac{P_0}{1 - P_0} \frac{T_e(\sigma_1)}{T_e(\sigma_0)} e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_0^2}}}_{t \approx \text{const}} \right) \right] = \\ &= mn \Delta + \sum_{i=1}^m \sum_{j=1}^n \underbrace{-\log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right)}_{\hat{V}(e_{ij})} \quad (26) \end{aligned}$$

- note we are summing over all  $mn$  matches ( $m, n$  are constant!)

# ► The Action of the Robust Matching Model on Data

## Example for $\hat{V}(e)$ from (26):



- red – the usual (non-robust) error when  $t = 0$
- blue – the rejected correspondence penalty  $t$
- green – ‘robust energy’ (26)

- if the error of a correspondence exceeds a limit, it is ignored
- then  $\hat{V}(e) = \text{const}$  and we essentially count outliers in (26)
- $t$  controls the ‘turn-off’ point
- the inlier/outlier threshold is  $e_T$  – the error for which  $(1 - P_0) p_1(e_T) = P_0 p_0(e_T)$ : note that  $t \approx 0$

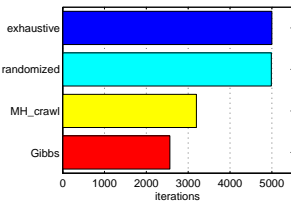
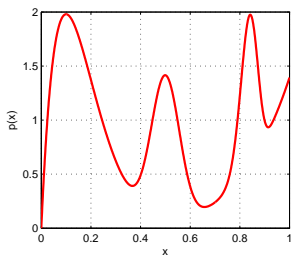
$$e_T = \sigma_1 \sqrt{-\log t^2} \quad (27)$$

## The full optimization problem (23) uses (26):

$$\mathbf{F}^* = \arg \max_{\mathbf{F}} \frac{\overbrace{p(E, D | \mathbf{F})}^{\text{data model}} \cdot \overbrace{p(\mathbf{F})}^{\text{prior}}}{\underbrace{p(E, D)}_{\text{evidence}}} \approx \arg \min_{\mathbf{F}} \left[ V(\mathbf{F}) + \sum_{i=1}^m \sum_{j=1}^n \log \left( e^{-\frac{e_{ij}^2(\mathbf{F})}{2\sigma_1^2}} + t \right) \right]$$

- typically we take  $V(\mathbf{F}) = -\log p(\mathbf{F}) = 0$  unless we need to stabilize a computation, e.g. when video camera moves smoothly (on a high-mass vehicle) and we have a prediction for  $\mathbf{F}$
- evidence is not needed unless we want to compare different models (eg. homography vs. epipolar geometry)

# How To Find the Global Maxima (Modes) of a PDF?



- averaged over  $10^4$  trials
- number of proposals before  $|x - x_{true}| \leq \text{step}$

• given the function  $p(x)$  at left  
**consider several methods:**

p.d.f. on  $[0, 1]$ , mode at 0.1

## 1. exhaustive search

```
step = 1/(iterations-1);  
for x = 0:step:1  
    if p(x) > bestp  
        bestx = x; bestp = p(x);  
    end  
end
```

- slow algorithm (definite quantization)
- fast to implement

## 2. randomized search with uniform sampling

```
while t < iterations  
    x = rand(1);  
    if p(x) > bestp  
        bestx = x; bestp = p(x);  
    end  
    t = t+1; % time  
end
```

- equally slow algorithm
- fast to implement

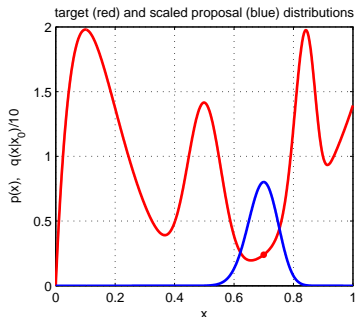
## 3. random sampling from $p(x)$ (Gibbs sampler)

- faster algorithm
- fast to implement but often infeasible (e.g. when  $p(x)$  is data dependent (our case in correspondence prob.))

## 4. Metropolis-Hastings sampling

- almost as fast (with care)
- not so fast to implement
- rarely infeasible
- RANSAC belongs here

# How To Generate Random Samples from a Complex Distribution?



- red: probability density function  $\pi(x)$  of the toy distribution on the unit interval **target distribution**

$$\pi(x) = \sum_{i=1}^4 \gamma_i \text{Be}(x; \alpha_i, \beta_i), \quad \sum_{i=1}^4 \gamma_i = 1, \quad \gamma_i \geq 0$$

$$\text{Be}(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot x^{\alpha-1} (1-x)^{\beta-1}$$

- note we can generate samples from this  $\pi(x)$  **how?**

- suppose we cannot sample from  $\pi(x)$  but we can sample from some 'simple' distribution  $q(x | x_0)$ , given the last sample  $x_0$  (blue) **proposal distribution**

$$q(x | x_0) = \begin{cases} U_{0,1}(x) & \text{(independent) uniform sampling} \\ \text{Be}(x; \frac{x_0}{T} + 1, \frac{1-x_0}{T} + 1) & \text{'beta' diffusion (crawler) } T - \text{temperature} \\ \pi(x) & \text{(independent) Gibbs sampler} \end{cases}$$

- note we have unified all the random sampling methods from the previous slide
- how to transform proposal samples  $q(x | x_0)$  to target distribution  $\pi(x)$  samples?

## ► Metropolis-Hastings (MH) Sampling

$C$  – configuration (of all variable values)

eg.  $C = x$  and  $\pi(C) = \pi(x)$  from →115

**Goal:** Generate a sequence of random samples  $\{C_t\}$  from target distribution  $\pi(C)$

- setup a Markov chain with a suitable transition probability to generate the sequence

### Sampling procedure

1. given  $C_t$ , draw a random sample  $S$  from  $q(S | C_t)$

$q$  may use some information from  $C_t$  (Hastings)

2. compute acceptance probability

the evidence term drops out

$$a = \min \left\{ 1, \frac{\pi(S)}{\pi(C_t)} \cdot \frac{q(C_t | S)}{q(S | C_t)} \right\}$$

3. draw a random number  $u$  from unit-interval uniform distribution  $U_{0,1}$
4. if  $u \leq a$  then  $C_{t+1} := S$  else  $C_{t+1} := C_t$

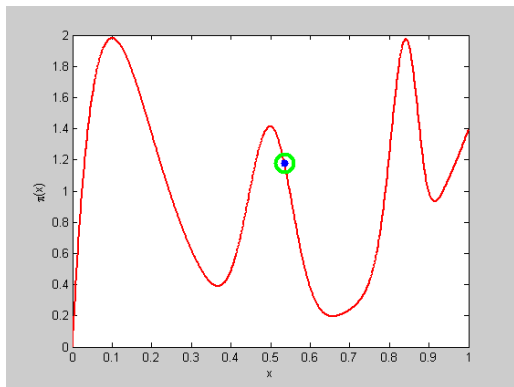
### 'Programming' an MH sampler

1. design a proposal distribution (mixture)  $q$  and a sampler from  $q$
2. write functions  $q(C_t | S)$  and  $q(S | C_t)$  that are proper distributions not always simple

### Finding the mode

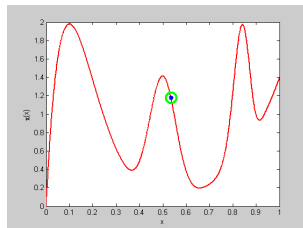
- remember the best sample fast implementation but must wait long to hit the mode
- use simulated annealing very slow
- start local optimization from the best sample good trade-off between speed and accuracy  
an optimal algorithm does not use just the best sample: a Stochastic EM Algorithm (e.g. SAEM)

# MH Sampling Demo

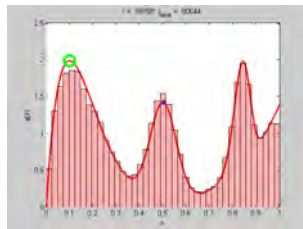


sampling process (video, 7:33, 100k samples)

- blue point: current sample
- green circle: best sample so far  $\text{quality} = \pi(x)$
- histogram: current distribution of visited states
- the vicinity of modes are the most often visited states



initial sample



final distribution  
of visited states

# Demo Source Code (Matlab)

```
function x = proposal_gen(x0)
% proposal generator q(x | x0)

T = 0.01; % temperature
x = betarnd(x0/T+1,(1-x0)/T+1);
end

function p = proposal_q(x, x0)
% proposal distribution q(x | x0)

T = 0.01;
p = betapdf(x, x0/T+1, (1-x0)/T+1);
end

function p = target_p(x)
% target distribution p(x)

% shape parameters:
a = [2 40 100 6];
b = [10 40 20 1];

% mixing coefficients:
w = [1 0.4 0.253 0.50]; w = w/sum(w);
p = 0;
for i = 1:length(a)
    p = p + w(i)*betapdf(x,a(i),b(i));
end
end
```

```
%% DEMO script

k = 10000; % number of samples
X = NaN(1,k); % list of samples

x0 = proposal_gen(0.5);
for i = 1:k
    x1 = proposal_gen(x0);
    a = target_p(x1)/target_p(x0) * ...
        proposal_q(x0,x1)/proposal_q(x1,x0);
    if rand(1) < a
        X(i) = x1; x0 = x1;
    else
        X(i) = x0;
    end
end

figure(1)
x = 0:0.001:1;
plot(x, target_p(x), 'r', 'linewidth',2);
hold on
binw = 0.025; % histogram bin width
n = histc(X, 0:binw:1);
h = bar(0:binw:1, n/sum(n)/binw, 'histc');
set(h, 'facecolor', 'r', 'facealpha', 0.3)
xlim([0 1]); ylim([0 2.5])
xlabel 'x'
ylabel 'p(x)'
title 'MH demo'
hold off
```

## ► Stripping MH Down

- when we are interested in the best sample only... and we need fast data exploration...

### Simplified sampling procedure

1. ~~given  $C_t$ , draw a random sample  $S$  from  $q(S|C_t)$~~   $q(S)$  independent sampling  
no use of information from  $C_t$

2. ~~compute acceptance probability~~

$$a = \min \left\{ 1, \frac{\pi(S)}{\pi(C_t)} \cdot \frac{q(C_t | S)}{q(S | C_t)} \right\}$$

3. ~~draw a random number  $u$  from unit-interval uniform distribution  $\mathbb{U}_{0,1}$~~
4. ~~if  $u \leq a$  then  $C_{t+1} := S$  else  $C_{t+1} := C_t$~~
5. if  $\pi(S) > \pi(C_{\text{best}})$  then remember  $C_{\text{best}} := S$

Steps 2–4 make no difference when waiting for the best sample

- ... but getting a good accuracy sample might take very long this way
- good overall exploration but slow convergence in the vicinity of a mode where  $C_t$  could serve as an attractor
- cannot use the past generated samples to estimate any parameters
- we will fix these problems by (possibly robust) 'local optimization'

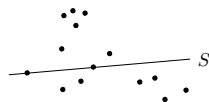


## ► Putting Some Clothes Back: RANSAC with Local Optimization

1. initialize the best sample as empty  $C_{\text{best}} := \emptyset$  and time  $t := 0$
2. estimate the number of needed iterations as  $N := \binom{mn}{s}$
3. while  $t \leq N$ :

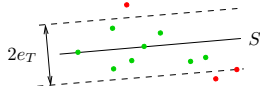
$s$  – minimal sample size

- a) draw a minimal random sample  $S$  of size  $s$  from  $q(S)$
- b) if  $\pi(S) > \pi(C_{\text{best}})$  then



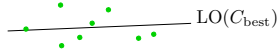
- i) update the best sample  $C_{\text{best}} := S$
- ii) threshold-out inliers using (27)

$\pi(S)$  marginalized as in (26);  $\pi(S)$  includes a prior  $\Rightarrow$  MAP



- iii) start local optimization from the inliers of  $C_{\text{best}}$

LM optimization with robustified ( $\rightarrow$ 112) Sampson error possibly weighted by posterior  $\pi(m_{ij})$  [Chum et al. 2003]



- iv) update  $C_{\text{best}}$ , update inliers using (27), re-estimate  $N$  from inlier counts

$\rightarrow$ 121 for derivation

$$N = \frac{\log(1 - P)}{\log(1 - \epsilon^s)}, \quad \epsilon = \frac{|\text{inliers}(C_{\text{best}})|}{mn},$$

- c)  $t := t + 1$

4. output  $C_{\text{best}}$

• see [MPV course](#) for RANSAC details

see also [Fischler & Bolles 1981], [25 years of RANSAC]

## ► Stopping RANSAC

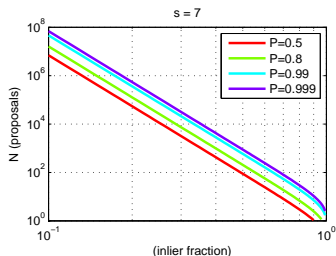
**Principle:** what is the number of proposals  $N$  that are needed to hit an all-inlier sample?  
this will tell us nothing about the accuracy of the result

$P$  ... probability that at least one sample is an all-inlier     $1 - P$  ... all previous  $N$  samples were bad  
 $\varepsilon$  ... the fraction of inliers among tentative correspondences,  $\varepsilon \leq 1$   
 $s$  ... sample size (7 in 7-point algorithm)

$$N \geq \frac{\log(1 - P)}{\log(1 - \varepsilon^s)}$$

- $\varepsilon^s$  ... proposal does not contain an outlier
- $1 - \varepsilon^s$  ... proposal contains at least one outlier
- $(1 - \varepsilon^s)^N$  ...  $N$  previous proposals contained an outlier =  $1 - P$

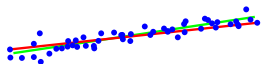
N for s = 7		
	P	
$\varepsilon$	0.8	0.99
0.5	205	590
0.2	$1.3 \cdot 10^5$	$3.5 \cdot 10^5$
0.1	$1.6 \cdot 10^7$	$4.6 \cdot 10^7$



- $N$  can be re-estimated using the current estimate for  $\varepsilon$  (if there is LO, then after LO)  
the quasi-posterior estimate for  $\varepsilon$  is the average over all samples generated so far
- this shows we have a good reason to limit all possible matches to tentative matches only
- for  $\varepsilon \rightarrow 0$  we gain nothing over the standard MH-sampler stopping criterion

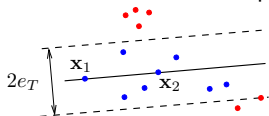
# The Core Ideas in RANSAC [Fischler & Bolles 1981]

1. configuration =  $s$ -tuple of inlier correspondences
  - the minimization will be over a discrete set of epipolar geometries proposable from 7-tuples
2. proposal distribution  $q(\cdot)$  is given by the empirical distribution of data samples:
  - a) select  $s$ -tuple from data independently  $q(S | C_t) = q(S)$ 
    - i)  $q$  uniform  $q(S) = \binom{mn}{s}^{-1}$  MAPSAC ( $p(S)$  includes the prior)
    - ii)  $q$  dependent on descriptor similarity PROSAC (similar pairs are proposed more often)
  - b) solve the minimal geometric problem  $\mapsto$  parameter proposal e.g.  $\mathbf{F}$  from  $s = 7$



- pairs of points define line distribution from  $p(\mathbf{n} | X)$  (left)
- random correspondence tuples drawn uniformly propose samples of  $\mathbf{F}$  from a data-driven distribution  $q(\mathbf{F} | E)$

3. independent sampling & looking for the best sample  $\Rightarrow$  no need to filter proposals by  $a$
4. standard RANSAC replaces probability maximization with consensus maximization

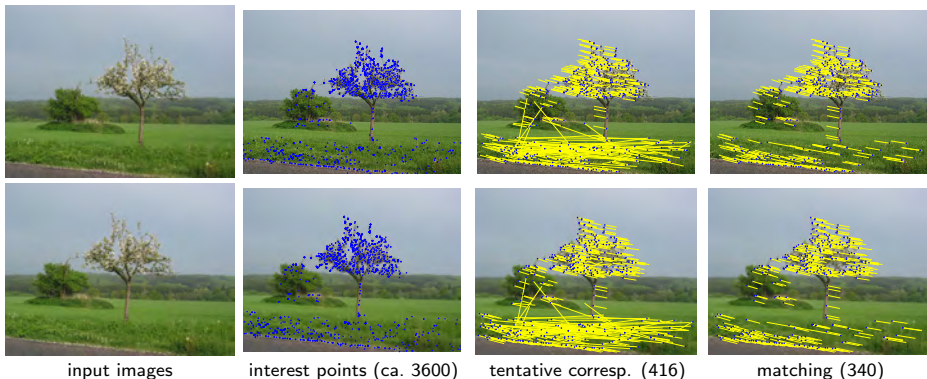


the  $e_T$  is the inlier/outlier threshold from (27)

5. stopping based on the probability of mode-hitting

$\rightarrow 121$

# Example Matching Results for the 7-point Algorithm with RANSAC



- notice some wrong matches (they have wrong depth, even negative)
- they cannot be rejected without additional constraints or scene knowledge
- without local optimization the minimization is over a discrete set of epipolar geometries proposable from 7-tuples

# Beyond RANSAC

By marginalization in (23) we have lost constraints on  $M$  (eg. uniqueness). One can choose a better model when not marginalizing:

$$\pi(M, \mathbf{F}, E, D) = \underbrace{p(E | M, \mathbf{F})}_{\text{geometric error}} \cdot \underbrace{p(D | M)}_{\text{similarity}} \cdot \underbrace{p(\mathbf{F})}_{\text{prior}} \cdot \underbrace{P(M)}_{\text{constraints}}$$

this is a global model: decisions on  $m_{ij}$  are no longer independent!

## In the MH scheme

- one can work with full  $p(M, \mathbf{F} | E, D)$ , then  $S = (M, \mathbf{F})$

- explicit labeling  $m_{ij}$  can be done by, e.g. sampling from

$$q(m_{ij} | \mathbf{F}) \sim ((1 - P_0) p_1(e_{ij} | \mathbf{F}), P_0 p_0(e_{ij} | \mathbf{F}))$$

when  $P(M)$  uniform then always accepted,  $a = 1$

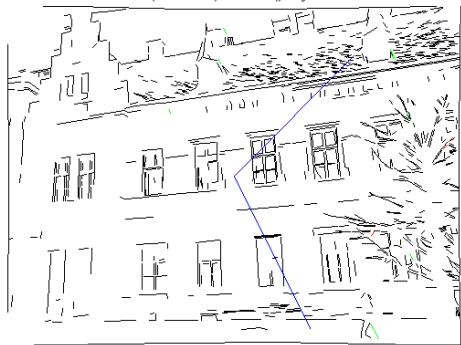
⊗ derive

- we can compute the posterior probability of each match  $p(m_{ij})$  by histogramming  $m_{ij}$  from  $\{S_i\}$
- local optimization can then use explicit inliers and  $p(m_{ij})$
- error can be estimated for elements of  $\mathbf{F}$  from  $\{S_i\}$  does not work in RANSAC!
- large error indicates problem degeneracy this is not directly available in RANSAC
- good conditioning is not a requirement we work with the entire distribution  $p(\mathbf{F})$
- one can find the most probable number of epipolar geometries by reversible jump MCMC and model selection  
(homographies or other models) if there are multiple models explaining data, RANSAC will return one of them randomly

# Example: MH Sampling for a More Complex Problem

**Task:** Find two vanishing points from line segments detected in input image.

iter: 10 (acc TOT=0.0%, HMC=NaN%); Eavg = 14.597



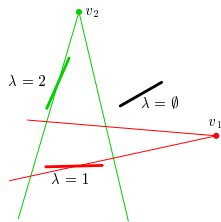
video

## simplifications

- vanishing points restricted to the set of all pairwise segment intersections
- mother lines fixed by segment centroid (then  $\theta_L$  uniquely given by  $\lambda_i$ )

## Model

- principal point known, square pixel
- latent variables
  1. each line has a vanishing point label  $\lambda_i \in \{\emptyset, 1, 2\}$ ,  $\emptyset$  represents an outlier
- explicit variables
  1. two unknown vanishing points  $v_1, v_2$
  2. 'mother line' parameters  $\theta_L$  (they pass through their vanishing points)



$$\arg \min_{v_1, v_2, \Lambda, \theta_L} V(v_1, v_2, \Lambda, L | S)$$

## 3D Structure and Camera Motion

6.1 Introduction

6.2 Reconstructing Camera Systems

6.3 Bundle Adjustment

**covered by**

[1] [H&Z] Secs: 9.5.3, 10.1, 10.2, 10.3, 12.1, 12.2, 12.4, 12.5, 18.1

[2] Triggs, B. et al. Bundle Adjustment—A Modern Synthesis. In *Proc ICCV Workshop on Vision Algorithms*. Springer-Verlag. pp. 298–372, 1999.

