

► Degenerate (Critical) Configurations for Camera Resection

Let $\mathcal{X} = \{X_i; i = 1, \dots\}$ be a set of points and $\mathbf{P}_1 \neq \mathbf{P}_j$ be two regular (rank-3) cameras. Then two configurations $(\mathbf{P}_1, \mathcal{X})$ and $(\mathbf{P}_j, \mathcal{X})$ are image-equivalent if

$$\mathbf{P}_1 \underline{\mathbf{X}}_i \simeq \mathbf{P}_j \underline{\mathbf{X}}_i \quad \text{for all } X_i \in \mathcal{X}$$

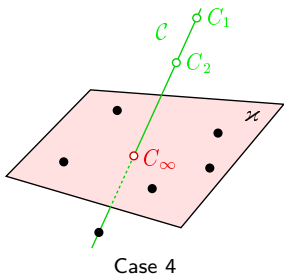
there is a non-trivial set of other cameras that see the same image

- **importantly:** If all calibration points $X_i \in \mathcal{X}$ lie on a plane κ then camera resection is non-unique and all image-equivalent camera centers lie on a spatial line \mathcal{C} with the $C_\infty = \kappa \cap \mathcal{C}$ excluded

this also means we cannot resect if all X_i are infinite

- by adding points $X_i \in \mathcal{X}$ to \mathcal{C} we gain nothing
- there are additional image-equivalent configurations, see next

proof sketch in [H&Z, Sec. 22.1.2]

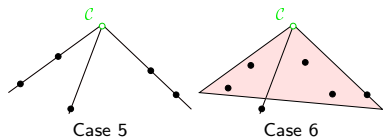


Case 4

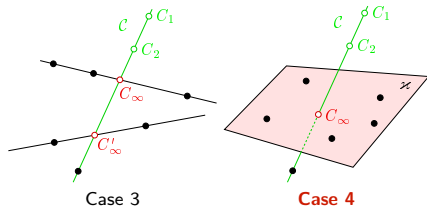
Note that if \mathbf{Q}, \mathbf{T} are suitable homographies then $\mathbf{P}_1 \simeq \mathbf{Q}\mathbf{P}_0\mathbf{T}$, where \mathbf{P}_0 is canonical and the analysis can be made with $\hat{\mathbf{P}}_j \simeq \mathbf{Q}^{-1}\mathbf{P}_j$

$$\mathbf{P}_0 \underbrace{\underline{\mathbf{T}}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \simeq \hat{\mathbf{P}}_j \underbrace{\underline{\mathbf{T}}\underline{\mathbf{X}}_i}_{\underline{\mathbf{Y}}_i} \quad \text{for all } Y_i \in \mathcal{Y}$$

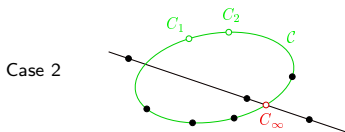
cont'd (all cases)



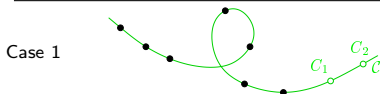
- cameras C_1, C_2 co-located at point C
- points on three optical rays or one optical ray and one optical plane
- Case 5: camera sees 3 isolated point images
- Case 6: cam. sees a line of points and an isolated point



- cameras lie on a line $C \setminus \{C_\infty, C'_\infty\}$
- points lie on C and
 1. on two lines meeting C at C_∞, C'_∞
 2. or on a plane meeting C at C_∞
- Case 3: camera sees 2 lines of points



- cameras lie on a planar conic $C \setminus \{C_\infty\}$
not necessarily an ellipse
- points lie on C and an additional line meeting the conic at C_∞
- Case 2: camera sees 2 lines of points



- cameras and points all lie on a twisted cubic C
- Case 1: camera sees a conic

► Three-Point Exterior Orientation Problem (P3P)

Calibrated camera rotation and translation from Perspective images of 3 reference Points.

