▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$ full rank (if finite perspective camera)

$\mathbf{K} \in \mathbb{R}^{3,3}$ upper triangular with positive diagonal entries

$\mathbf{R} \in \mathbb{R}^{3,3}$ rotation: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{I} & \mathbf{Q}^{-1}\mathbf{q}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & -\mathbf{R}\mathbf{C}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$ also →36
2. RQ decomposition of $\mathbf{Q} = \mathbf{K}\mathbf{R}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q}\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g. (see next slide for derivation details)

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix} \text{ gives } \begin{matrix} c^2+s^2=1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33}\end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2+q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2+q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{K}\mathbf{R} = \mathbf{K}\mathbf{T}^{-1}\mathbf{T}\mathbf{R}$, where $\mathbf{T} = \operatorname{diag}(-1,-1,1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive

  'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array [q#1,#2 &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q.R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{ c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3} q_{3,2} + q_{1,2} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2} q_{3,2} + q_{1,3} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3} q_{3,2} + q_{2,2} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2} q_{3,2} + q_{2,3} q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

►Center of Projection

Observation: finite $\mathbf{P}$ has a non-trivial right null-space   rank 3 but 4 columns

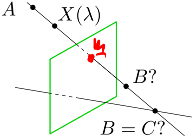
Theorem

Let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equal to the projection center $\underline{\mathbf{C}}$ (homogeneous, in world coordinate frame).

Proof.

1. Consider spatial line $A\,B$ (B is given). We can write

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. it projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + (1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}} \simeq \mathbf{P}\,\underline{\mathbf{A}}$$

- the entire line projects to a single point $\Rightarrow$ it must pass through the optical center of $\mathbf{P}$
- this holds for all choices of A $\Rightarrow$ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \;\Rightarrow\; \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. consider line

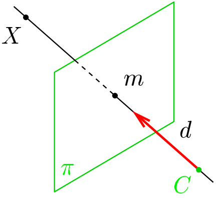
   $\mathbf{d}$ unit line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X} = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the projection of the (finite) point X is   QC + q = 0

$$\underline{\mathbf{m}} \simeq \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$$



…which is also the image of a point at infinity in $\mathbb{P}^3$

- optical ray line corresponding to image point m is the set

$$\mathbf{X} = \mathbf{C} + (\lambda\mathbf{Q})^{-1}\underline{\mathbf{m}}, \qquad \lambda \in \mathbb{R}$$

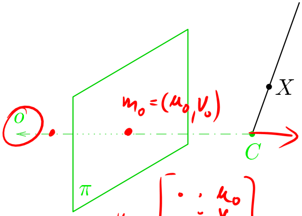
- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π

1. points on a line parallel to π project to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

$$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$$

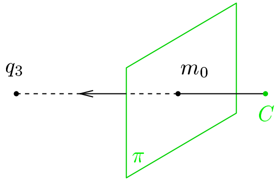
if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ [H&Z, p. 161]

- the axis is expressed in world coordinate frame

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to principal point m_0



3. then

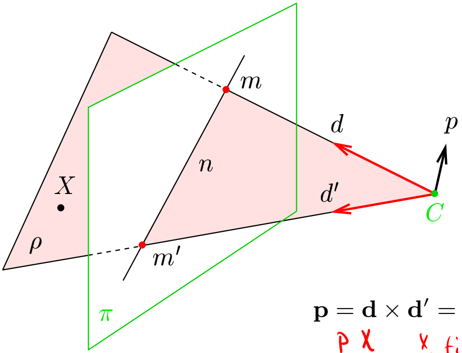
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p passing through optical center C and a given image line n.



optical ray given by m $\quad \mathbf{d} = \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' = \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} = \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

p $\underline{X}$ X finite

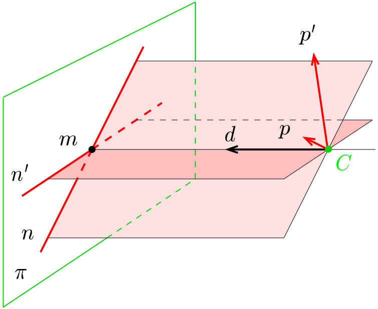
- note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top}\underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\rightarrow 32} = \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \boldsymbol{\rho} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}$

$\rho_1\, x + \rho_2\, y + \rho_3\, z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad \mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad \mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\underline{\mathbf{m}}}$$

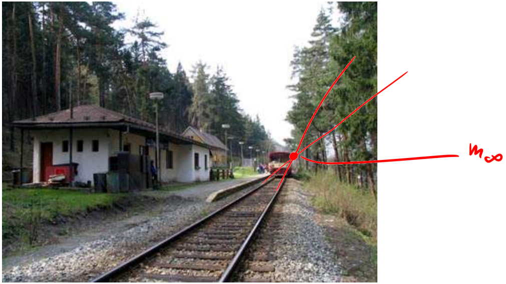
▶Summary: Optical Center, Ray, Axis, Plane