**additional references**



D. Martinec and T. Pajdla. Robust Rotation and Translation Estimation in Multiview Reconstruction. In *Proc CVPR*, 2007



M. I. A. Lourakis and A. A. Argyros. SBA: A Software Package for Generic Sparse Bundle Adjustment. *ACM Trans Math Software* 36(1):1–30, 2009.

## ► Constructing Cameras from the Fundamental Matrix

Given  $\mathbf{F}$ , construct some cameras  $\mathbf{P}_1, \mathbf{P}_2$  such that  $\mathbf{F}$  is their fundamental matrix.

**Solution**

$$\mathbf{P}_1 = [\mathbf{I} \quad \mathbf{0}]$$

See [H&Z, p. 256]

$$\mathbf{P}_2 = [[\mathbf{e}_2]_{\times} \mathbf{F} + \mathbf{e}_2 \underline{\mathbf{v}}^{\top} \quad \lambda \mathbf{e}_2]$$

where

- $\underline{\mathbf{v}}$  is any 3-vector, e.g.  $\underline{\mathbf{v}} = \mathbf{e}_1 = \text{null}(\mathbf{F})$ , i.e.  $\mathbf{F} \mathbf{e}_1 = 0$ , to make the camera finite
- $\lambda \neq 0$  is a scalar,
- $\mathbf{e}_2 = \text{null}(\mathbf{F}^{\top})$ , i.e.  $\mathbf{e}_2^{\top} \mathbf{F} = 0$

**Proof**

1.  $\mathbf{S}$  is skew-symmetric iff  $\mathbf{x}^{\top} \mathbf{S} \mathbf{x} = 0$  for all  $\mathbf{x}$  look-up the proof!
2. we have  $\underline{\mathbf{x}} \simeq \mathbf{P} \underline{\mathbf{X}}$
3. a non-zero  $\mathbf{F}$  is a f.m. of  $(\mathbf{P}_1, \mathbf{P}_2)$  iff  $\mathbf{P}_2^{\top} \mathbf{F} \mathbf{P}_1$  is skew-symmetric
4. if  $\mathbf{P}_1 = [\mathbf{I} \quad \mathbf{0}]$  and  $\mathbf{P}_2 = [\mathbf{S} \mathbf{F} \quad \mathbf{e}_2]$  then  $\mathbf{F}$  corresponds to  $(\mathbf{P}_1, \mathbf{P}_2)$  by Step 3
5. we can write  $\mathbf{S} = [\mathbf{s}]_{\times}$
6. a suitable choice is  $\mathbf{s} = \mathbf{e}_2$  [Luong96]
7. for the full the class including  $\underline{\mathbf{v}}$ , see [H&Z, Sec. 9.5]

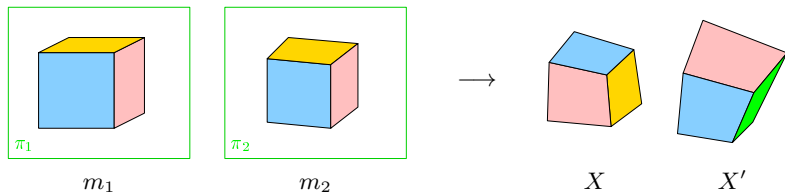


## ► The Projective Reconstruction Theorem

**Observation:** Unless  $\mathbf{P}_i$  are constrained, then for any number of cameras  $i = 1, \dots, k$

$$\underline{\mathbf{m}}_i \simeq \mathbf{P}_i \underline{\mathbf{X}} = \underbrace{\mathbf{P}_i \mathbf{H}^{-1}}_{\mathbf{P}'_i} \underbrace{\mathbf{H} \underline{\mathbf{X}}}_{\underline{\mathbf{X}'}} = \mathbf{P}'_i \underline{\mathbf{X}'}$$

- when  $\mathbf{P}_i$  and  $\underline{\mathbf{X}}$  are both determined from correspondences (including calibrations  $\mathbf{K}_i$ ), they are given up to a common 3D homography  $\mathbf{H}$   
(translation, rotation, scale, shear, pure perspectivity)

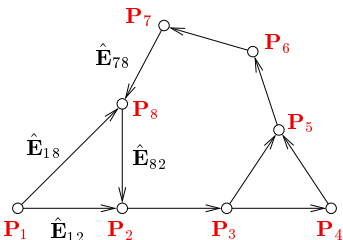


- when cameras are internally calibrated ( $\mathbf{K}_i$  known) then  $\mathbf{H}$  is restricted to a similarity since it must preserve the calibrations  $\mathbf{K}_i$  [H&Z, Secs. 10.2, 10.3], [Longuet-Higgins 1981]  
(translation, rotation, scale)

## ► Reconstructing Camera Systems

**Problem:** Given a set of  $p$  decomposed pairwise essential matrices  $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_{\times} \hat{\mathbf{R}}_{ij}$  and calibration matrices  $\mathbf{K}_i$  reconstruct the camera system  $\mathbf{P}_i, i = 1, \dots, k$

→79 and →144 on representing  $\mathbf{E}$



We construct calibrated camera pairs  $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$  →127

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1} \hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1} \hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons  $i, j$  correspond to graph nodes  $k$  nodes
- pairs  $ij$  correspond to graph edges  $p$  edges

$\hat{\mathbf{P}}_{ij}$  are in different coordinate systems but these are related by similarities  $\hat{\mathbf{P}}_{ij} \mathbf{H}_{ij} = \mathbf{P}_{ij}$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^{\top} & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \stackrel{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\mathbb{R}^{6,4}} \quad (28)$$

- (28) is a linear system of  $24p$  eqs. in  $7p + 6k$  unknowns  $7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij}), 6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each  $\mathbf{P}_i$  appears on the right side as many times as is the degree of node  $\mathbf{P}_i$  eg.  $P_5$  3-times

## ► cont'd

Eq. (28) implies 
$$\begin{bmatrix} \mathbf{R}_{ij} \\ \hat{\mathbf{R}}_{ij} \mathbf{R}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i \\ \mathbf{R}_j \end{bmatrix} \quad \begin{bmatrix} \mathbf{t}_{ij} \\ \hat{\mathbf{R}}_{ij} \mathbf{t}_{ij} + s_{ij} \hat{\mathbf{t}}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{t}_j \end{bmatrix}$$

- $\mathbf{R}_{ij}$  and  $\mathbf{t}_{ij}$  can be eliminated:

$$\hat{\mathbf{R}}_{ij} \mathbf{R}_i = \mathbf{R}_j, \quad \hat{\mathbf{R}}_{ij} \mathbf{t}_i + s_{ij} \hat{\mathbf{t}}_{ij} = \mathbf{t}_j, \quad s_{ij} > 0 \quad (29)$$

- note transformations that do not change these equations assuming no error in  $\hat{\mathbf{R}}_{ij}$

1.  $\mathbf{R}_i \mapsto \mathbf{R}_i \mathbf{R}$ ,    2.  $\mathbf{t}_i \mapsto \sigma \mathbf{t}_i$  and  $s_{ij} \mapsto \sigma s_{ij}$ ,    3.  $\mathbf{t}_i \mapsto \mathbf{t}_i + \mathbf{R}_i \mathbf{t}$

- the global frame is fixed, e.g. by selecting

$$\mathbf{R}_1 = \mathbf{I}, \quad \sum_{i=1}^k \mathbf{t}_i = \mathbf{0}, \quad \frac{1}{p} \sum_{i,j} s_{ij} = 1 \quad (30)$$

- rotation equations are decoupled from translation equations
- in principle,  $s_{ij}$  could correct the sign of  $\hat{\mathbf{t}}_{ij}$  from essential matrix decomposition →79  
but  $\mathbf{R}_i$  cannot correct the  $\alpha$  sign in  $\hat{\mathbf{R}}_{ij}$

⇒ therefore make sure all points are in front of cameras and constrain  $s_{ij} > 0$ ; →81

+ pairwise correspondences are sufficient

- suitable for well-distributed cameras only (dome-like configurations)

otherwise intractable or numerically unstable

## ► Solving Eq. (29) by Stepwise Gluing

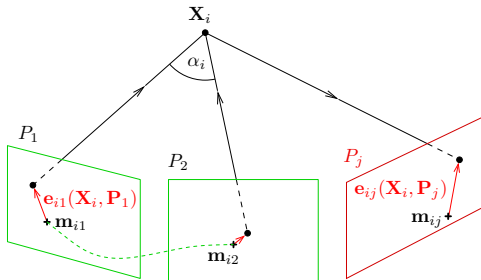
**Given:** Calibration matrices  $\mathbf{K}_j$  and tentative correspondences per camera triples.

### Initialization

1. initialize camera cluster  $\mathcal{C}$  with  $P_1, P_2$ ,
2. find essential matrix  $\mathbf{E}_{12}$  and matches  $M_{12}$  by the 5-point algorithm →86
3. construct camera pair

$$\mathbf{P}_1 = \mathbf{K}_1 [\mathbf{I} \quad \mathbf{0}], \quad \mathbf{P}_2 = \mathbf{K}_2 [\mathbf{R} \quad \mathbf{t}]$$

4. compute 3D reconstruction  $\{X_i\}$  per match from  $M_{12}$  →103
5. initialize point cloud  $\mathcal{X}$  with  $\{X_i\}$  satisfying chirality constraint  $z_i > 0$  and apical angle constraint  $|\alpha_i| > \alpha_T$



### Attaching camera $P_j \notin \mathcal{C}$

1. select points  $\mathcal{X}_j$  from  $\mathcal{X}$  that have matches to  $P_j$
2. estimate  $\mathbf{P}_j$  using  $\mathcal{X}_j$ , RANSAC with the 3-pt alg. (P3P), projection errors  $e_{ij}$  in  $\mathcal{X}_j$  →68
3. reconstruct 3D points from all tentative matches from  $P_j$  to all  $P_l, l \neq k$  that are not in  $\mathcal{X}$
4. filter them by the chirality and apical angle constraints and add them to  $\mathcal{X}$
5. add  $P_j$  to  $\mathcal{C}$
6. perform bundle adjustment on  $\mathcal{X}$  and  $\mathcal{C}$

coming next →135

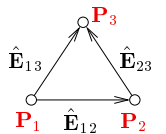
# Finding The Rotation Component in Eq. (29): A Global Algorithm

**Task:** Solve  $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$ ,  $i, j \in V$ ,  $(i, j) \in E$  where  $\mathbf{R}$  are a  $3 \times 3$  rotation matrix each. Per columns  $c = 1, 2, 3$  of  $\mathbf{R}_j$ :

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \quad \text{for all } i, j \quad (31)$$

- fix  $c$  and denote  $\mathbf{r}^c = [\mathbf{r}_1^c, \mathbf{r}_2^c, \dots, \mathbf{r}_k^c]^\top$   $c$ -th columns of all rotation matrices stacked;  $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (31) becomes  $\mathbf{D}\mathbf{r}^c = \mathbf{0}$   $\mathbf{D} \in \mathbb{R}^{3p, 3k}$
- $3p$  equations for  $3k$  unknowns  $\rightarrow p \geq k$  in a 1-connected graph we have to fix  $\mathbf{r}_1^c = [1, 0, 0]$

**Ex:** ( $k = p = 3$ )



$\hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c = \mathbf{0}$   
 $\hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c = \mathbf{0}$   
 $\hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c = \mathbf{0}$

$$\rightarrow \mathbf{D}\mathbf{r}^c = \begin{bmatrix} \hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I} \\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^c \\ \mathbf{r}_2^c \\ \mathbf{r}_3^c \end{bmatrix} = \mathbf{0}$$

- must hold for any  $c$

**Idea:**

[Martinec & Pajdla CVPR 2007]

1. find the space of all  $\mathbf{r}^c \in \mathbb{R}^{3k}$  that solve (31)  $\mathbf{D}$  is sparse, use  $[V, E] = \text{eigs}(\mathbf{D}^* \mathbf{D}, 3, 0)$ ; (Matlab)
  2. choose 3 unit orthogonal vectors in this space 3 smallest eigenvectors
  3. find closest rotation matrices per cam. using SVD because  $\|\mathbf{r}^c\| = 1$  is necessary but insufficient  
 $\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$ , where  $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$
- global world rotation is arbitrary

# Finding The Translation Component in Eq. (29)

From (29) and (30):

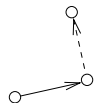
$d \leq 3$  – rank of camera center set,  $p$  – #pairs,  $k$  – #cameras

$$\hat{\mathbf{R}}_{ij} \mathbf{t}_i + s_{ij} \hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}, \quad \sum_{i=1}^k \mathbf{t}_i = \mathbf{0}, \quad \sum_{i,j} s_{ij} = p, \quad s_{ij} > 0, \quad \mathbf{t}_i \in \mathbb{R}^d$$

- in rank  $d$ :  $d \cdot p + d + 1$  equations for  $d \cdot k + p$  unknowns  $\rightarrow p \geq \frac{d(k-1)-1}{d-1} \stackrel{\text{def}}{=} Q(d, k)$

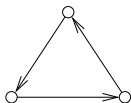
**Ex: Chains and circuits** construction from sticks of known orientation and unknown length?

$p = k - 1$



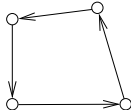
$k \leq 2$  for any  $d$

$k = p = 3$



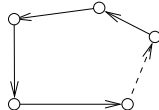
$3 \geq d \geq 2$ : non-collinear ok

$k = p = 4$



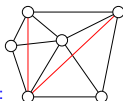
$3 \geq d \geq 3$ : non-planar ok

$k = p > 4$



$3 \geq d \geq k - 1$ : impossible

- equations insufficient for chains, trees, or when  $d = 1$  collinear cameras
- 3-connectivity implies sufficient equations for  $d = 3$  cams. in general pos. in 3D
  - $s$ -connected graph has  $p \geq \lceil \frac{sk}{2} \rceil$  edges for  $s \geq 2$ , hence  $p \geq \lceil \frac{3k}{2} \rceil \geq Q(3, k) = \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any  $k$  when  $d = 2$  coplanar cams
  - since  $p \geq \lceil 2k \rceil \geq Q(2, k) = 2k - 3$
  - maximal planar triangulated graphs have  $p = 3k - 6$  maximal planar triangulated graph example:
  - and give a solution for  $k \geq 3$



Linear equations in (29) and (30) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \quad \mathbf{t} = [\mathbf{t}_1^\top, \mathbf{t}_2^\top, \dots, \mathbf{t}_k^\top, s_{12}, \dots, s_{ij}, \dots]^\top$$

for  $d = 3$ :  $\mathbf{t} \in \mathbb{R}^{3k+p}$ ,  $\mathbf{D} \in \mathbb{R}^{3p, 3k+p}$  is sparse

$$\mathbf{t}^* = \arg \min_{\mathbf{t}, s_{ij} > 0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D} \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

## ► Bundle Adjustment

### Given:

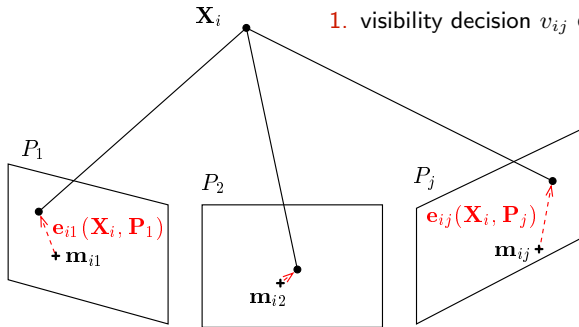
1. set of 3D points  $\{\mathbf{X}_i\}_{i=1}^P$
2. set of cameras  $\{\mathbf{P}_j\}_{j=1}^C$
3. fixed tentative projections  $\mathbf{m}_{ij}$

### Required:

1. corrected 3D points  $\{\mathbf{X}'_i\}_{i=1}^P$
2. corrected cameras  $\{\mathbf{P}'_j\}_{j=1}^C$

### Latent:

1. visibility decision  $v_{ij} \in \{0, 1\}$  per  $\mathbf{m}_{ij}$



- for simplicity,  $\mathbf{X}$ ,  $\mathbf{m}$  are considered Cartesian (not homogeneous)
- we have projection error  $\mathbf{e}_{ij}(\mathbf{X}_i, \mathbf{P}_j) = \mathbf{x}_i - \mathbf{m}_i$  per image feature, where  $\mathbf{x}_i = \mathbf{P}_j \mathbf{X}_i$
- for simplicity, we will work with scalar error  $e_{ij} = \|\mathbf{e}_{ij}\|$



# Robust Objective Function for Bundle Adjustment

The data model is

constructed by marginalization, as in Robust Matching Model →111

$$p(\{e\} | \{\mathbf{P}, \mathbf{X}\}) = \prod_{\text{pts}:i=1}^p \prod_{\text{cams}:j=1}^c \left( (1 - P_0)p_1(e_{ij} | \mathbf{X}_i, \mathbf{P}_j) + P_0p_0(e_{ij} | \mathbf{X}_i, \mathbf{P}_j) \right)$$

marginalized negative log-density is (→112)

$$-\log p(\{e\} | \{\mathbf{P}, \mathbf{X}\}) = \sum_i \sum_j \underbrace{-\log \left( e^{-\frac{e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)}{2\sigma_1^2}} + t \right)}_{\rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)) = \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)} \stackrel{\text{def}}{=} \sum_i \sum_j \nu_{ij}^2(\mathbf{X}_i, \mathbf{P}_j)$$

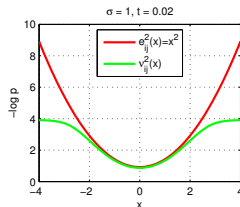
- $e_{ij}$  is the projection error (not Sampson error)
- $\nu_{ij}$  is a 'robust' error fcn.; it is non-robust ( $\nu_{ij} = e_{ij}$ ) when  $t = 0$
- $\rho(\cdot)$  is a 'robustification function' we often find in M-estimation
- the  $\mathbf{L}_{ij}$  in Levenberg-Marquardt changes to vector

$$(\mathbf{L}_{ij})_l = \frac{\partial \nu_{ij}}{\partial \theta_l} = \underbrace{\frac{1}{1 + t e^{e_{ij}^2(\theta)/(2\sigma_1^2)}}}_{\text{small for big } e_{ij}} \cdot \frac{1}{\nu_{ij}(\theta)} \cdot \frac{1}{4\sigma_1^2} \cdot \frac{\partial e_{ij}^2(\theta)}{\partial \theta_l} \quad (32)$$

but the LM method stays the same as before →105–106

- outliers: almost no impact on  $\mathbf{d}_s$  in normal equations because the red term in (32) scales contributions to both sums down for the particular  $ij$

$$-\sum_{i,j} \mathbf{L}_{ij}^\top \nu_{ij}(\theta^s) = \left( \sum_{i,j} \mathbf{L}_{ij}^\top \mathbf{L}_{ij} \right) \mathbf{d}_s$$