Problem: Given \mathbf{K} and three corresponding pairs $\{(m_i, X_i)\}_{i=1}^3$, find \mathbf{R} , \mathbf{C} by solving

$$\lambda_i \underline{\mathbf{m}}_i = \mathbf{K}\mathbf{R}(\mathbf{X}_i - \mathbf{C}), \quad i = 1, 2, 3$$

1. Transform $\underline{\mathbf{v}}_i \stackrel{\text{def}}{=} \mathbf{K}^{-1}\underline{\mathbf{m}}_i$. Then

$$\lambda_i \underline{\mathbf{v}}_i = \mathbf{R}(\mathbf{X}_i - \mathbf{C}). \quad (10)$$

2. Eliminate \mathbf{R} by taking rotation preserves length: $\|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$

$$|\lambda_i| \cdot \|\underline{\mathbf{v}}_i\| = \|\mathbf{X}_i - \mathbf{C}\| \stackrel{\text{def}}{=} z_i \quad (11)$$

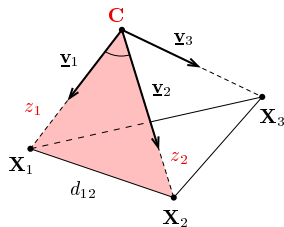
3. Consider only angles among $\underline{\mathbf{v}}_i$ and apply Cosine Law per triangle $(\mathbf{C}, \mathbf{X}_i, \mathbf{X}_j)$ $i, j = 1, 2, 3, i \neq j$

$$d_{ij}^2 = z_i^2 + z_j^2 - 2 z_i z_j c_{ij},$$

$$z_i = \|\mathbf{X}_i - \mathbf{C}\|, \quad d_{ij} = \|\mathbf{X}_j - \mathbf{X}_i\|, \quad c_{ij} = \cos(\angle \underline{\mathbf{v}}_i \underline{\mathbf{v}}_j)$$

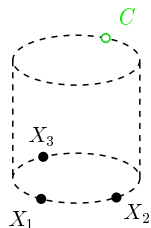
4. Solve system of 3 quadratic eqs in 3 unknowns z_i [Fischler & Bolles, 1981]
there may be no real root; there are up to 4 solutions that cannot be ignored (verify on additional points)
5. Compute \mathbf{C} by trilateration (3-sphere intersection) from \mathbf{X}_i and z_i ; then λ_i from (11) and \mathbf{R} from (10)

configuration w/o rotation in (11)



Similar problems (P4P with unknown f) at <http://cmp.felk.cvut.cz/minimal/> (with code)

Degenerate (Critical) Configurations for Exterior Orientation



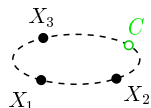
unstable solution

- center of projection C located on the orthogonal circular cylinder with base circumscribing the three points X_i

unstable: a small change of X_i results in a large change of C
can be detected by error propagation

degenerate

- camera C is coplanar with points (X_1, X_2, X_3) but is not on the circumscribed circle of (X_1, X_2, X_3) camera sees a line



no solution

- C cocyclic with (X_1, X_2, X_3) camera sees a line

- additional critical configurations depend on the method to solve the quadratic equations

► Populating A Little ZOO of Minimal Geometric Problems in CV

problem	given	unknown	slide
camera resection	6 world–img correspondences $\{(X_i, m_i)\}_{i=1}^6$	P	64
exterior orientation	K , 3 world–img correspondences $\{(X_i, m_i)\}_{i=1}^3$	R, C	68

- camera resection and exterior orientation are similar problems in a sense:
 - we do resectioning when our camera is uncalibrated
 - we do orientation when our camera is calibrated
- more problems to come

Computing with a Camera Pair

- 4.1 Camera Motions Inducing Epipolar Geometry
- 4.2 Estimating Fundamental Matrix from 7 Correspondences
- 4.3 Estimating Essential Matrix from 5 Correspondences
- 4.4 Triangulation: 3D Point Position from a Pair of Corresponding Points

covered by

- [1] [H&Z] Secs: 9.1, 9.2, 9.6, 11.1, 11.2, 11.9, 12.2, 12.3, 12.5.1
- [2] H. Li and R. Hartley. Five-point motion estimation made easy. In *Proc ICPR 2006*, pp. 630–633

additional references

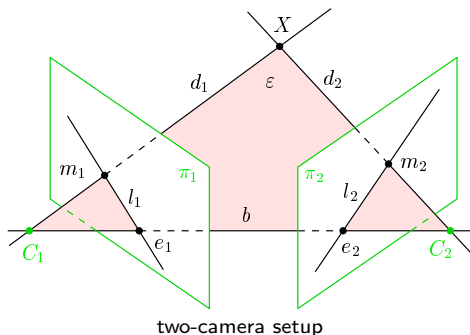


H. Longuet-Higgins. A computer algorithm for reconstructing a scene from two projections. *Nature*, 293 (5828):133–135, 1981.