General finite camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14}\\ \mathbf{q}_2^\top & q_{24}\\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \mathrm{rnull}(\mathbf{P})$	optical center (world coords.)
$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$	optical ray direction (world coords.)
$\det(\mathbf{Q})\,\mathbf{q}_3$	outward optical axis (world coords.)
$\mathbf{Q}\,\mathbf{q}_3$	principal point (in image plane)
$\boldsymbol{\rho} = \mathbf{P}^\top\,\underline{\mathbf{n}}$	optical plane (world coords.)
$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0\\ 0 & f/(a\sin\theta) & v_0\\ 0 & 0 & 1\end{bmatrix}$	camera (calibration) matrix (f, u_0, v_0 in pixels)
$\mathbf{R}$	camera rotation matrix (cam coords.)
$\mathbf{t}$	camera translation vector (cam coords.)

What Can We Do with An 'Uncalibrated' Perspective Camera?



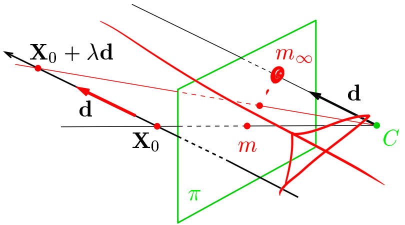
How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, image resolution is too low

We will review some life-saving theory...
...and build a bit of geometric intuition...

▶Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



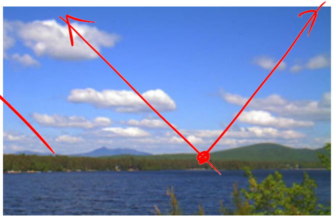
$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots \simeq \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

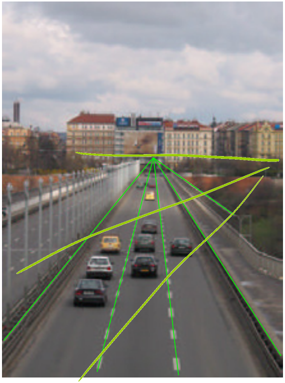
- the V.P. of a spatial line with directional vector $\mathbf{d}$ is $\underline{\mathbf{m}}_\infty \simeq \mathbf{Q}\,\mathbf{d}$
- V.P. is independent on line position $\mathbf{X}_0$, it depends on its directional vector only
- all parallel lines share the same V.P., including the optical ray defined by m_∞

Some Vanishing Point "Applications"

where is the sun?



what is the wind direction?
(must have video)

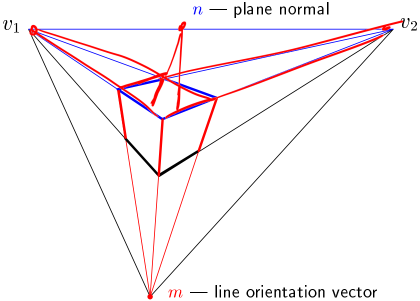


fly above the lane,
at constant altitude!

▶Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane
and in all parallel planes



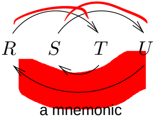
- V.L. n corresponds to spatial plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

  because this is the normal vector of a parallel optical plane (!) →40
- a spatial plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top} \mathbf{p}$.

►Cross Ratio

Four distinct collinear spatial points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{UR}|}\,\frac{|\overrightarrow{SU}|}{|\overrightarrow{TS}|}$$

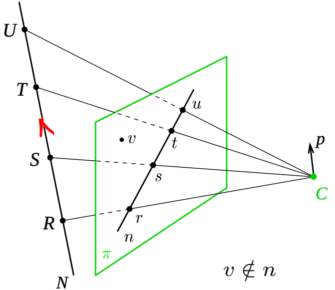


$$|\overleftarrow{RT}| = -|\overrightarrow{RT}|$$

$|\overrightarrow{RT}|$ – signed distance from R to T

(w.r.t. a fixed line orientation)

$$[SRUT] = [RSTU],\ [RSUT] = \frac{1}{[RSTU]},\ [RTSU] = 1 - [RSTU]$$



Obs: $$[RSTU] = \frac{|\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}|}{|\underline{\mathbf{r}}\ \ \underline{\mathbf{u}}\ \ \underline{\mathbf{v}}|} \cdot \frac{|\underline{\mathbf{s}}\ \ \underline{\mathbf{u}}\ \ \underline{\mathbf{v}}|}{|\underline{\mathbf{s}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}|}, \qquad |\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}| = \det\left[\underline{\mathbf{r}}\ \ \underline{\mathbf{t}}\ \ \underline{\mathbf{v}}\right] = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \qquad (1)$$

Corollaries:

- cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will “see” the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points R, S, T, U may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

Thank You