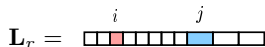


## ► Sparsity in Bundle Adjustment

We have  $q = 3p + 11k$  parameters:  $\theta = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p; \mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_k)$  points, cameras  
 We will use a running index  $r = 1, \dots, z$ ,  $z = p \cdot k$ . Then each  $r$  corresponds to some  $i, j$

$$\theta^* = \arg \min_{\theta} \sum_{r=1}^z \nu_r^2(\theta), \quad \theta^{s+1} := \theta^s + \mathbf{d}_s, \quad - \sum_{r=1}^z \mathbf{L}_r^\top \nu_r(\theta^s) = \left( \sum_{r=1}^z \mathbf{L}_r^\top \mathbf{L}_r + \lambda \text{diag} \mathbf{L}_r^\top \mathbf{L}_r \right) \mathbf{d}_s$$

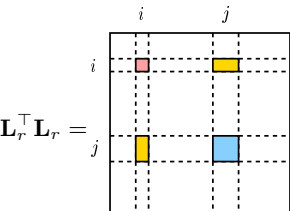
The block form of  $\mathbf{L}_r$  in Levenberg-Marquardt ( $\rightarrow 105$ ) is zero except in columns  $i$  and  $j$ :  
 $r$ -th error term is  $\nu_r^2 = \rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j))$



$r = (i, j)$  blocks:

■:  $\mathbf{X}_i, 1 \times 3$

■:  $\mathbf{P}_j, 1 \times 11$

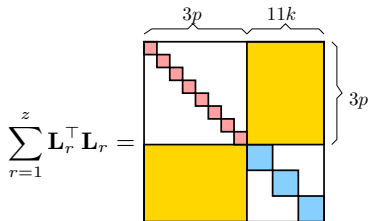


blocks:

■:  $\mathbf{X}_i - \mathbf{X}_i, 3 \times 3$

■:  $\mathbf{X}_i - \mathbf{P}_j, 3 \times 11$

■:  $\mathbf{P}_j - \mathbf{P}_j, 11 \times 11$



- “points first, then cameras” scheme
- standard bundle adjustment eliminates points and solves cameras, then back-substitutes

## ► Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{d}_s \text{ such that } - \sum_{r=1}^z \mathbf{L}_r^\top \nu_r(\theta^s) = \left( \sum_{r=1}^z \mathbf{L}_r^\top \mathbf{L}_r + \lambda \text{diag } \mathbf{L}_r^\top \mathbf{L}_r \right) \mathbf{d}_s$$

This is a linear set of equations  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , where

- $\mathbf{A}$  is very large approx.  $3 \cdot 10^4 \times 3 \cdot 10^4$  for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$  is sparse and symmetric,  $\mathbf{A}^{-1}$  is dense direct matrix inversion is prohibitive

Choleski: Every symmetric positive definite matrix  $\mathbf{A}$  can be decomposed to  $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ , where  $\mathbf{L}$  is lower triangular. If  $\mathbf{A}$  is sparse then  $\mathbf{L}$  is sparse, too.

1. decompose  $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$  transforms the problem to solving  $\underbrace{\mathbf{L}\mathbf{L}^\top}_{\mathbf{c}} \mathbf{x} = \mathbf{b}$

2. solve for  $\mathbf{x}$  in two passes:

$$\mathbf{L}\mathbf{c} = \mathbf{b} \quad \mathbf{c}_i := \mathbf{L}_{ii}^{-1} \left( \mathbf{b}_i - \sum_{j < i} \mathbf{L}_{ij} \mathbf{c}_j \right) \quad \text{forward substitution, } i = 1, \dots, q$$

$$\mathbf{L}^\top \mathbf{x} = \mathbf{c} \quad \mathbf{x}_i := \mathbf{L}_{ii}^{-1} \left( \mathbf{c}_i - \sum_{j > i} \mathbf{L}_{ji} \mathbf{x}_j \right) \quad \text{back-substitution}$$

- Choleski decomposition is fast (does not touch zero blocks)  
non-zero elements are  $9p + 121k + 66pk \approx 3.4 \cdot 10^6$ ; ca.  $250 \times$  fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse  $\mathbf{A}$  and diagonal pivoting for semi-definite  $\mathbf{A}$  [Triggs et al. 1999]
- $\lambda$  controls the definiteness

## Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%   L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%   for sparse square symmetric positive definite matrix A,
%   especially useful for arrowhead sparse matrices.

% (c) 2010 Radim Sara (sara@cmp.felk.cvut.cz)