► Geometric Model of a Camera Pair

Epipolar geometry:

- brings constraints necessary for inter-image matching
- its parametric form encapsulates information about the relative pose of two cameras



Description

- baseline b joins projection centers C_1, C_2
$$\mathbf{b} = \mathbf{C}_2 - \mathbf{C}_1$$
- epipole $e_i \in \pi_i$ is the image of C_j :
$$\mathbf{e}_1 \simeq \mathbf{P}_1 \mathbf{C}_2, \quad \mathbf{e}_2 \simeq \mathbf{P}_2 \mathbf{C}_1$$
- $l_i \in \pi_i$ is the image of epipolar plane
$$\varepsilon = (C_2, X, C_1)$$
- l_j is the epipolar line in image π_j induced by m_i in image π_i

Epipolar constraint: corresponding d_2, b, d_1 are coplanar a necessary condition, see →87

Epipolar Geometry Example: Forward Motion

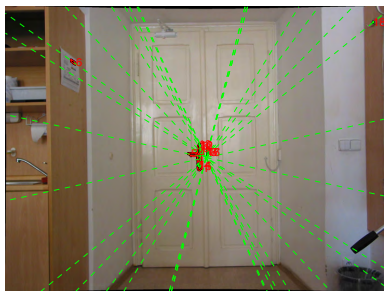


image 1

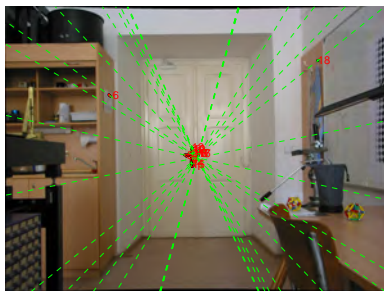
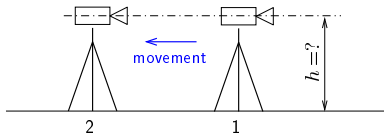


image 2

- red: correspondences
- green: epipolar line pairs per correspondence

click on the image to see their IDs
same ID in both images

How high was the camera above the floor?



► Cross Products and Maps by Skew-Symmetric 3×3 Matrices

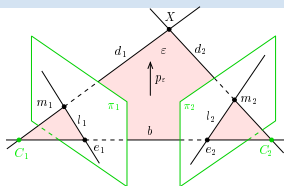
- There is an equivalence $\mathbf{b} \times \mathbf{m} = [\mathbf{b}]_{\times} \mathbf{m}$, where $[\mathbf{b}]_{\times}$ is a 3×3 skew-symmetric matrix

$$[\mathbf{b}]_{\times} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \quad \text{assuming } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Some properties

- $[\mathbf{b}]_{\times}^{\top} = -[\mathbf{b}]_{\times}$ the general antisymmetry property
- \mathbf{A} is skew-symmetric iff $\mathbf{x}^{\top} \mathbf{A} \mathbf{x} = 0$ for all \mathbf{x} skew-sym mtx generalizes cross products
- $[\mathbf{b}]_{\times}^3 = -\|\mathbf{b}\|^2 \cdot [\mathbf{b}]_{\times}$
- $\|[\mathbf{b}]_{\times}\|_F = \sqrt{2} \|\mathbf{b}\|$ Frobenius norm ($\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^{\top} \mathbf{A})} = \sqrt{\sum_{i,j} |a_{ij}|^2}$)
- $[\mathbf{b}]_{\times} \mathbf{b} = \mathbf{0}$
- $\text{rank} [\mathbf{b}]_{\times} = 2$ iff $\|\mathbf{b}\| > 0$ check minors of $[\mathbf{b}]_{\times}$
- eigenvalues of $[\mathbf{b}]_{\times}$ are $(0, \lambda, -\lambda)$
- for any regular \mathbf{B} : $[\mathbf{Bz}]_{\times} \mathbf{B} = \det \mathbf{B} \cdot \mathbf{B}^{-\top} [\mathbf{z}]_{\times}$ follows from the factoring on $\rightarrow 40$
- special case: if $\mathbf{R} \mathbf{R}^{\top} = \mathbf{I}$ then $[\mathbf{Rb}]_{\times} = \mathbf{R} [\mathbf{b}]_{\times} \mathbf{R}^{\top}$
 - note that if \mathbf{R}_b is rotation about \mathbf{b} then $\mathbf{R}_b \mathbf{b} = \mathbf{b}$
 - note $[\mathbf{b}]_{\times}$ is not a homography; it is not a rotation matrix it is a logarithm of a rotation mtx

► Expressing Epipolar Constraint Algebraically



$$\mathbf{P}_i = [\mathbf{Q}_i \quad \mathbf{q}_i] = \mathbf{K}_i [\mathbf{R}_i \quad \mathbf{t}_i], \quad i = 1, 2$$