[p,q] = size(A);
if p ~= q, error 'Matrix must be square'; end

L = sparse(q,q);
F = ones(q,1);
for i=1:q
    F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
    for j = F(i):i-1
        k = max(F(i),F(j));
        a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
        L(i,j) = a/L(j,j);
    end
    a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
    if a < 0, error 'Matrix A must be positive definite'; end
    L(i,i) = sqrt(a);
end
end
```

## ► Gauge Freedom

1. The external frame is not fixed: See Projective Reconstruction Theorem →128

$$\underline{\mathbf{m}}_{ij} \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.
- $\mathbf{P}$  is 12 numbers for 11 parameters
  - we may represent  $\mathbf{P}$  in decomposed form  $\mathbf{K}, \mathbf{R}, \mathbf{t}$
  - but  $\mathbf{R}$  is 9 numbers representing the 3 parameters of rotation

### As a result

- there is no unique solution
- matrix  $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$  is singular

### Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints
- 2a. either imposing constraints on projective entities
- cameras, e.g.  $\mathbf{P}_{3,4} = 1$
  - points, e.g.  $\|\underline{\mathbf{X}}_i\|^2 = 1$
- this excludes affine cameras  
this way we can represent points at infinity
- 2b. or using minimal representations
- points in their Euclidean representation  $\mathbf{X}_i$  but finite points may be an unrealistic model
  - rotation matrix can be represented by axis-angle or the Cayley transform see next

# Implementing Simple Constraints

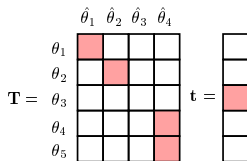
## What for?

1. fixing external frame as in  $\theta_i = \mathbf{t}_i$  'trivial gauge'
2. representing additional knowledge as in  $\theta_i = \theta_j$  e.g. cameras share calibration matrix  $\mathbf{K}$

Introduce reduced parameters  $\hat{\theta}$  and replication matrix  $\mathbf{T}$ :

$$\theta = \mathbf{T} \hat{\theta} + \mathbf{t}, \quad \mathbf{T} \in \mathbb{R}^{p, \hat{p}}, \quad \hat{p} \leq p$$

then  $\mathbf{L}_r$  in LM changes to  $\mathbf{L}_r \mathbf{T}$  and everything else stays the same  $\rightarrow$ 105



these  $\mathbf{T}$ ,  $\mathbf{t}$  represent

$\theta_1 = \hat{\theta}_1$	no change
$\theta_2 = \hat{\theta}_2$	no change
$\theta_3 = t_3$	constancy
$\theta_4 = \theta_5 = \hat{\theta}_4$	equality

- $\mathbf{T}$  deletes columns of  $\mathbf{L}_r$  that correspond to fixed parameters it reduces the problem size
- consistent initialisation:  $\theta^0 = \mathbf{T} \hat{\theta}^0 + \mathbf{t}$  or filter the init by pseudoinverse  $\theta^0 \mapsto \mathbf{T}^\dagger \theta^0$
- no need for computing derivatives for  $\theta_j$  corresponding to all-zero rows of  $\mathbf{T}$  fixed  $\theta$
- constraining projective entities  $\rightarrow$ 143–144
- more complex constraints tend to make normal equations dense
- implementing constraints is safer than explicit renaming of the parameters, gives a flexibility to experiment
- other methods are much more involved, see [Triggs et al. 1999]
- BA resource: <http://www.ics.forth.gr/~lourakis/sba/> [Lourakis 2009]

# Matrix Exponential

- for any square matrix we define

$$\expm \mathbf{A} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \quad \text{note: } \mathbf{A}^0 = \mathbf{I}$$

- some properties

$$\expm \mathbf{0} = \mathbf{I}, \quad \expm(-\mathbf{A}) = (\expm \mathbf{A})^{-1}, \quad \expm(\mathbf{A} + \mathbf{B}) \neq \expm(\mathbf{A}) \expm(\mathbf{B})$$
$$\expm(\mathbf{A}^\top) = (\expm \mathbf{A})^\top \text{ hence if } \mathbf{A} \text{ is skew symmetric then } \expm \mathbf{A} \text{ is orthogonal:}$$

$$(\expm(\mathbf{A}))^\top = \expm(\mathbf{A}^\top) = \expm(-\mathbf{A}) = (\expm(\mathbf{A}))^{-1}$$

$$\expm(a \mathbf{A}) \expm(b \mathbf{A}) = \expm((a + b)\mathbf{A}), \quad \det \expm \mathbf{A} = \expm(\text{tr } \mathbf{A})$$

## Ex:

- homography can be represented via exponential map with 8 numbers e.g. as

$$\mathbf{H} = \expm \mathbf{Z} \quad \text{such that} \quad \text{tr } \mathbf{Z} = 0, \quad \text{eg. } \mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & -(z_{11} + z_{22}) \end{bmatrix}$$

## ► Minimal Representations for Rotation

- $\mathbf{o}$  – rotation axis,  $\|\mathbf{o}\| = 1$ ,  $\varphi$  – rotation angle
- **wanted**: simple mapping to/from rotation matrices

1. Matrix exponential. Let  $\boldsymbol{\omega} = \varphi \mathbf{o}$ ,  $0 < \varphi < \pi$ , then

$$\mathbf{R} = \expm[\boldsymbol{\omega}]_{\times} = \sum_{n=0}^{\infty} \frac{[\boldsymbol{\omega}]_{\times}^n}{n!} = \cdots \stackrel{\textcircled{*}}{=} \mathbf{1} = \mathbf{I} + \frac{\sin \varphi}{\varphi} [\boldsymbol{\omega}]_{\times} + \frac{1 - \cos \varphi}{\varphi^2} [\boldsymbol{\omega}]_{\times}^2$$

- for  $\varphi = 0$  we take the limit and  $\mathbf{R} = \mathbf{I}$
- this is the Rodrigues' formula for rotation
- inverse (the principal logarithm of  $\mathbf{R}$ ) from

$$0 \leq \varphi < \pi, \quad \cos \varphi = \frac{1}{2}(\text{tr } \mathbf{R} - 1), \quad [\boldsymbol{\omega}]_{\times} = \frac{\varphi}{2 \sin \varphi} (\mathbf{R} - \mathbf{R}^{\top}),$$

2. Cayley's representation; let  $\mathbf{a} = \mathbf{o} \tan \frac{\varphi}{2}$ , then

$$\mathbf{R} = (\mathbf{I} + [\mathbf{a}]_{\times})(\mathbf{I} - [\mathbf{a}]_{\times})^{-1}, \quad [\mathbf{a}]_{\times} = (\mathbf{R} + \mathbf{I})^{-1}(\mathbf{R} - \mathbf{I})$$

$$\mathbf{a}_1 \circ \mathbf{a}_2 = \frac{\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_1 \times \mathbf{a}_2}{1 - \mathbf{a}_1^{\top} \mathbf{a}_2} \quad \text{composition of rotations } \mathbf{R} = \mathbf{R}_1 \mathbf{R}_2$$

- again, cannot represent rotations for  $\phi \geq \pi$
- no trigonometric functions
- explicit composition formula



## ► Minimal Representations for Other Entities

with the help of rotation we can minimally represent

1. fundamental matrix

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top, \quad \mathbf{D} = \text{diag}(1, d^2, 0), \quad \mathbf{U}, \mathbf{V} \text{ are rotations,} \quad 3 + 1 + 3 = 7 \text{ DOF}$$

2. essential matrix

$$\mathbf{E} = [-\mathbf{t}]_{\times} \mathbf{R}, \quad \mathbf{R} \text{ is rotation,} \quad \|\mathbf{t}\| = 1, \quad 3 + 2 = 5 \text{ DOF}$$

3. camera

$$\mathbf{P} = \mathbf{K} [\mathbf{R} \quad \mathbf{t}], \quad 5 + 3 + 3 = 11 \text{ DOF}$$

Interestingly, let

[Eade 2017]

$$\mathbf{B} = \begin{bmatrix} [\boldsymbol{\omega}]_{\times} & \mathbf{u} \\ \mathbf{0}^\top & 0 \end{bmatrix}, \quad \mathbf{B} \in \mathbb{R}^{4,4}$$

then, assuming  $\|\boldsymbol{\omega}\| = \phi > 0$

for  $\phi = 0$  we take the limits

$$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \text{expm } \mathbf{B} = \mathbf{I}_4 + \mathbf{B} + h_2(\phi) \mathbf{B}^2 + h_3(\phi) \mathbf{B}^3 = \begin{bmatrix} \text{expm} [\boldsymbol{\omega}]_{\times} & \mathbf{V} \mathbf{u} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{V} = \mathbf{I}_3 + h_2(\phi) [\boldsymbol{\omega}]_{\times} + h_3(\phi) [\boldsymbol{\omega}]_{\times}^2, \quad \mathbf{V}^{-1} = \mathbf{I}_3 - \frac{1}{2} [\boldsymbol{\omega}]_{\times} + h_4(\phi) [\boldsymbol{\omega}]_{\times}^2$$

$$h_1(\phi) = \frac{\sin \phi}{\phi}, \quad h_2(\phi) = \frac{1 - \cos \phi}{\phi^2}, \quad h_3(\phi) = \frac{\phi - \sin \phi}{\phi^3}, \quad h_4(\phi) = \frac{1}{\phi^2} \left( 1 - \frac{1}{2} \phi \cot \frac{\phi}{2} \right)$$

## Stereovision

- 7.1 Introduction
- 7.2 Epipolar Rectification
- 7.3 Binocular Disparity and Matching Table
- 7.4 Image Similarity
- 7.5 Marroquin's Winner Take All Algorithm
- 7.6 Maximum Likelihood Matching
- 7.7 Uniqueness and Ordering as Occlusion Models

### mostly covered by

Šára, R. How To Teach Stereoscopic Vision. Proc. ELMAR 2010 [referenced as \[SP\]](#)

### additional references



C. Geyer and K. Daniilidis. Conformal rectification of omnidirectional stereo pairs. In *Proc Computer Vision and Pattern Recognition Workshop*, p. 73, 2003.



J. Gluckman and S. K. Nayar. Rectifying transformations that minimize resampling effects. In *Proc IEEE CS Conf on Computer Vision and Pattern Recognition*, vol. 1:111–117. 2001.



M. Pollefeys, R. Koch, and L. V. Gool. A simple and efficient rectification method for general motion. In *Proc Int Conf on Computer Vision*, vol. 1:496–501, 1999.

# What Are The Relative Distances?



- monocular vision already gives a rough 3D sketch because we understand the scene

# What Are The Relative Distances?



Centrum för teknikstudier at Malmö Högskola, Sweden



The Vyšehrad Fortress, Prague

- left: we have no help from image interpretation
- right: ambiguous interpretation due to a combination of missing texture and occlusion

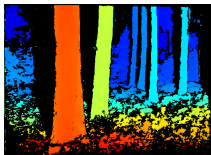
## ► How Difficult Is Stereo?



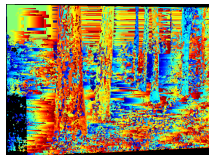
- when we do not recognize the scene and cannot use high-level constraints the problem seems difficult (right, less so in the center)
- most stereo matching algorithms do not require scene understanding prior to matching
- the success of a model-free stereo matching algorithm is unlikely:



left image



a good disparity map



disparity map from WTA

### WTA Matching:

for every left-image pixel  
find the most similar  
right-image pixel  
along the  
corresponding epipolar  
line [Marroquin 83]

# A Summary of Our Observations and an Outlook

1. simple matching algorithms do not work
2. stereopsis requires image interpretation in sufficiently complex scenes  
or another-modality measurement

we have a tradeoff: model strength  $\leftrightarrow$  universality

## Outlook:

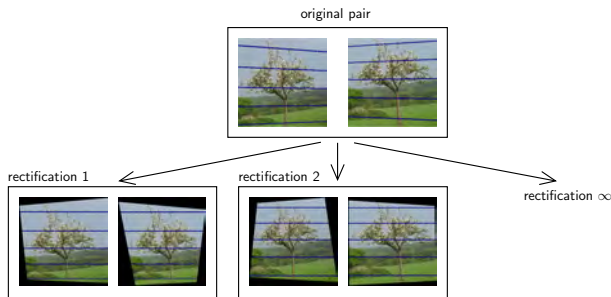
1. represent the occlusion constraint: correspondences are not independent due to occlusions
  - epipolar rectification
  - disparity
  - uniqueness as an occlusion constraint
2. represent piecewise continuity the weakest of interpretations; piecewise: object boundaries
  - ordering as a weak continuity model
3. use a consistent framework
  - looking for the most probable solution (MAP)

## ► Linear Epipolar Rectification for Easier Correspondence Search

**Problem:** Given fundamental matrix  $\mathbf{F}$  or camera matrices  $\mathbf{P}_1, \mathbf{P}_2$ , transform images by a pair of homographies so that epipolar lines become horizontal with the same row coordinate. The result is a standard stereo pair.

### Procedure:

1. find a pair of rectification homographies  $\mathbf{H}_1$  and  $\mathbf{H}_2$ .
2. warp images using  $\mathbf{H}_1$  and  $\mathbf{H}_2$  and modify the fundamental matrix  $\mathbf{F} \mapsto \mathbf{H}_2^{-T} \mathbf{F} \mathbf{H}_1^{-1}$  or the cameras  $\mathbf{P}_1 \mapsto \mathbf{H}_1 \mathbf{P}_1, \mathbf{P}_2 \mapsto \mathbf{H}_2 \mathbf{P}_2$ .

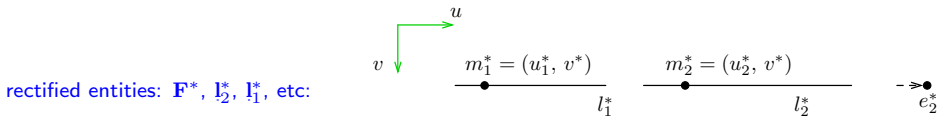


- binocular rectification: there is a 9-parameter family of rectification homographies, see next
- trinocular rectification: has 9 or 6 free parameters (depending on additional constrains)
- in general, linear rectification is not possible for more than three cameras

## ► Rectification Homographies

**Assumption:** Cameras  $(\mathbf{P}_1, \mathbf{P}_2)$  are rectified by a homography pair  $(\mathbf{H}_1, \mathbf{H}_2)$ :

$$\mathbf{P}_i^* = \mathbf{H}_i \mathbf{P}_i = \mathbf{H}_i \mathbf{K}_i \mathbf{R}_i [\mathbf{I} \quad -\mathbf{C}_i], \quad i = 1, 2$$



- the rectified location difference  $d = u_1^* - u_2^*$  is called disparity

**corresponding epipolar lines must be:**

1. parallel to image rows  $\Rightarrow$  epipoles become  $e_1^* = e_2^* = (1, 0, 0)$
2. equivalent  $l_2^* = l_1^* \Rightarrow$  (a)  $\mathbf{l}_2^* \simeq \mathbf{l}_1^* \simeq \mathbf{e}_1^* \times \mathbf{m}_1 = [\mathbf{e}_1^*]_{\times} \mathbf{m}_1$ , (b)  $\mathbf{l}_2^* \simeq \mathbf{F}^* \mathbf{m}_1$

- therefore the canonical fundamental matrix is

$$\mathbf{F}^* \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

**A two-step rectification procedure**

1. find some pair of primitive rectification homographies  $\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2$
2. upgrade to a pair of optimal rectification homographies while preserving  $\mathbf{F}^*$



## ► Geometric Interpretation of Linear Rectification

What pair of physical cameras is compatible with  $\mathbf{F}^*$ ?

- we know that  $\mathbf{F} = (\mathbf{Q}_1 \mathbf{Q}_2^{-1})^\top [\mathbf{e}_1]_\times$
- we choose  $\mathbf{Q}_1^* = \mathbf{K}_1^*$ ,  $\mathbf{Q}_2^* = \mathbf{K}_2^* \mathbf{R}^*$ ; then

→77

$$(\mathbf{Q}_1^* \mathbf{Q}_2^{*-1})^\top [\mathbf{e}_1^*]_\times = (\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*-1})^\top \mathbf{F}^*$$

- we look for  $\mathbf{R}^*$ ,  $\mathbf{K}_1^*$ ,  $\mathbf{K}_2^*$  compatible with

$$(\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*-1})^\top \mathbf{F}^* = \lambda \mathbf{F}^*, \quad \mathbf{R}^* \mathbf{R}^{*\top} = \mathbf{I}, \quad \mathbf{K}_1^*, \mathbf{K}_2^* \text{ upper triangular}$$

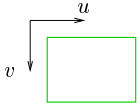
- we also want  $\mathbf{b}^*$  from  $\mathbf{e}_1^* \simeq \mathbf{P}_1^* \mathbf{C}_2^* = \mathbf{K}_1^* \mathbf{b}^*$   $\mathbf{b}^*$  in cam. 1 frame
- result:

$$\mathbf{R}^* = \mathbf{I}, \quad \mathbf{b}^* = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_1^* = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2^* = \begin{bmatrix} k_{21} & k_{22} & k_{23} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \quad (33)$$



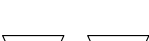

- rectified cameras are in canonical position with respect to each other  
not rotated, canonical baseline
- rectified calibration matrices can differ in the first row only
- when  $\mathbf{K}_1^* = \mathbf{K}_2^*$  then the rectified pair is called the standard stereo pair and the homographies standard rectification homographies
- this does not mean that the images are not distorted after rectification

## ► The Degrees of Freedom in Epipolar Rectification

**Proposition 1** Homographies  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are rectification-preserving if the images stay rectified, i.e. if  $\mathbf{A}_2^{-\top} \mathbf{F}^* \mathbf{A}_1^{-1} \simeq \mathbf{F}^*$ , which gives

$$\mathbf{A}_1 = \begin{bmatrix} l_1 & l_2 & l_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix},$$


where  $s_v \neq 0$ ,  $t_v$ ,  $l_1 \neq 0$ ,  $l_2$ ,  $l_3$ ,  $r_1 \neq 0$ ,  $r_2$ ,  $r_3$ ,  $q$  are 9 free parameters.

general	transformation		standard	type
$l_1, r_1$	horizontal scales		$l_1 = r_1$	geometric
$l_2, r_2$	horizontal shears		$l_2 = r_2$	algebraic
$l_3, r_3$	horizontal shifts		$l_3 = r_3$	algebraic
$q$	common special projective			geometric
$s_v$	common vertical scale			geometric
$t_v$	common vertical shift			algebraic
9 DoF			$9 - 3 = 6$ DoF	

- $q$  is rotation about the baseline
- $s_v$  changes the focal length

proof: find a rotation  $\mathbf{G}$  that brings  $\mathbf{K}$  to upper triangular form via RQ decomposition:  $\mathbf{A}_1 \mathbf{K}_1^* = \hat{\mathbf{K}}_1 \mathbf{G}$  and  $\mathbf{A}_2 \mathbf{K}_2^* = \hat{\mathbf{K}}_2 \mathbf{G}$

**Corollary for Proposition 1** Let  $\bar{\mathbf{H}}_1$  and  $\bar{\mathbf{H}}_2$  be (primitive or other) rectification homographies. Then  $\mathbf{H}_1 = \mathbf{A}_1 \bar{\mathbf{H}}_1$ ,  $\mathbf{H}_2 = \mathbf{A}_2 \bar{\mathbf{H}}_2$  are also rectification homographies.

**Proposition 2** Pairs of rectification-preserving homographies  $(\mathbf{A}_1, \mathbf{A}_2)$  form a group with group operation  $(\mathbf{A}'_1, \mathbf{A}'_2) \circ (\mathbf{A}_1, \mathbf{A}_2) = (\mathbf{A}'_1 \mathbf{A}_1, \mathbf{A}'_2 \mathbf{A}_2)$ .

**Proof:**

- closure by Proposition 1
- associativity by matrix multiplication
- identity belongs to the set
- inverse element belongs to the set by  $\mathbf{A}_2^\top \mathbf{F}^* \mathbf{A}_1 \simeq \mathbf{F}^* \Leftrightarrow \mathbf{F}^* \simeq \mathbf{A}_2^{-\top} \mathbf{F}^* \mathbf{A}_1^{-1}$

## ► Primitive Rectification

**Goal:** Given fundamental matrix  $\mathbf{F}$ , derive some simple rectification homographies  $\mathbf{H}_1, \mathbf{H}_2$

1. Let the SVD of  $\mathbf{F}$  be  $\mathbf{UDV}^\top = \mathbf{F}$ , where  $\mathbf{D} = \text{diag}(1, d^2, 0)$ ,  $1 \geq d^2 > 0$
2. Write  $\mathbf{D}$  as  $\mathbf{D} = \mathbf{A}^\top \mathbf{F}^* \mathbf{B}$  for some regular  $\mathbf{A}, \mathbf{B}$ . For instance ( $\mathbf{F}^*$  is given  $\rightarrow 151$ )

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -d & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$

3. Then

$$\mathbf{F} = \mathbf{UDV}^\top = \underbrace{\mathbf{UA}^\top}_{\hat{\mathbf{H}}_2^\top} \mathbf{F}^* \underbrace{\mathbf{BV}^\top}_{\hat{\mathbf{H}}_1}$$

and the primitive rectification homographies are

$$\hat{\mathbf{H}}_2 = \mathbf{AU}^\top, \quad \hat{\mathbf{H}}_1 = \mathbf{BV}^\top$$

⊛ P1; 1pt: derive some other admissible  $\mathbf{A}, \mathbf{B}$

- rectification homographies do exist  $\rightarrow 151$
- there are other primitive rectification homographies, these suggested are just simple to obtain

## ► Primitive Rectification Suffices for Calibrated Cameras

**Obs:** calibrated cameras:  $d = 1 \Rightarrow \hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2$  are orthogonal

1. determine primitive rectification homographies  $(\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2)$  from the essential matrix
2. choose a suitable common calibration matrix  $\mathbf{K}$ , e.g.

$$\mathbf{K} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \frac{1}{2}(f^1 + f^2), \quad u_0 = \frac{1}{2}(u_0^1 + u_0^2), \quad \text{etc.}$$

3. the final rectification homographies applied as  $\mathbf{P}_i \mapsto \mathbf{H}_i \mathbf{P}_i$  are

$$\mathbf{H}_1 = \mathbf{K} \hat{\mathbf{H}}_1 \mathbf{K}_1^{-1}, \quad \mathbf{H}_2 = \mathbf{K} \hat{\mathbf{H}}_2 \mathbf{K}_2^{-1}$$

- we got a standard stereo pair ( $\rightarrow 152$ ) and non-negative disparity

$$\text{let } \mathbf{K}_i^{-1} \mathbf{P}_i = \mathbf{R}_i [\mathbf{I} \quad -\mathbf{C}_i], \quad i = 1, 2 \quad \text{note we started from } \mathbf{E}, \text{ not } \mathbf{F}$$

$$\mathbf{H}_1 \mathbf{P}_1 = \mathbf{K} \hat{\mathbf{H}}_1 \mathbf{K}_1^{-1} \mathbf{P}_1 = \mathbf{K} \underbrace{\mathbf{B} \mathbf{V}^\top \mathbf{R}_1}_{\mathbf{R}^*} [\mathbf{I} \quad -\mathbf{C}_1] = \mathbf{K} \mathbf{R}^* [\mathbf{I} \quad -\mathbf{C}_1]$$

$$\mathbf{H}_2 \mathbf{P}_2 = \mathbf{K} \hat{\mathbf{H}}_2 \mathbf{K}_2^{-1} \mathbf{P}_2 = \mathbf{K} \underbrace{\mathbf{A} \mathbf{U}^\top \mathbf{R}_2}_{\mathbf{R}^*} [\mathbf{I} \quad -\mathbf{C}_2] = \mathbf{K} \mathbf{R}^* [\mathbf{I} \quad -\mathbf{C}_2]$$

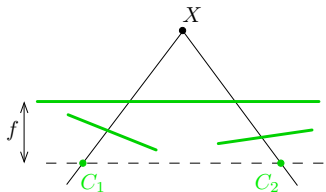
- one can prove that  $\mathbf{B} \mathbf{V}^\top \mathbf{R}_1 = \mathbf{A} \mathbf{U}^\top \mathbf{R}_2$  with the help of essential matrix decomposition (13)
- points at infinity project to  $\mathbf{K} \mathbf{R}^*$  in both images  $\Rightarrow$  they have zero disparity

$\rightarrow 159$

## ► Summary

- rectification is a pair of homographies (one per image) →150  
⇒ rectified camera centers are equal to the original ones
- rectified cameras are in canonical orientation →152  
⇒ rectified image projection planes are coplanar
- equal rectified calibration matrices give standard rectification →152  
⇒ rectified image projection planes are equal
- primitive rectification is standard in calibrated cameras →156

standard rectification homographies reproject onto a common image plane parallel to the baseline



### Corollary

- standard rectified pair: disparity vanishes when corresponding 3D points are at infinity
  - known  $\mathbf{F}$  used alone gives no constraints on standard rectification homographies
  - for that we need either of these:
    1. projection matrices, or
    2. calibrated cameras, or
    3. a few points at infinity calibrating  $k_{1i}$ ,  $k_{2i}$ ,  $i = 1, 2, 3$  in (33)

## Optimal choice for the free parameters

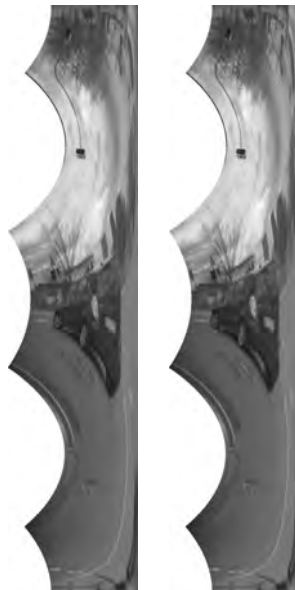
- by minimization of residual image distortion, eg. [Gluckman & Nayar 2001]

$$\mathbf{A}_1^* = \arg \min_{\mathbf{A}_1} \iint_{\Omega} (\det J(\mathbf{A}_1 \hat{\mathbf{H}}_1 \mathbf{x}) - 1)^2 d\mathbf{x}$$

- by minimization of image information loss [Matoušek, ICIG 2004]
- non-linear rectification suitable for forward motion  
non-parametric: [Pollefeys et al. 1999]  
analytic: [Geyer & Daniilidis 2003]



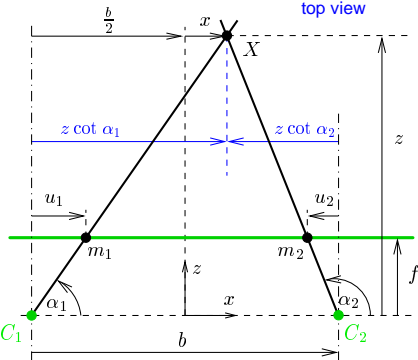
forward egomotion



rectified images, Pollefeys' method

## ► Binocular Disparity in Standard Stereo Pair

top view



- Assumptions: single image line, standard camera pair

$$b = z \cot \alpha_1 - z \cot \alpha_2$$

$$u_1 = f \cot \alpha_1$$

$$u_2 = f \cot \alpha_2$$

$$b = \frac{b}{2} + x - z \cot \alpha_2$$

$X = (x, z)$  from **disparity**  $d = u_1 - u_2$ :

$$z = \frac{b f}{d}, \quad x = \frac{b}{d} \frac{u_1 + u_2}{2}, \quad y = \frac{b v}{d}$$

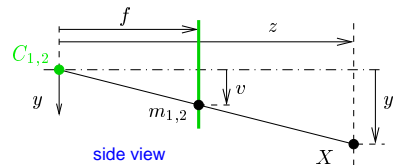
$f, d, u, v$  in pixels,  $b, x, y, z$  in meters

### Observations

- constant disparity surface is a frontoparallel plane
- distant points have small disparity
- relative error in  $z$  is large for small disparity

$$\frac{1}{z} \frac{dz}{dd} = -\frac{1}{d}$$

- increasing the baseline or the focal length increases disparity and reduces the error



side view



# Structural Ambiguity in Stereovision

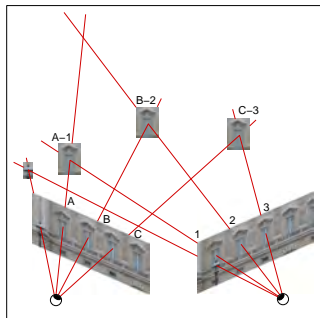
- we can recognize matches but have no scene model
  - lack of an occlusion model
  - lack of a continuity model
- $\Rightarrow$  structural ambiguity in the presence of repetitions (or lack of texture)



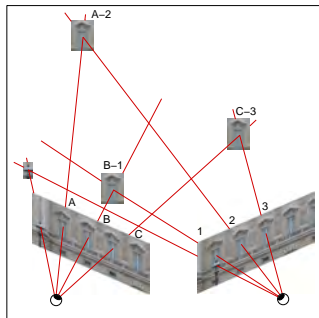
left image



right image

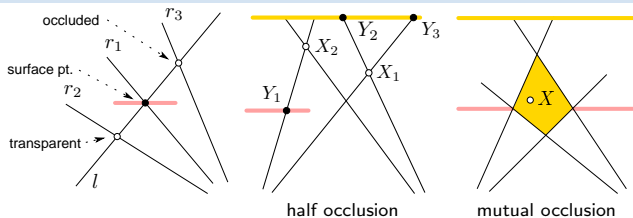


interpretation 1



interpretation 2

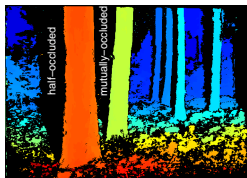
## ► Understanding Basic Occlusion Types



- surface point at the intersection of rays  $l$  and  $r_1$  occludes a world point at the intersection  $(l, r_3)$  and implies the world point  $(l, r_2)$  is transparent, therefore

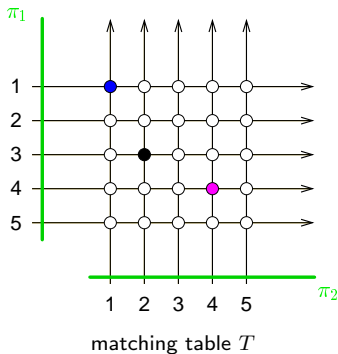
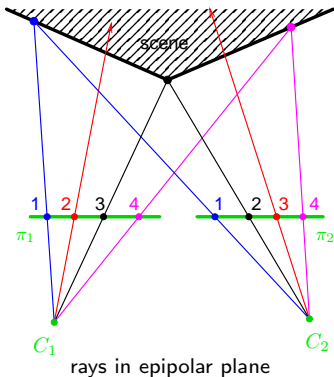
$(l, r_3)$  and  $(l, r_2)$  are excluded by  $(l, r_1)$

- in half-occlusion, every world point such as  $X_1$  or  $X_2$  is excluded by a binocularly visible surface point such as  $Y_1, Y_2, Y_3$   
 $\Rightarrow$  decisions on correspondences are not independent
- in mutual occlusion this is no longer the case: any  $X$  in the yellow zone is not excluded  
 $\Rightarrow$  decisions in the zone are independent on the rest



## ► Matching Table

Based on the observation on mutual exclusion we expect each pixel to match at most once.



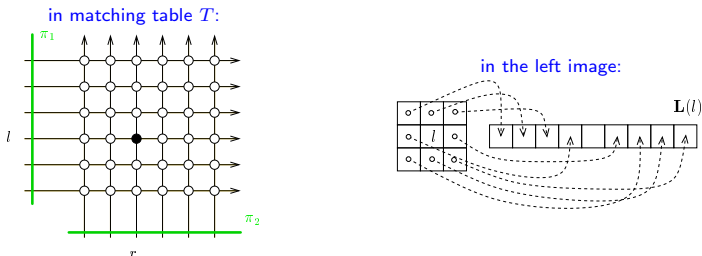
### matching table

- rows and columns represent optical rays
- nodes: possible correspondence pairs
- full nodes: matches
- numerical values associated with nodes: descriptor similarities

[see next](#)

## ► Constructing A Suitable Image Similarity Statistic

- let  $p_i = (l, r)$  and  $\mathbf{L}(l)$ ,  $\mathbf{R}(r)$  be (left, right) image descriptors (vectors) constructed from local image neighborhood windows



- a simple similarity is  $\text{SAD}(l, r) = \|\mathbf{L}(l) - \mathbf{R}(r)\|_1$   $L_1$  metric (sum of absolute differences)
- a scaled-descriptor similarity is  $\text{sim}(l, r) = \frac{\|\mathbf{L}(l) - \mathbf{R}(r)\|^2}{\sigma_I^2(l, r)}$
- $\sigma_I^2$  – the difference scale; a suitable (plug-in) estimate is  $\frac{1}{2} [\text{var}(\mathbf{L}(l)) + \text{var}(\mathbf{R}(r))]$ , giving

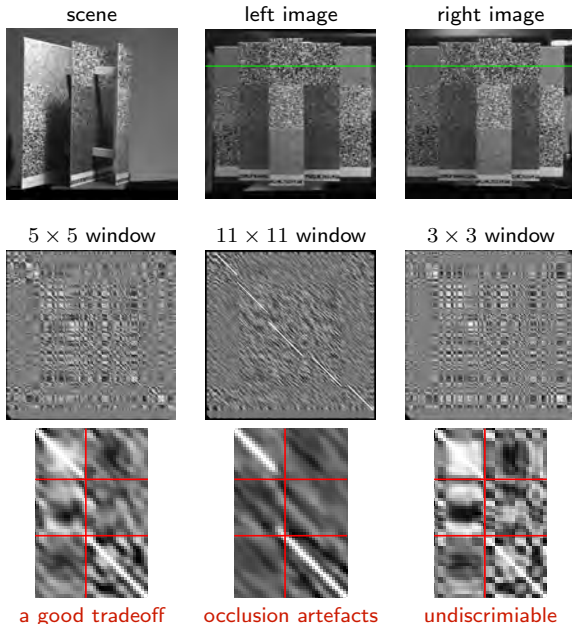
$$\text{sim}(l, r) = 1 - \frac{2 \text{cov}(\mathbf{L}(l), \mathbf{R}(r))}{\underbrace{\text{var}(\mathbf{L}(l)) + \text{var}(\mathbf{R}(r))}_{\rho(\mathbf{L}(l), \mathbf{R}(r))}} \quad \text{var}(\cdot), \text{cov}(\cdot) \text{ is sample (co-)variance} \quad (34)$$

- $\rho$  – MNCC – Moravec's Normalized Cross-Correlation statistic

[Moravec 1977]

$$\rho^2 \in [0, 1], \quad \text{sign } \rho \sim \text{'phase'}$$

# How A Scene Looks in The Filled-In Matching Table



- MNCC  $\rho$  used ( $\alpha = 1.5, \beta = 1$ )
- high-correlation structures correspond to scene objects

## constant disparity

- a diagonal in matching table
- zero disparity is the main diagonal

## depth discontinuity

- horizontal or vertical jump in matching table

## large image window

- better correlation
- worse occlusion localization

## repeated texture

- horizontal and vertical block repetition

# Image Point Descriptors And Their Similarity

**Descriptors:** Image points are tagged by their (viewpoint-invariant) physical properties:

- texture window
- a descriptor like DAISY
- learned descriptors
- reflectance profile under a moving illuminant
- photometric ratios
- dual photometric stereo
- polarization signature
- ...

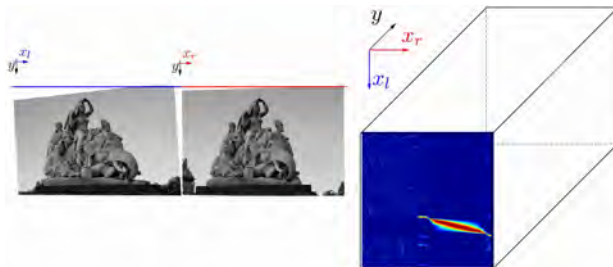
[Moravec 77]

[Tola et al. 2010]

[Wolff & Angelopoulou 93-94]

[Ikeuchi 87]

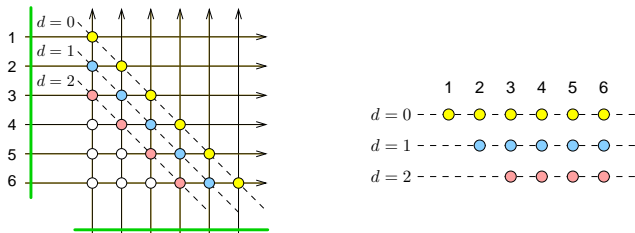
- similar points are more likely to match
- image similarity values for all 'match candidates' give the 3D matching table



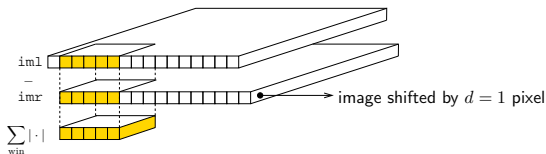
video

## ► Marroquin's Winner Take All (WTA) Matching Algorithm

1. per left-image pixel: find the most similar right-image pixel using SAD →163
2. select disparity range this is a critical weak point
3. represent the matching table diagonals in a compact form



4. use an 'image sliding & cost aggregation algorithm'



5. threshold results by maximal allowed dissimilarity

# A Matlab Code for WTA