\mathbf{R}_{21} – relative camera rotation, $\mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1^T$

\mathbf{t}_{21} – relative camera translation, $\mathbf{t}_{21} = \mathbf{t}_2 - \mathbf{R}_{21} \mathbf{t}_1 = -\mathbf{R}_2 \mathbf{b}$

\mathbf{b} – baseline (world coordinate system)

remember: $\mathbf{C} = -\mathbf{Q}^{-1} \mathbf{q} = -\mathbf{R}^T \mathbf{t}$

→34 and 36

$$0 = \mathbf{d}_2^T \underbrace{\mathbf{p}_\varepsilon}_{\text{normal of } \varepsilon} \simeq \underbrace{(\mathbf{Q}_2^{-1} \mathbf{m}_2)^T}_{\text{optical ray}} \underbrace{\mathbf{Q}_1^T \mathbf{l}_1}_{\text{optical plane}} = \mathbf{m}_2^T \underbrace{\mathbf{Q}_2^{-T} \mathbf{Q}_1^T (\mathbf{e}_1 \times \mathbf{m}_1)}_{\text{image of } \varepsilon \text{ in } \pi_2} = \mathbf{m}_2^T \underbrace{(\mathbf{Q}_2^{-T} \mathbf{Q}_1^T [\mathbf{e}_1]_\times)}_{\text{fundamental matrix } \mathbf{F}} \mathbf{m}_1$$

Epipolar constraint $\mathbf{m}_2^T \mathbf{F} \mathbf{m}_1 = 0$ is a point-line incidence constraint

- point \mathbf{m}_2 is incident on epipolar line $\mathbf{l}_2 \simeq \mathbf{F} \mathbf{m}_1$
- point \mathbf{m}_1 is incident on epipolar line $\mathbf{l}_1 \simeq \mathbf{F}^T \mathbf{m}_2$
- $\mathbf{F} \mathbf{e}_1 = \mathbf{F}^T \mathbf{e}_2 = \mathbf{0}$ (non-trivially)
- all epipolars meet at the epipole

$$\mathbf{e}_1 \simeq \mathbf{Q}_1 \mathbf{C}_2 + \mathbf{q}_1 = \mathbf{Q}_1 \mathbf{C}_2 - \mathbf{Q}_1 \mathbf{C}_1 = \mathbf{K}_1 \mathbf{R}_1 \mathbf{b} = -\mathbf{K}_1 \mathbf{R}_1 \mathbf{R}_2^T \mathbf{t}_{21} = -\mathbf{K}_1 \mathbf{R}_{21}^T \mathbf{t}_{21}$$

$$\mathbf{F} = \mathbf{Q}_2^{-T} \mathbf{Q}_1^T [\mathbf{e}_1]_\times = \mathbf{Q}_2^{-T} \mathbf{Q}_1^T [\mathbf{K}_1 \mathbf{R}_1 \mathbf{b}]_\times = \overset{\textcircled{*} \mathbf{1}}{\dots} \simeq \mathbf{K}_2^{-T} [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} \mathbf{K}_1^{-1} \quad \text{fundamental}$$

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = \underbrace{[\mathbf{R}_2 \mathbf{b}]_\times}_{\text{baseline in Cam 2}} \mathbf{R}_{21} = \mathbf{R}_{21} \underbrace{[\mathbf{R}_1 \mathbf{b}]_\times}_{\text{baseline in Cam 1}} = \mathbf{R}_{21} [-\mathbf{R}_{21} \mathbf{t}_{21}]_\times \quad \text{essential}$$

► The Structure and the Key Properties of the Fundamental Matrix

$$\mathbf{F} = \underbrace{(\mathbf{Q}_2 \mathbf{Q}_1^{-1})^{-\top}}_{\text{epipolar homography } \mathbf{H}} [\mathbf{e}_1]_{\times} = \underbrace{\mathbf{K}_2^{-\top} \mathbf{R}_{21} \mathbf{K}_1^{\top}}_{\mathbf{H}^{-\top}} \overbrace{[\mathbf{e}_1]_{\times}}^{\text{left epipole}} \simeq \overbrace{[\mathbf{H} \mathbf{e}_1]_{\times}}^{\text{right epipole}} \mathbf{H} = \mathbf{K}_2^{-\top} \underbrace{[-\mathbf{t}_{21}]_{\times} \mathbf{R}_{21}}_{\text{essential matrix } \mathbf{E}} \mathbf{K}_1^{-1}$$