```
function dmap = marroquin(impl,imr,disparityRange)
%       impl, imr - rectified gray-scale images
% disparityRange - non-negative disparity range

% (c) Radim Sara (sara@cmp.felk.cvut.cz) FEE CTU Prague, 10 Dec 12

thr = 20;           % bad match rejection threshold
r = 2;
winsize = 2*r+[1 1]; % 5x5 window (neighborhood) for r=2

% the size of each local patch; it is  $N=(2r+1)^2$  except for boundary pixels
N = boxing(ones(size(impl)), winsize);

% computing dissimilarity per pixel (unscaled SAD)
for d = 0:disparityRange
    slice = abs(imr(:,1:end-d) - impl(:,d+1:end)); % cycle over all disparities
    V(:,d+1:end,d+1) = boxing(slice, winsize)./N; % pixelwise dissimilarity
    % window aggregation
end

% collect winners, threshold, and output disparity map
[cmap,dmap] = min(V,[],3);
dmap(cmap > thr) = NaN; % mask-out high dissimilarity pixels
end % of marroquin

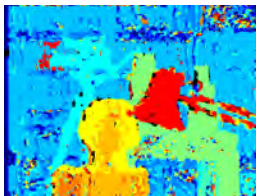
function c = boxing(im, wsz)
% if the mex is not found, run this slow version:
c = conv2(ones(1,wsz(1)), ones(wsz(2),1), im, 'same');
end % of boxing
```



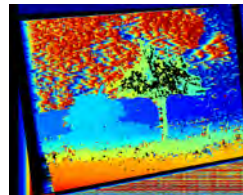
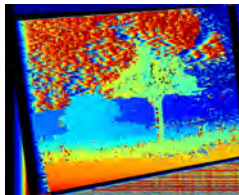
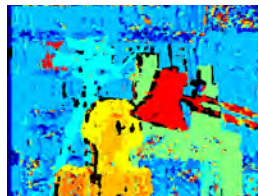
# WTA: Some Results



thr = 20



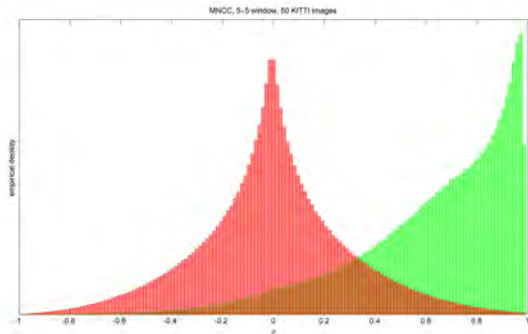
thr = 10



- results are fairly bad
- false matches in textureless image regions and on repetitive structures (book shelf)
- a more restrictive threshold (thr = 10) does not work as expected
- we searched the true disparity range, results get worse if the range is set wider
- chief failure reasons:
  - unnormalized image dissimilarity does not work well
  - no occlusion model

## ► A Principled Approach to Similarity

Empirical Distribution of MNCC  $\rho$  for Matches and Non-Matches



- histograms of  $\rho$  computed over  $5 \times 5$  correlation window
- KITTI dataset
  - $4.2 \cdot 10^6$  ground-truth (LiDAR) matches for  $p_1(\rho)$  (green),
  - $4.2 \cdot 10^6$  random non-matches for  $p_0(\rho)$  (red)

Obs:

- non-matches (red) may have arbitrarily large  $\rho$
- matches (green) may have arbitrarily low  $\rho$
- $\rho = 1$  is improbable for matches

# Match Likelihood

- $\rho$  is just a statistic
- we need a probability distribution on  $[0, 1]$ , e.g. Beta distribution

$$p_1(\rho(l, r)) = \frac{1}{B(\alpha, \beta)} \rho^{2(\alpha-1)} (1 - \rho^2)^{\beta-1}$$

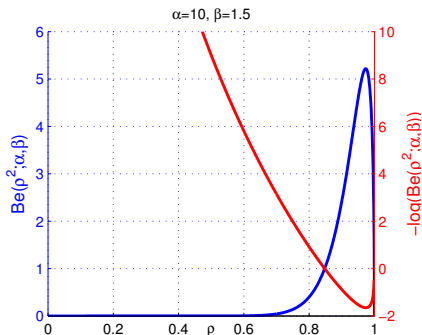
- note that uniform distribution is obtained for  $\alpha = \beta = 1$
- when  $\alpha = 3/2$  and  $\beta = 1$  then  $p_1(\cdot) = \frac{2}{3}|\rho|$

- the mode is at  $\sqrt{\frac{\alpha-1}{\alpha+\beta-2}} \approx 0.9733$  for  $\alpha = 10, \beta = 1.5$
- if we chose  $\beta = 1$  then the mode was at  $\rho = 1$
- perfect similarity is 'suspicious' (depends on expected camera noise level)
- from now on we will work with negative log-likelihood

$$V_1(\rho(l, r)) = -\log p_1(\rho(l, r)) \quad (35)$$

smaller is better

- we may also define similarity (and negative log-likelihood  $V_0(\rho(l, r))$ ) for non-matches



## ► A Principled Approach to Matching

- given matching  $M$  what is the likelihood of observed data  $D$ ?
- data – all pairwise costs in matching table  $T$
- matches – pairs  $p_i = (l_i, r_i)$ ,  $i = 1, \dots, n$
- matching: partitioning matching table  $T$  to matched  $M$  and excluded  $E$  pairs

$$T = M \cup E, \quad M \cap E = \emptyset$$

- matching cost (negative log-likelihood, smaller is better)

$$V(D | M) = \sum_{p \in M} V_1(D | p) + \sum_{p \in E} V_0(D | p)$$

$V_1(D | p)$  – negative log-probability of data  $D$  at matched pixel  $p$  (35)

$V_0(D | p)$  – ditto at unmatched pixel  $p$

→169 and →170

- matching problem

$$M^* = \arg \min_{M \in \mathcal{M}(T)} V(D | M)$$

$\mathcal{M}(T)$  – the set of all matchings in table  $T$

- symmetric: formulated over pairs, invariant to left  $\leftrightarrow$  right image swap

## ►(cont'd) Log-Likelihood Ratio

- we need to reduce matching to a standard polynomial-complexity problem
- we convert the matching cost to an 'easier' sum

$$\begin{aligned} V(D | M) &= \sum_{p \in M} V_1(D | p) + \sum_{p \in E} V_0(D | p) + \overbrace{\sum_{p \in M} V_0(D | p) - \sum_{p \in M} V_0(D | p)}^0 \\ &= \underbrace{\sum_{p \in M} (V_1(D | p) - V_0(D | p))}_{-L(D | p)} + \underbrace{\sum_{p \in E} V_0(D | p) + \sum_{p \in M} V_0(D | p)}_{\sum_{p \in T} V_0(D | p) = \text{const}} \end{aligned}$$

- hence

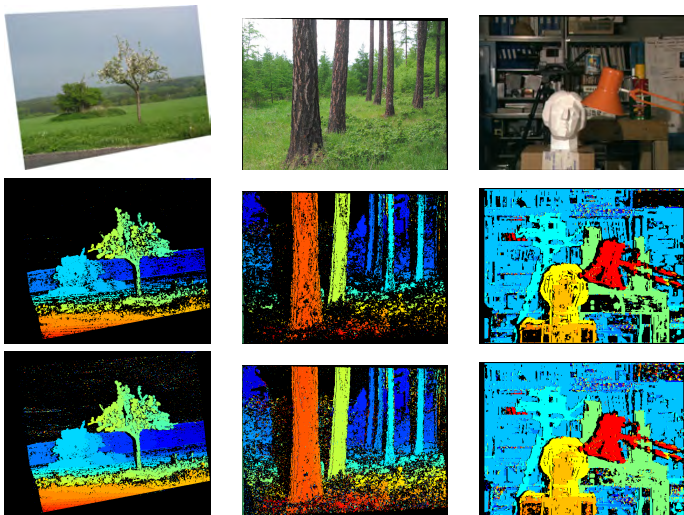
$$\arg \min_{M \in \mathcal{M}(T)} V(D | M) = \arg \max_{M \in \mathcal{M}(T)} \sum_{p \in M} L(D | p) \quad (36)$$

$L(D | p)$  – logarithm of matched-to-unmatched likelihood ratio (bigger is better)

why this way: we want to use maximum-likelihood but our measurement is all data  $D$

- (36) is max-cost matching (maximum assignment) for the maximum-likelihood (ML) matching problem
  - it must contain no pairs  $p$  with  $L(D | p) < 0$
  - use Hungarian (Munkres) algorithm and threshold the result based on  $L(D | p)$
  - or step back: sacrifice symmetry to speed and use dynamic programming

# Some Results for the Maximum-Likelihood (ML) Matching



- unlike the WTA we can efficiently control the density/accuracy tradeoff black = no match
- middle row:  $L(D | p)$  threshold set to achieve error rate of 3% (and 61% density results)
- bottom row:  $L(D | p)$  threshold set to achieve density of 76% (and 4.3% error rate results)

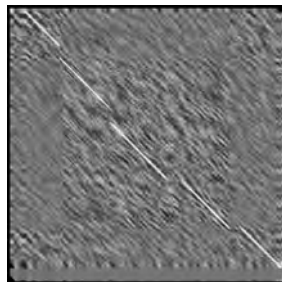
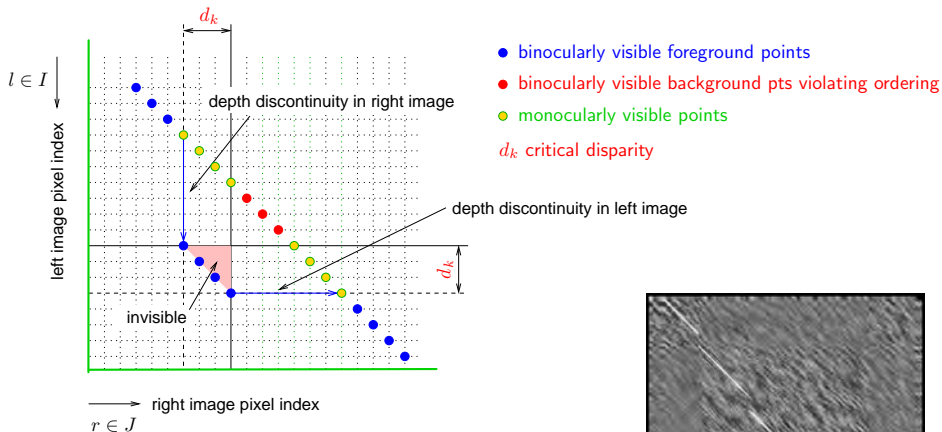
## ► Basic Stereoscopic Matching Models

- notice many small isolated errors in the ML matching
- we need a stronger model

### Potential models for $M$ (from weaker to stronger)

1. Uniqueness: Every image point matches at most once
  - excludes semi-transparent objects
  - used by the ML matching algorithm (but not by the WTA algorithm)
2. Monotonicity: Matched pixel ordering is preserved
  - For all  $(i, j) \in M, (k, l) \in M, k > i \Rightarrow l > j$   
Notation:  $(i, j) \in M$  or  $j = M(i)$  – left-image pixel  $i$  matches right-image pixel  $j$
  - excludes thin objects close to the cameras
3. Coherence: Objects occupy well-defined 3D volumes
  - concept by [Prazdny 85]
  - algorithms are based on image/disparity map segmentation
  - a popular model (segment-based, bilateral filtering and their successors)
4. Continuity: There are no occlusions or self-occlusions
  - too strong, except in some applications

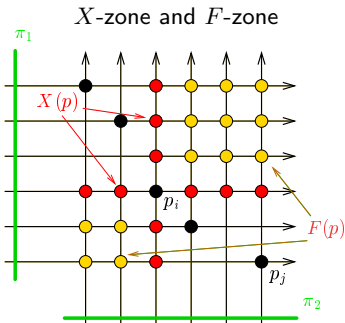
# Understanding Occlusion Structure in Matching Table



- this leads to the concept of 'forbidden zone'



## ► Formally: Uniqueness and Ordering in Matching Table $T$



$$p_j \notin X(p_i), \quad p_j \notin F(p_i)$$

- **Uniqueness Constraint:**

A set of pairs  $M = \{p_i\}_{i=1}^n, p_i \in T$  is a matching iff  
 $\forall p_i, p_j \in M : p_j \notin X(p_i).$

$X$ -zone,  $p_i \notin X(p_i)$

- **Ordering Constraint:**

Matching  $M$  is monotonic iff  
 $\forall p_i, p_j \in M : p_j \notin F(p_i).$

$F$ -zone,  $p_i \notin F(p_i)$

- ordering constraint: matched points form a monotonic set in both images
  - ordering is a powerful constraint: in  $n \times n$  table we have monotonic matchings  $O(4^n) \ll O(n!)$  all matchings
- ⊗ 2: how many are there maximal monotonic matchings? (e.g. 27 for  $n = 4$ ; hard!)