1. \mathbf{E} captures relative camera pose only (the change of the world coordinate system does not change \mathbf{E}) [Longuet-Higgins 1981]

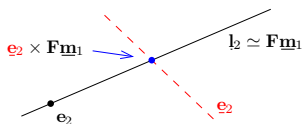
$$[\mathbf{R}'_i \quad \mathbf{t}'_i] = [\mathbf{R}_i \quad \mathbf{t}_i] \cdot \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix} = [\mathbf{R}_i \mathbf{R} \quad \mathbf{R}_i \mathbf{t} + \mathbf{t}_i],$$

then

$$\mathbf{R}'_{21} = \mathbf{R}'_2 \mathbf{R}'_1{}^{\top} = \dots = \mathbf{R}_{21}$$

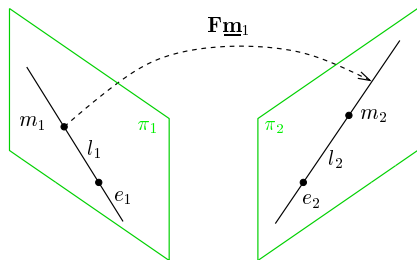
$$\mathbf{t}'_{21} = \mathbf{t}'_2 - \mathbf{R}'_{21} \mathbf{t}'_1 = \dots = \mathbf{t}_{21}$$

2. the translation length \mathbf{t}_{21} is lost since \mathbf{E} is homogeneous
3. \mathbf{F} maps points to lines and it is not a homography
4. $\mathbf{e}_2 \times (\mathbf{e}_2 \times \mathbf{F} \underline{\mathbf{m}}_1) \simeq \mathbf{F} \underline{\mathbf{m}}_1$, in general $\mathbf{F} \simeq [\mathbf{e}_2]_{\times}^{2a} \mathbf{F} [\mathbf{e}_1]_{\times}^{2b}$ for any $a, b \in \mathbb{N}$



- by point/line 'transmutation' (left)
- point \mathbf{e}_2 does not lie on line \mathbf{e}_2 (dashed): $\mathbf{e}_2^{\top} \mathbf{e}_2 \neq 0$
- application: $\mathbf{F}^{\top} (\mathbf{e}_2 \times \mathbf{l}_2) \simeq \mathbf{F}^{\top} (\mathbf{e}_2 \times \mathbf{F} \underline{\mathbf{m}}_1) \simeq \mathbf{F}^{\top} \underline{\mathbf{m}}_2 \simeq \mathbf{l}_1$
- $\mathbf{F}^{\top} [\mathbf{e}_2]_{\times}$ maps epipolar lines to epi. lines but it is not a homography

► Some Mappings by the Fundamental Matrix

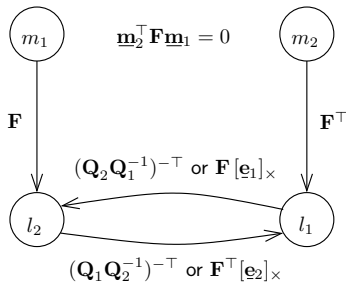


$$0 = \underline{\mathbf{m}}_2^\top \mathbf{F} \underline{\mathbf{m}}_1$$

$$\mathbf{e}_1 \simeq \text{null}(\mathbf{F}), \quad \mathbf{e}_2 \simeq \text{null}(\mathbf{F}^\top)$$

$$\mathbf{l}_2 = \mathbf{F} \underline{\mathbf{m}}_1 \quad \mathbf{l}_1 = \mathbf{F}^\top \underline{\mathbf{m}}_2$$

$$\mathbf{l}_2 = \mathbf{F}[\mathbf{e}_1]_{\times} \mathbf{l}_1 \quad \mathbf{l}_1 = \mathbf{F}^\top[\mathbf{e}_2]_{\times} \mathbf{l}_2$$



- $\mathbf{l}_2 \simeq \mathbf{F}[\mathbf{e}_1]_{\times} \mathbf{l}_1$: by 'transmutation' →78
- $\mathbf{F}[\mathbf{e}_1]_{\times}$ maps lines to lines but it is not a homography
- $\mathbf{H} = \mathbf{Q}_2 \mathbf{Q}_1^{-1}$ is the epipolar homography →78 mapping epipolar lines to epipolar lines, hence

$$\mathbf{H} = \mathbf{Q}_2 \mathbf{Q}_1^{-1} = \mathbf{K}_2 \mathbf{R}_{21} \mathbf{K}_1^{-1}$$

you have seen this →61

Thank You