- uniqueness constraint is a basic occlusion model
- ordering constraint is a weak continuity model and partly also an occlusion model
- monotonic matching can be found by **dynamic programming**

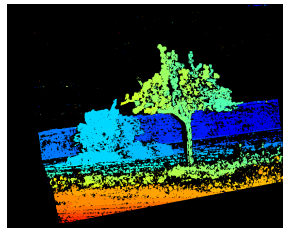
# Some Results: AppleTree



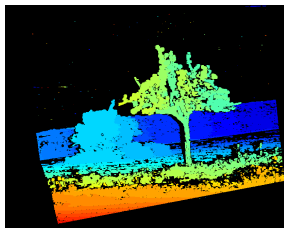
left image



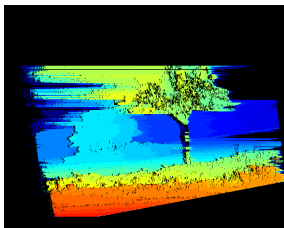
right image



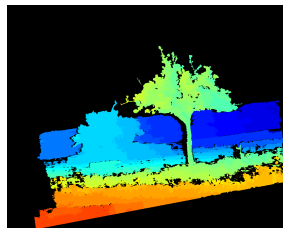
ML  $\rightarrow$ 172



3LDP w/ordering [SP]



naïve DP [Cox et al. 1992]



stable segmented 3LDP

- 3LDP parameters  $\alpha_i$ ,  $V_e$  learned on Middlebury stereo data <http://vision.middlebury.edu/stereo/>

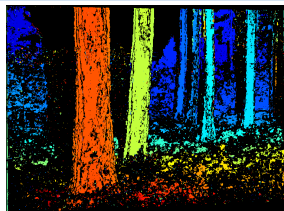
## Some Results: Larch



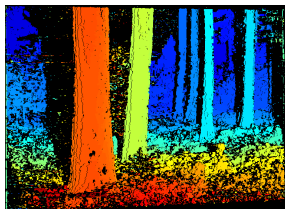
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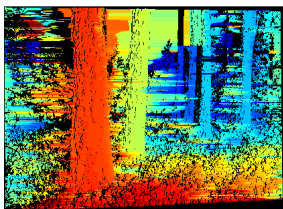
right image



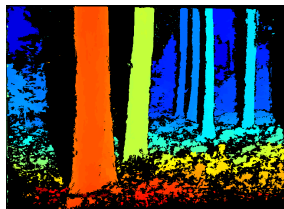
ML →172



3DLP w/ordering [SP]



naïve DP



stable segmented 3DLP

- naïve DP does not model mutual occlusion
- but even 3DLP has errors in mutually occluded region
- stable segmented 3DLP has few errors in mutually occluded region since it uses a coherence model

# Algorithm Comparison

## Marroquin's Winner-Take-All (WTA →166)

- the ur-algorithm very weak model
- dense disparity map
- $O(N^3)$  algorithm, simple but it rarely works

## Maximum Likelihood Matching (ML →172)

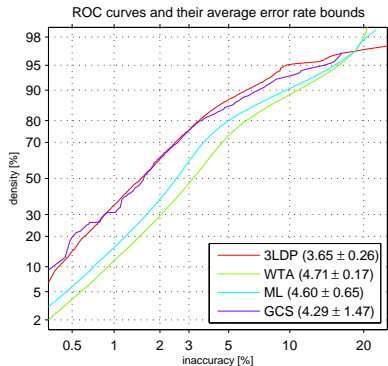
- semi-dense disparity map
- many small isolated errors
- models basic occlusion
- $O(N^3 \log(NV))$  algorithm max-flow by cost scaling

## MAP with Min-Cost Labeled Path (3LDP)

- semi-dense disparity map
- models occlusion in flat, piecewise continuous scenes
- has 'illusions' if ordering does not hold
- $O(N^3)$  algorithm

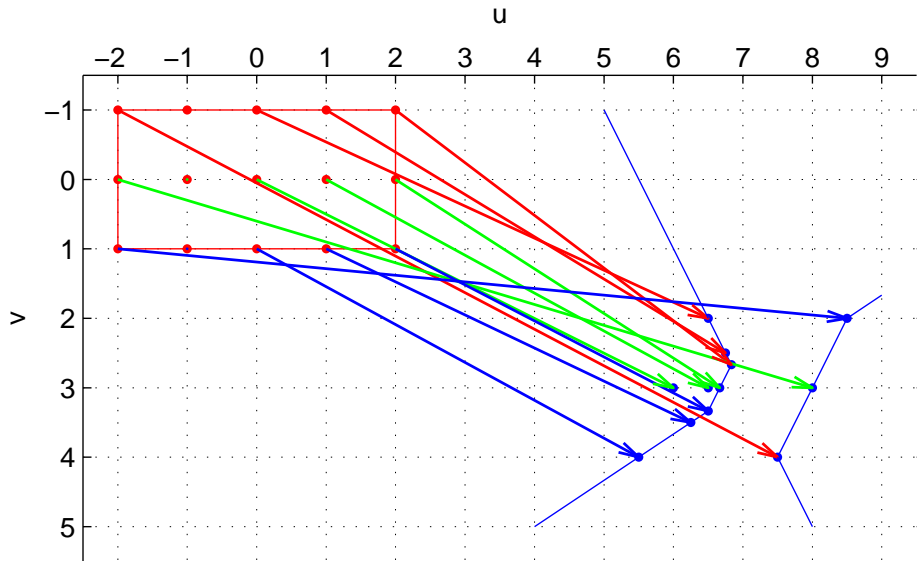
## Stable Segmented 3LDP

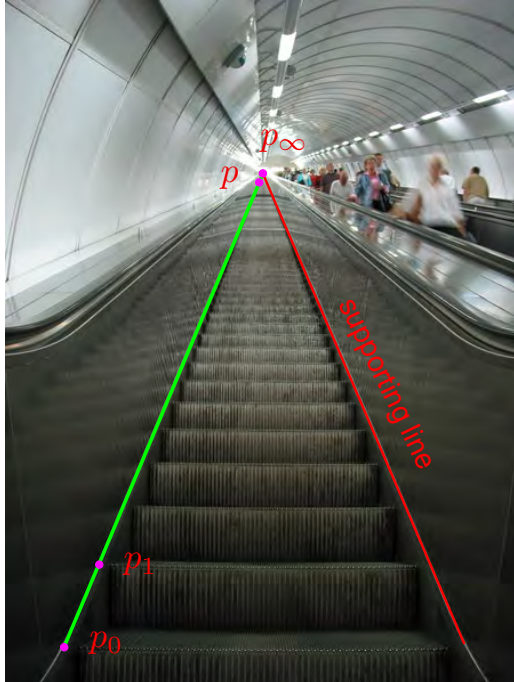
- better (fewer errors at any given density)
- $O(N^3 \log N)$  algorithm
- requires image segmentation itself a difficult task

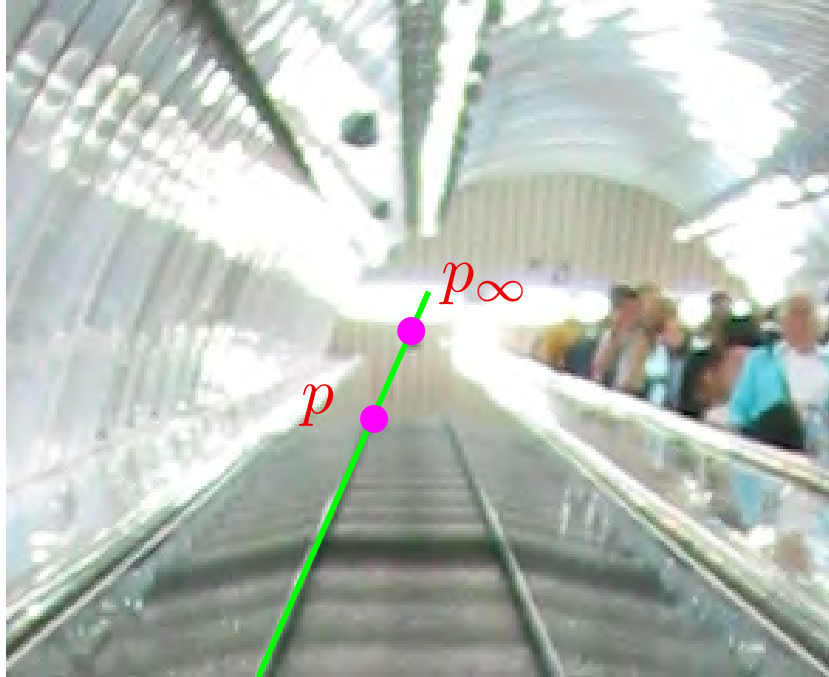


- ROC-like curve captures the density/accuracy tradeoff
- numbers: AUC (smaller is better)
- GCS is the one used in the exercises
- more algorithms at <http://vision.middlebury.edu/stereo/> (good luck!)

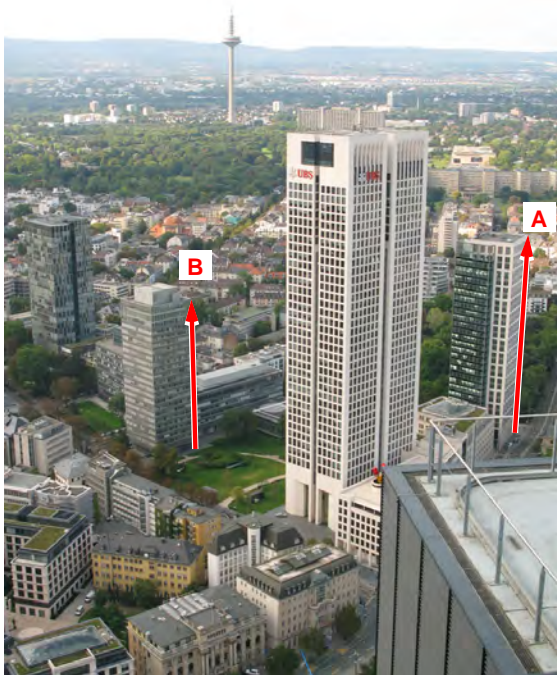
Thank You

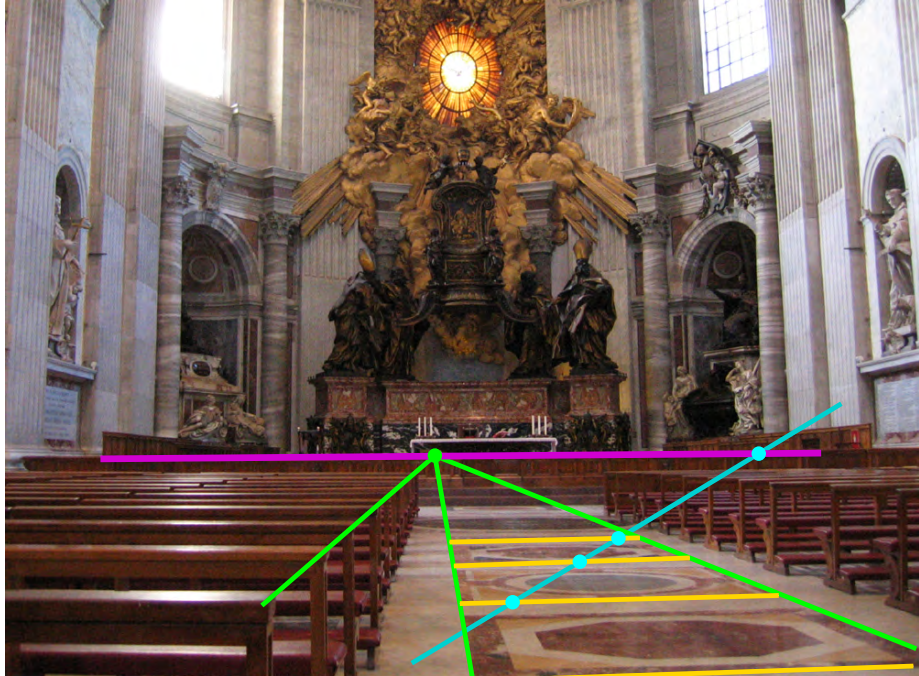






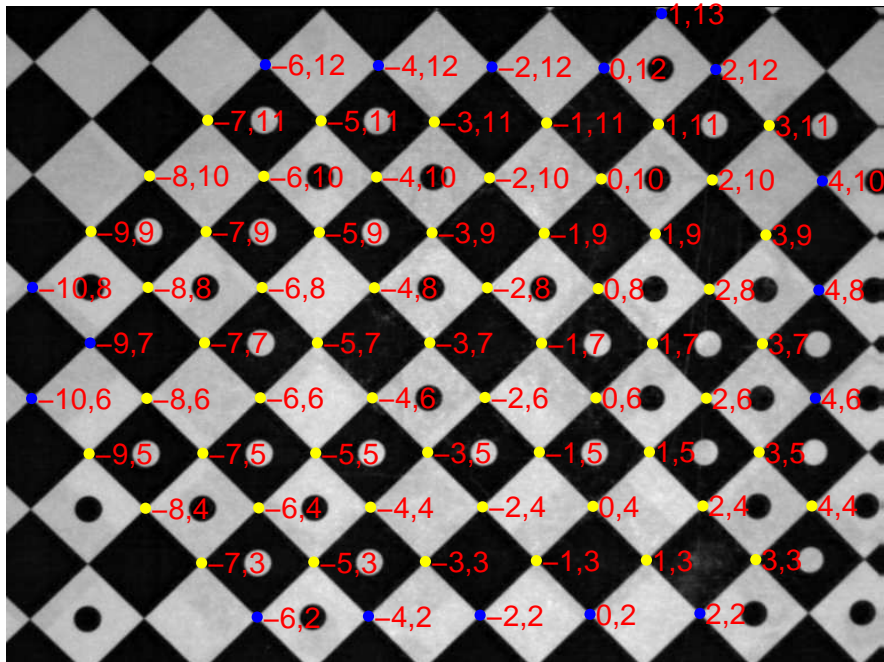


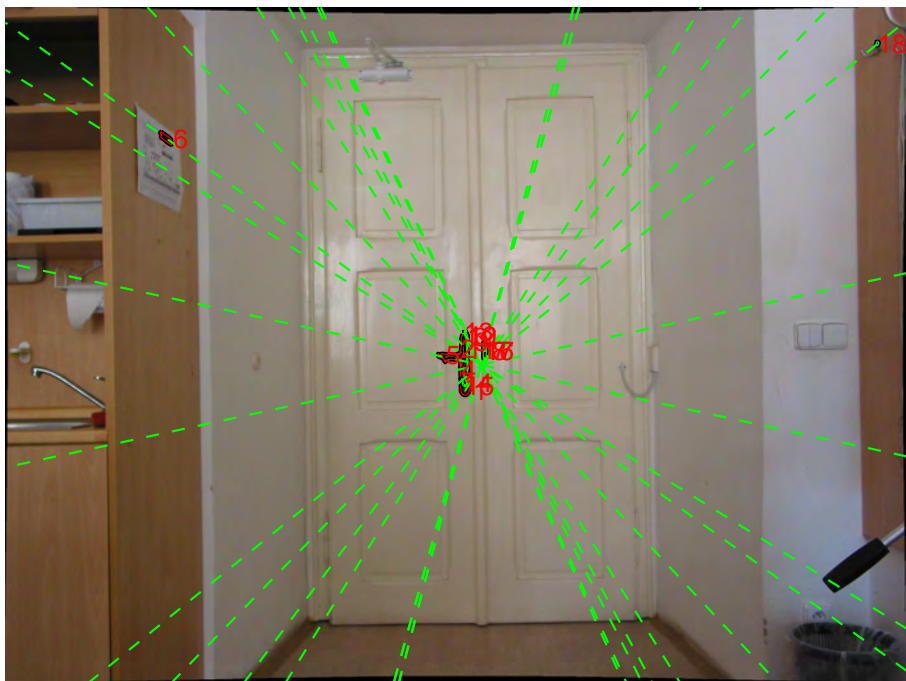


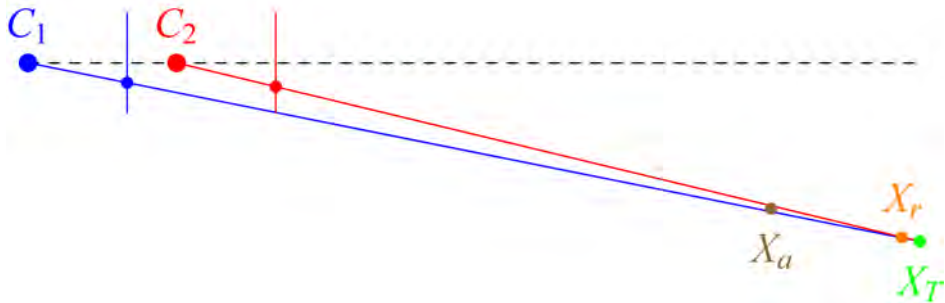








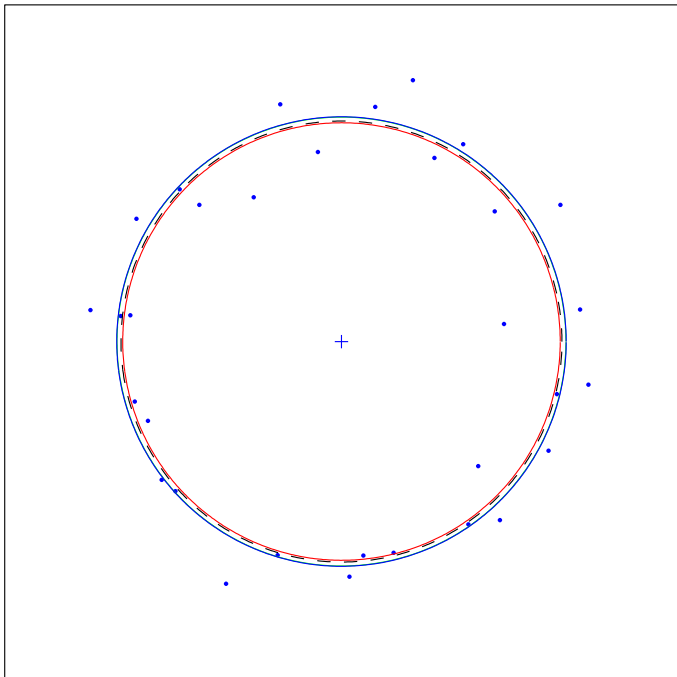


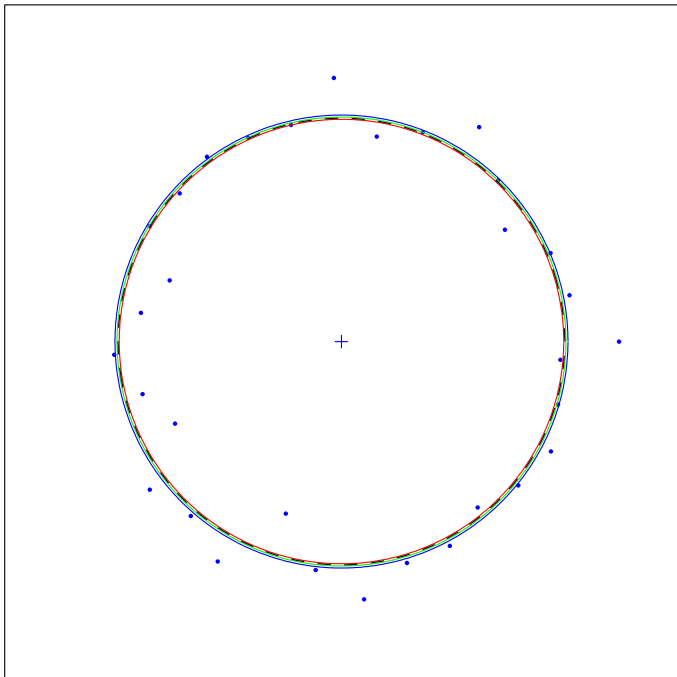


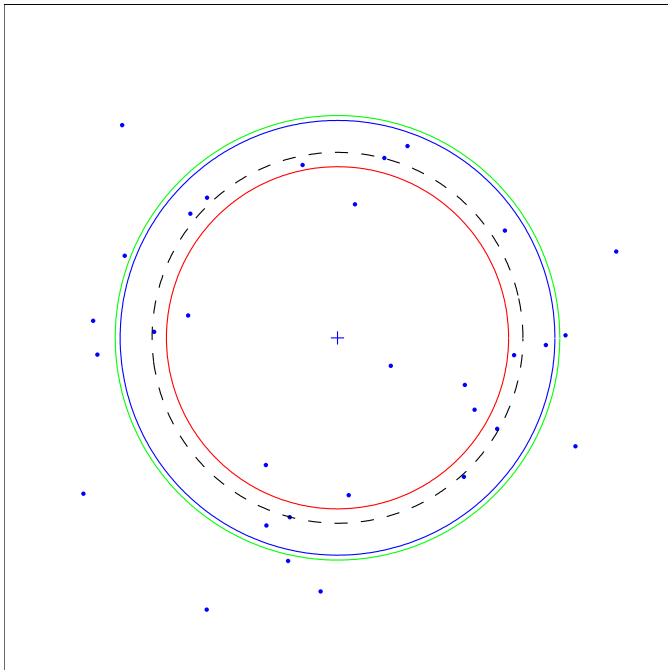
$C_1$ 

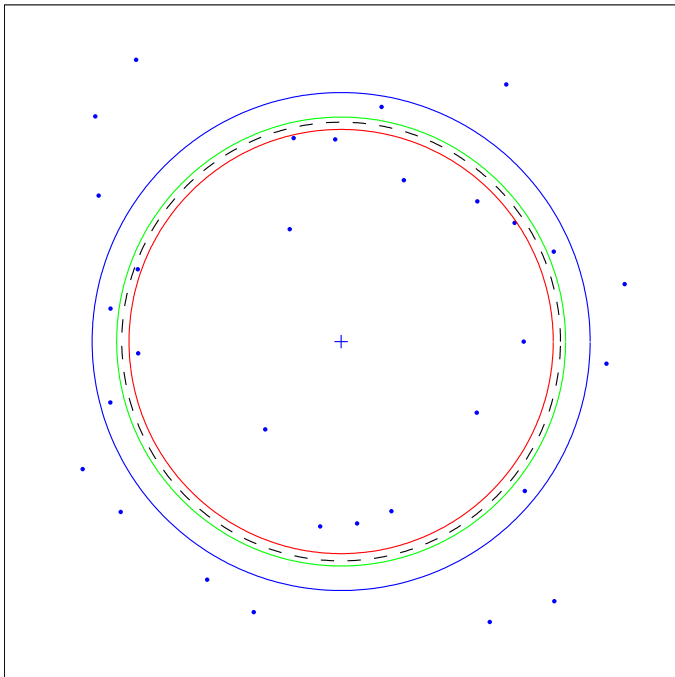


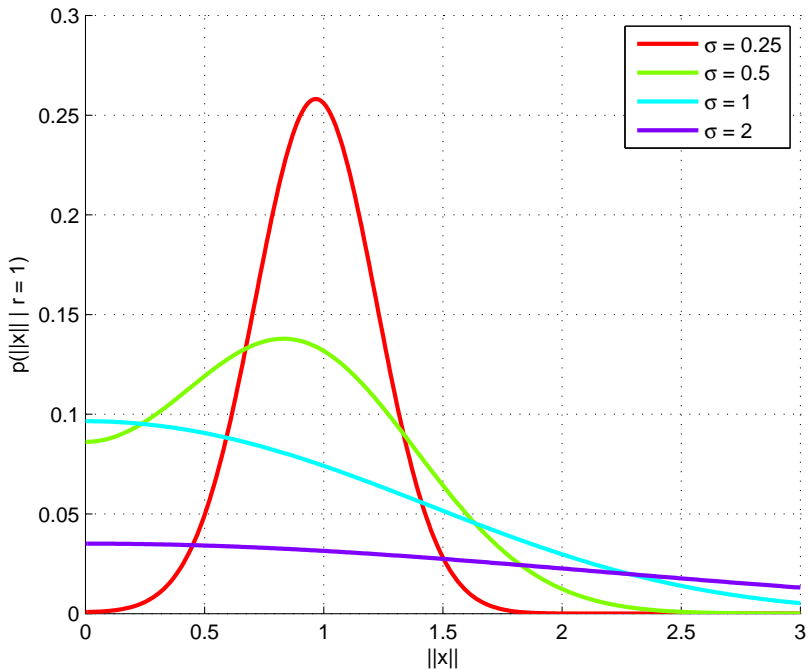


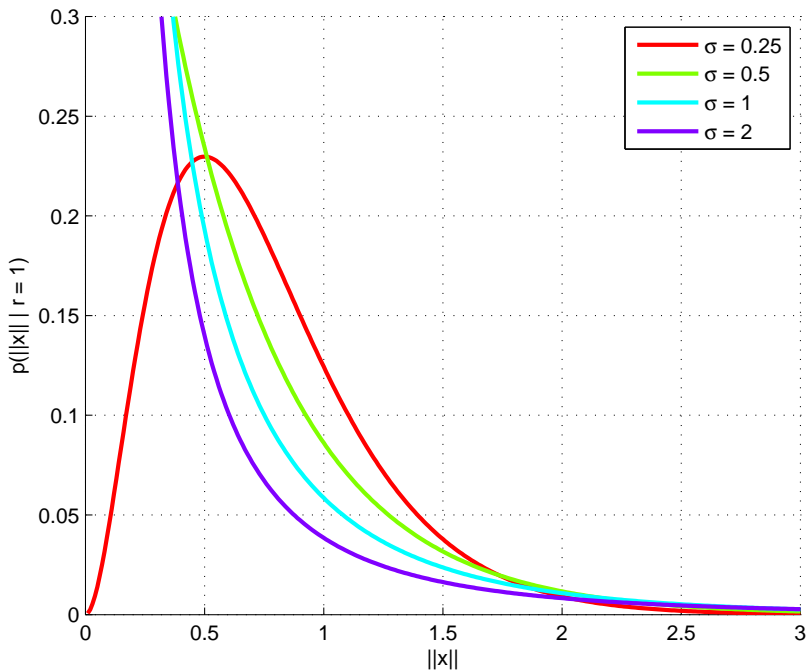


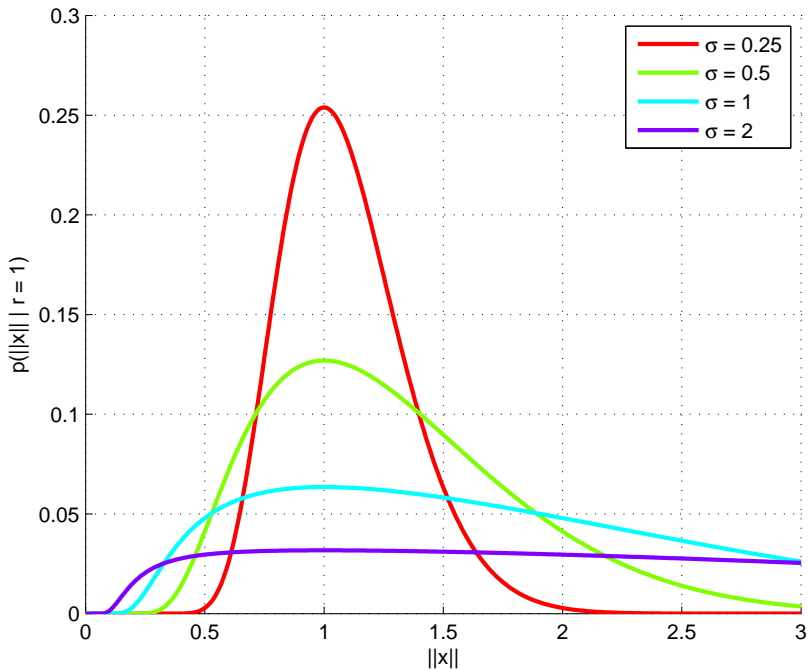




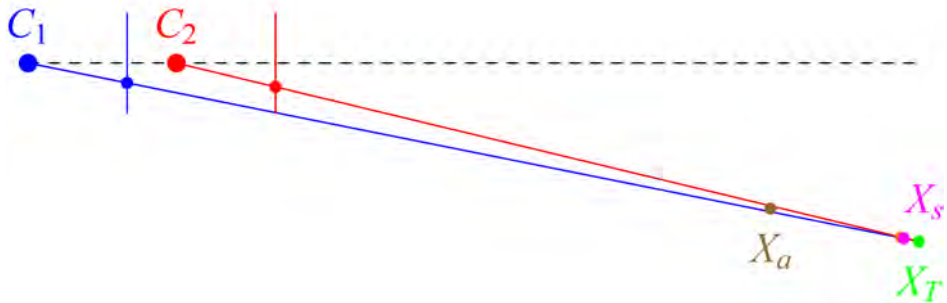




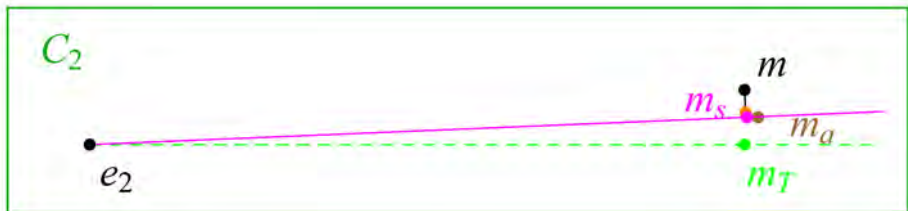




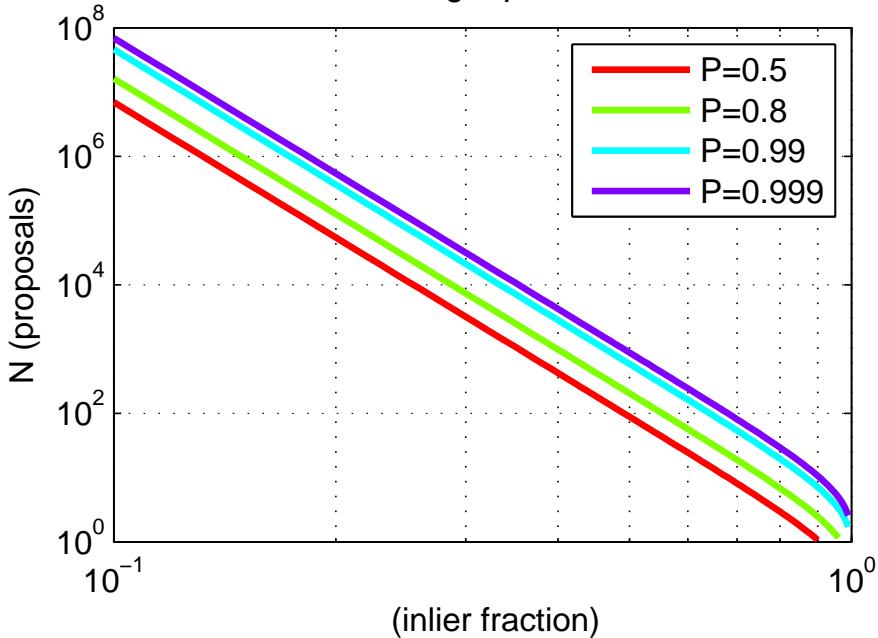




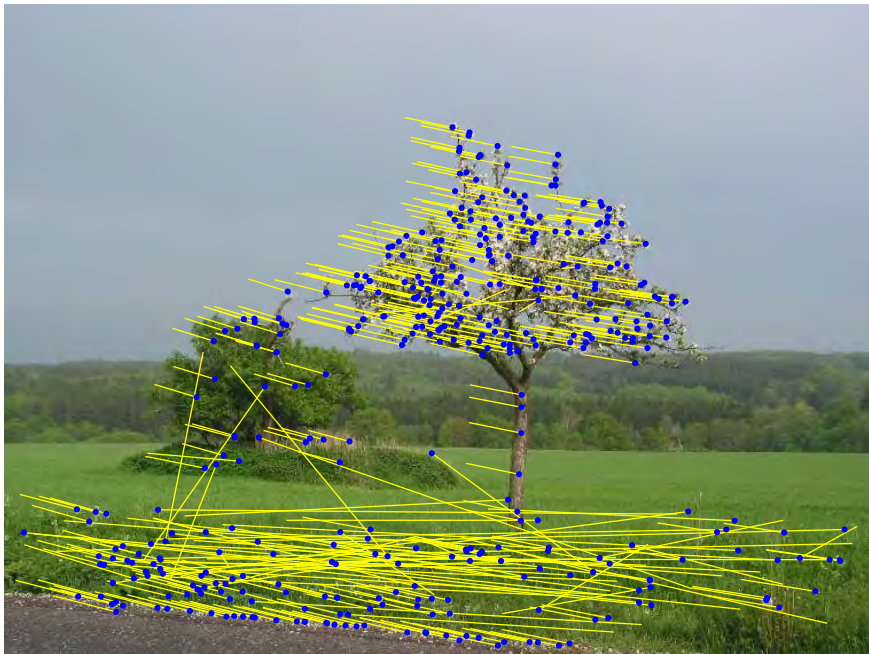




$s = 7$



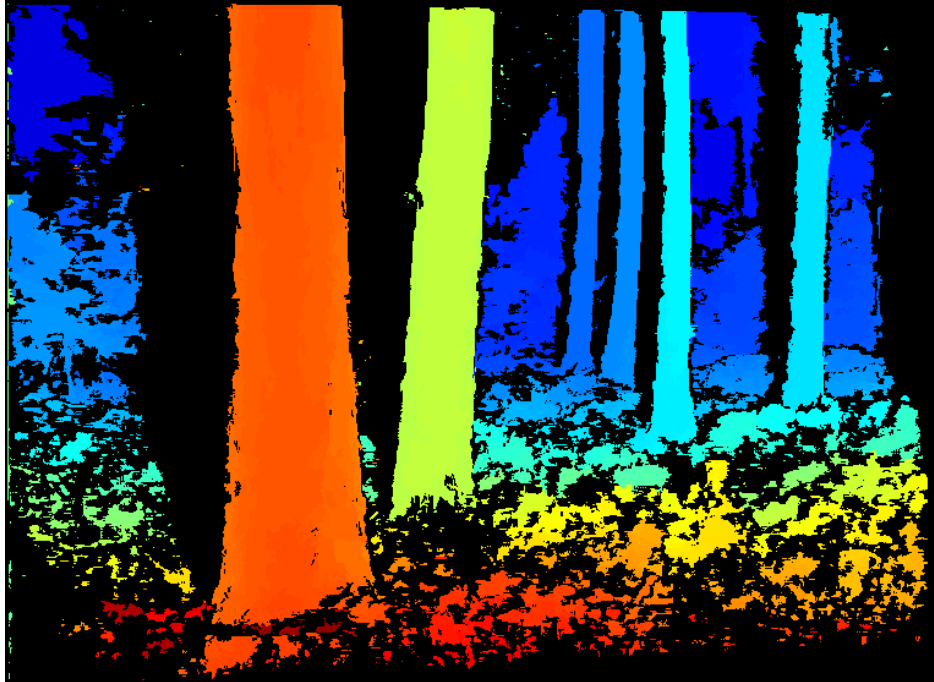


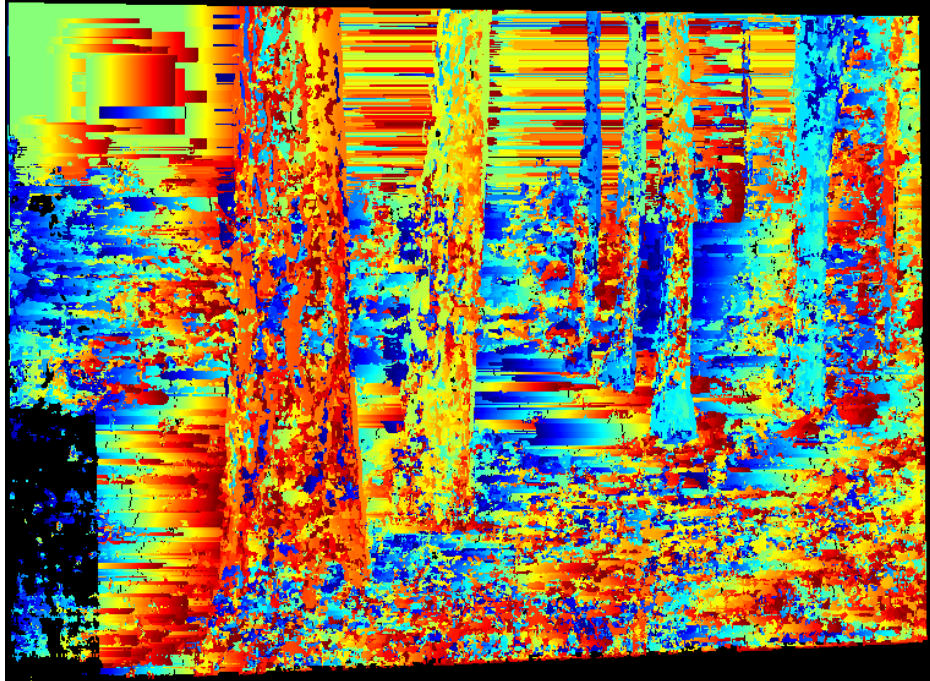




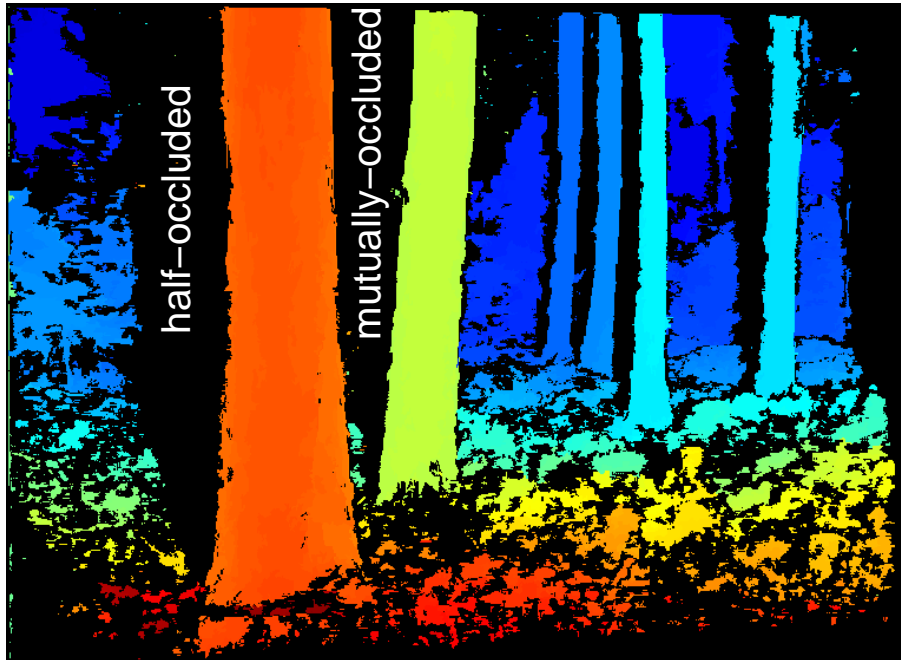


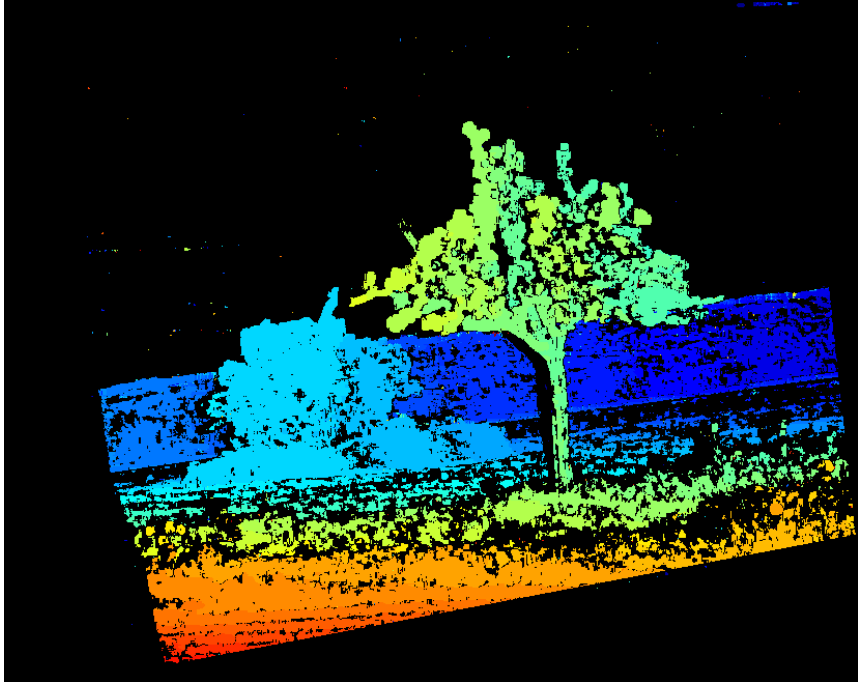


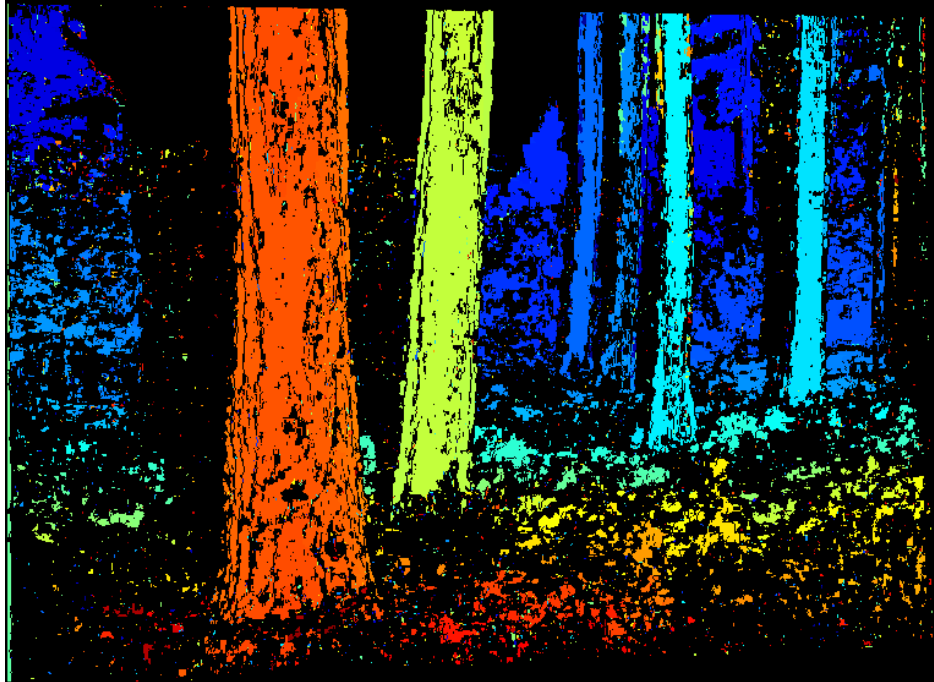




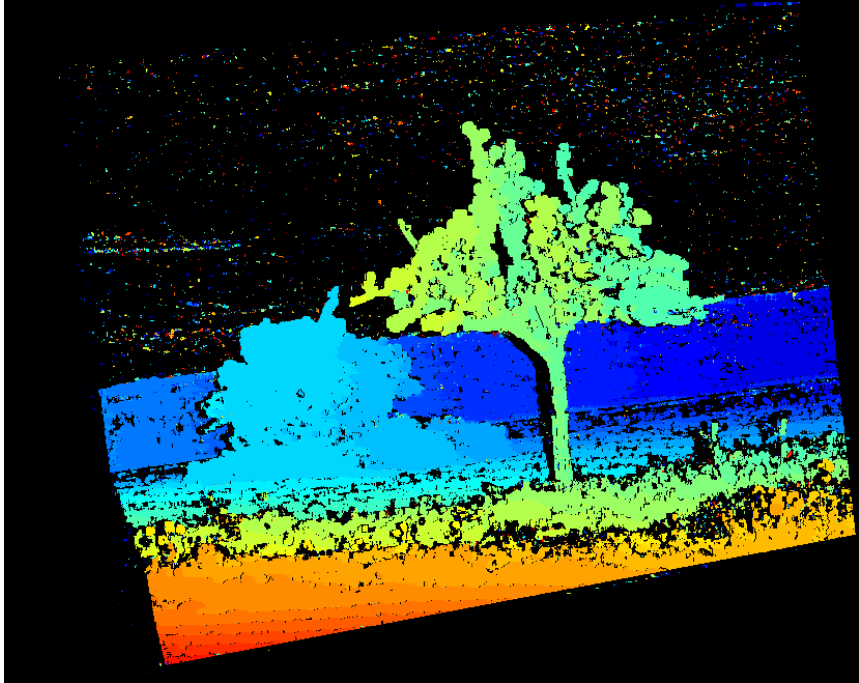




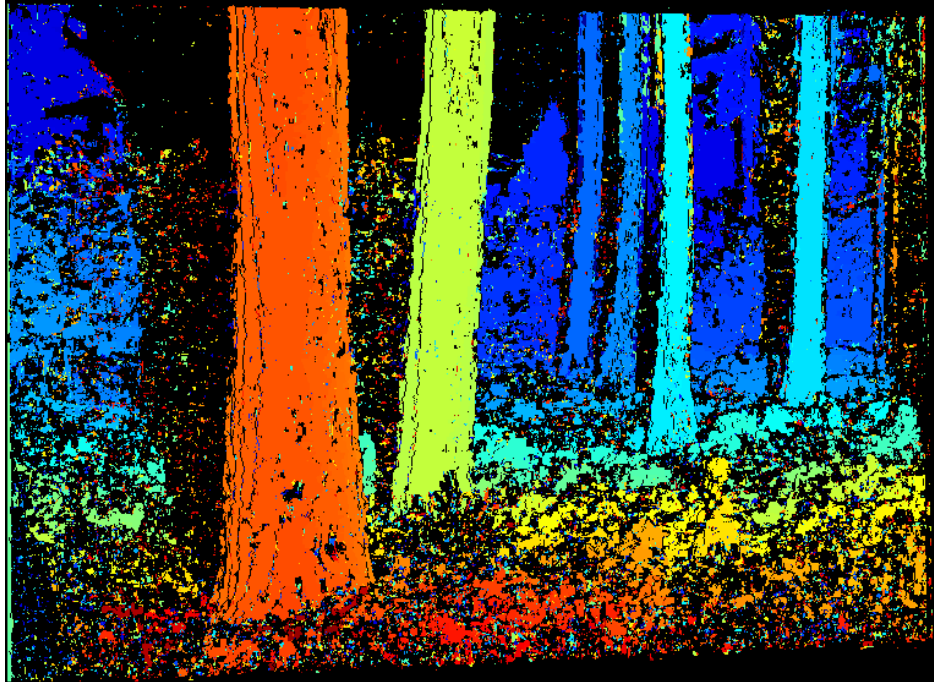








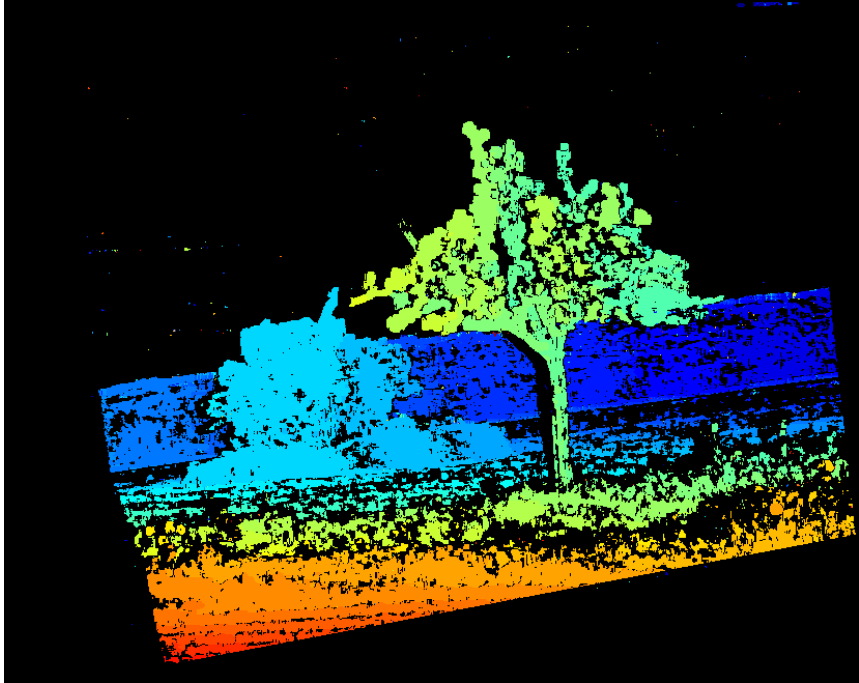


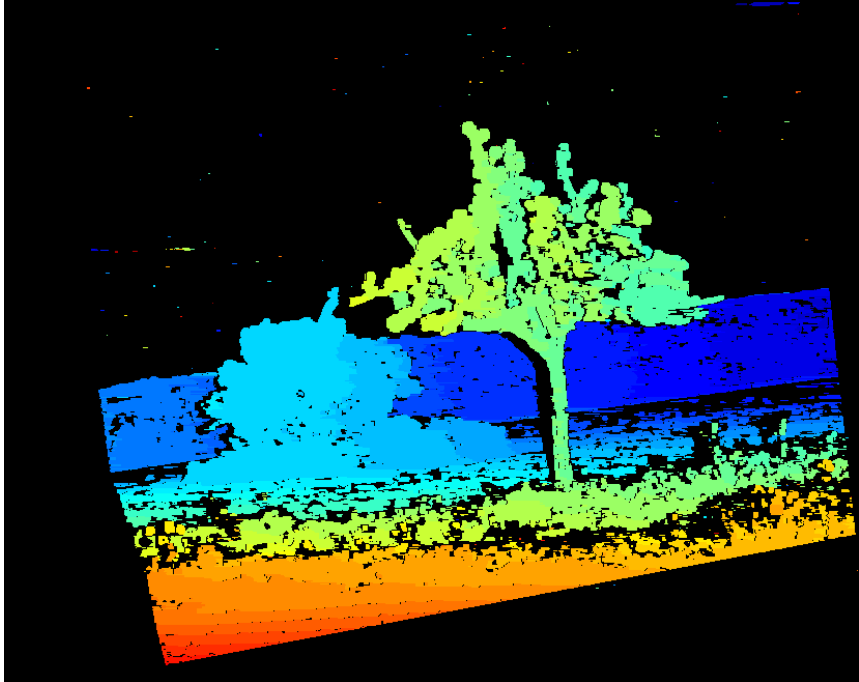


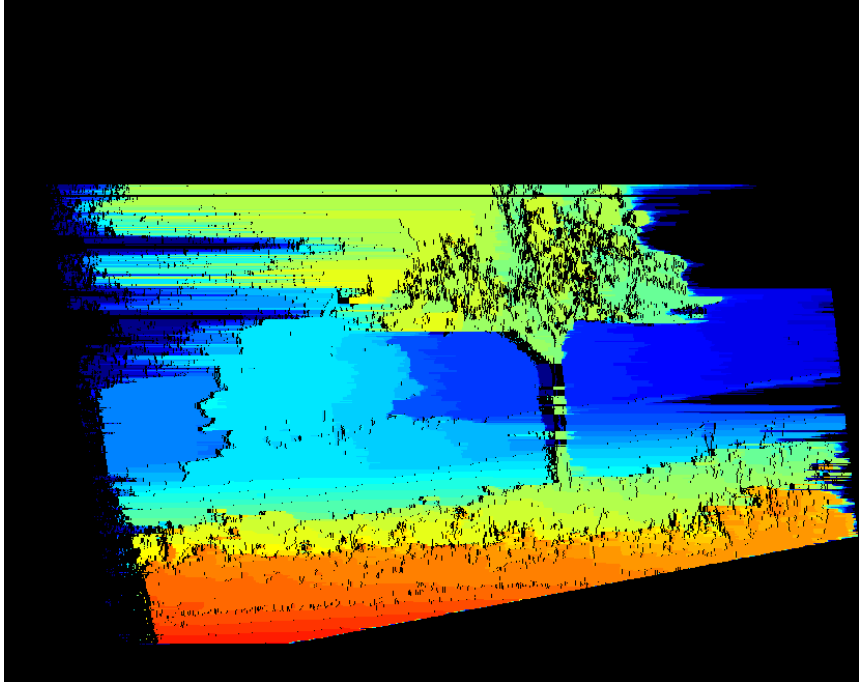


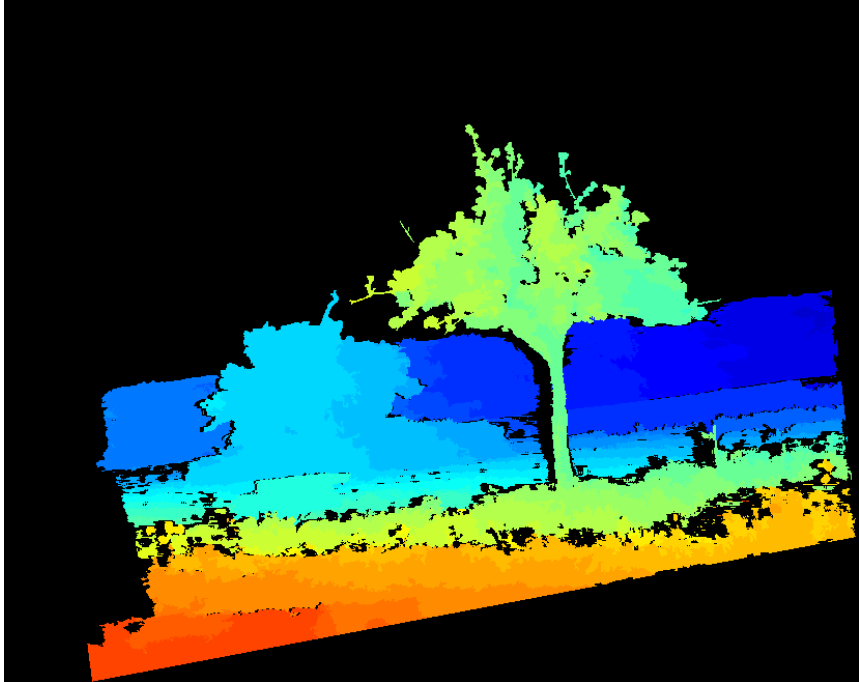








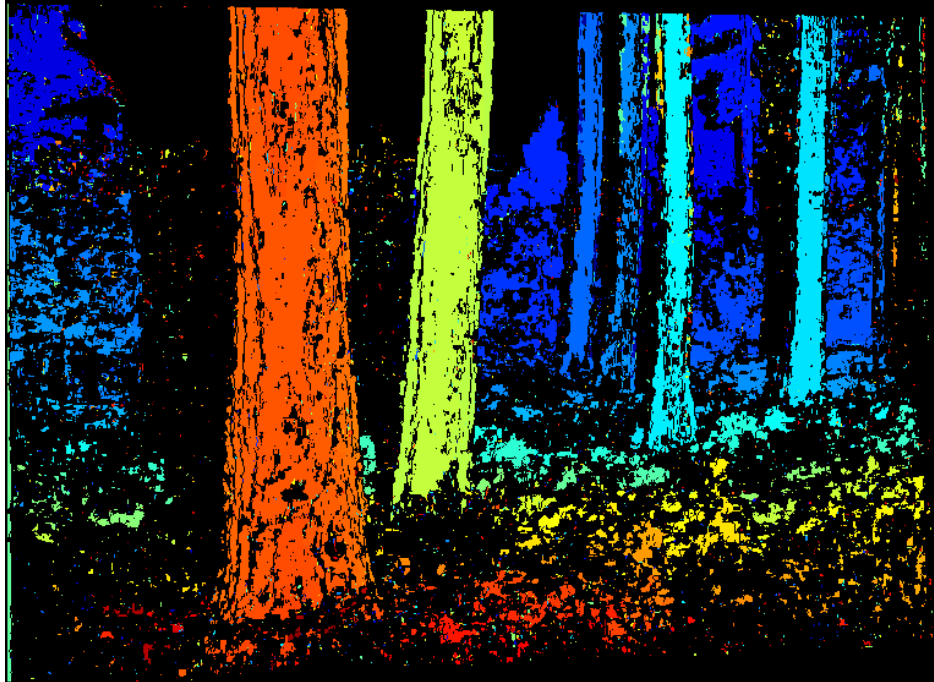


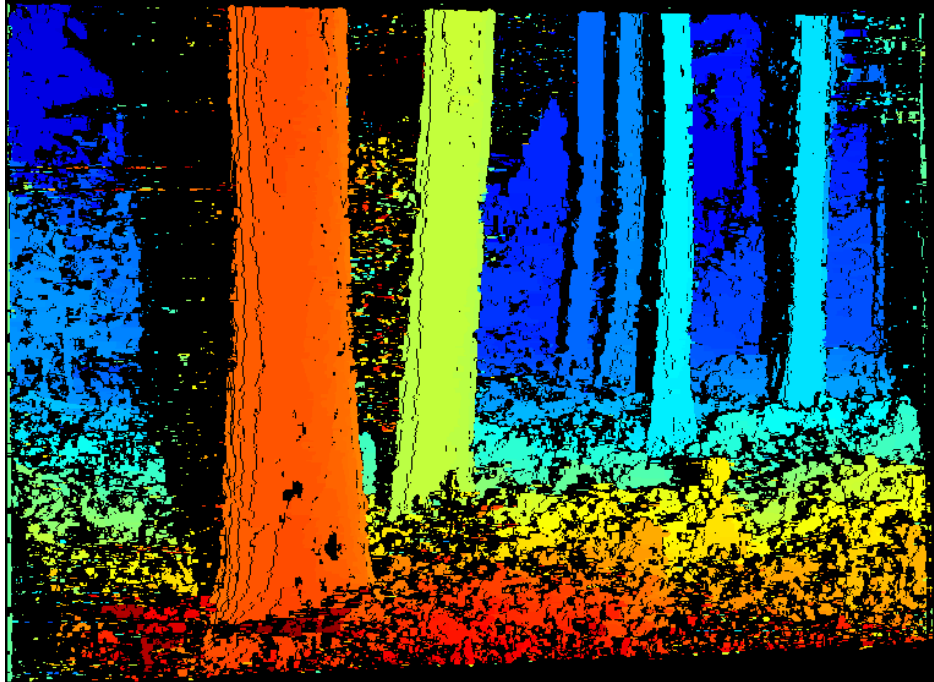


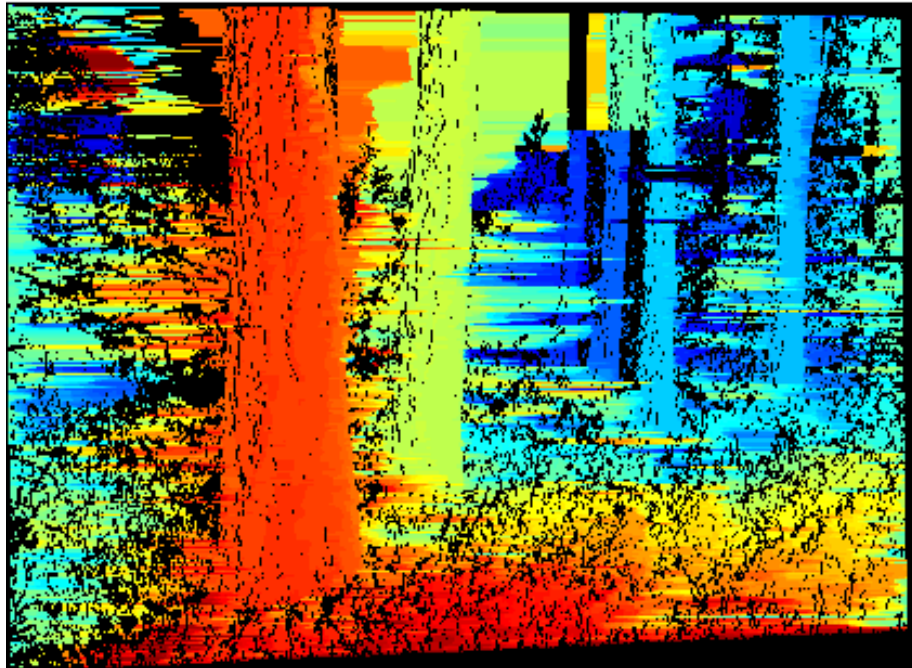


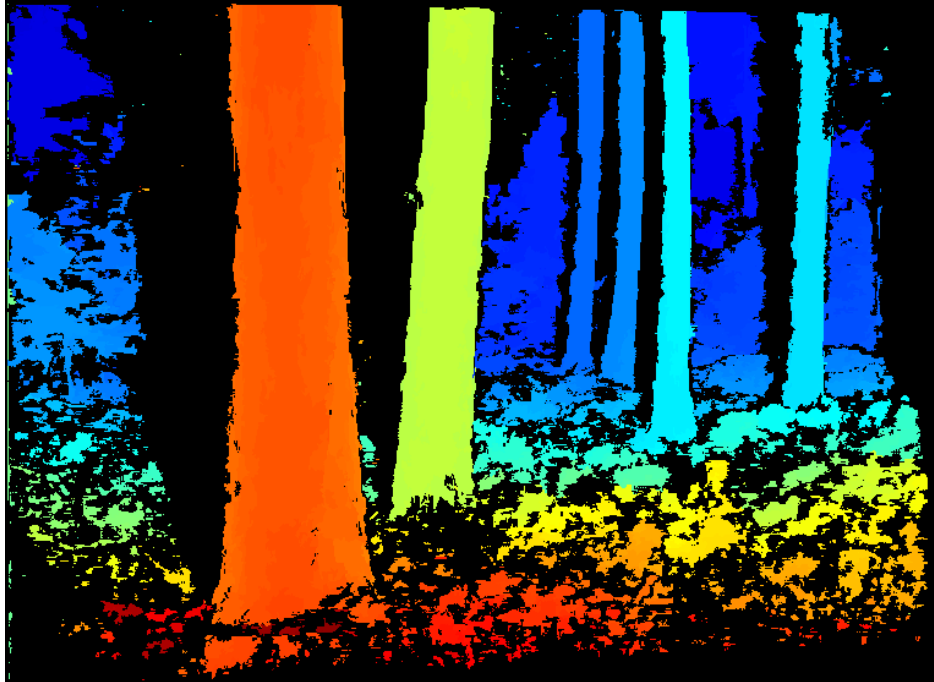












ROC curves and their average error rate bounds

