# Solution to Test-01-Examples

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1. Let us have two bases of linear space $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ and $\beta' = [2\,\vec{b}_1 + \vec{b}_2, -\vec{b}_1, \vec{c}]$ Write the coordinates of vector $\vec{c}$ in basis $\beta$, when

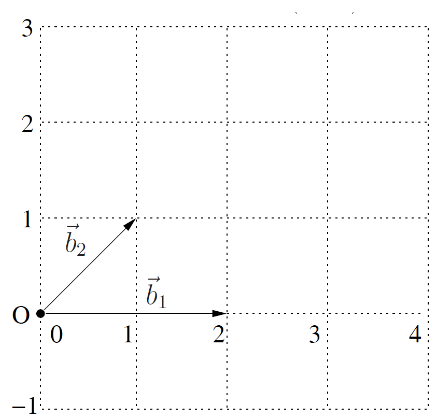
$$\text{for} \quad \vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad \vec{x}_{\beta'} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution

$$\vec{x}_\beta = R\,\vec{x}'_\beta \quad = \quad [\vec{b}'_{1_\beta}, \vec{b}'_{2_\beta}, \vec{b}'_{3_\beta}]\,\vec{x}'_\beta \quad = \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & c_1 \\ 1 & 0 & c_2 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{c}_\beta = \begin{bmatrix} 1/3 & 0 & 1/3 \end{bmatrix}^\top$$

2. The following ilustration shows a coordinate system $\sigma = (O, \beta)$ and a basis $\beta = (\vec{b}_1, \vec{b}_2)$.




(a) i. Find a coordinate system $\sigma' = (O', \beta')$, $\beta' = (\vec{b}'_1, \vec{b}'_2)$, whose basis vector $\vec{b}'_1$ has in basis $\beta$ coordinates

$$\vec{b}'_{1_\beta} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and its origin $O'$ is in the coordinate system $\sigma$ described by vector

$$\vec{O}'_\beta = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

and there exists point $X$ described by vector $\vec{X}$ in $\sigma$ and vector $\vec{X}'$ in $\sigma'$ with coordinates

$$\vec{X}_\beta = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, \quad \vec{X}'_{\beta'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and draw it on the picture.

ii. Write the coordinates of vector $\vec{b}'_2$ in basis $\beta$.

iii. Write the coordinates of vector $O$ in coordinate system $\sigma'$.

iv. Write the coordinates of basis vectors of $\beta$ in basis $\beta'$.

(b) i. Find a coordinate system $\sigma' = (O', \beta')$, $\beta' = (\vec{b}'_1, \vec{b}'_2)$, when you know that the basis vectors of basis $\beta$ have in basis $\beta'$ coordinates

$$\vec{b}_{1\beta'} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad \vec{b}_{2\beta'} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

and there exists point $X$ described by vector $\vec{X}$ in $\sigma$ and vector $\vec{X}'$ in $\sigma'$ with coordinates

$$\vec{X}_\beta = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix}, \quad \vec{X}'_{\beta'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

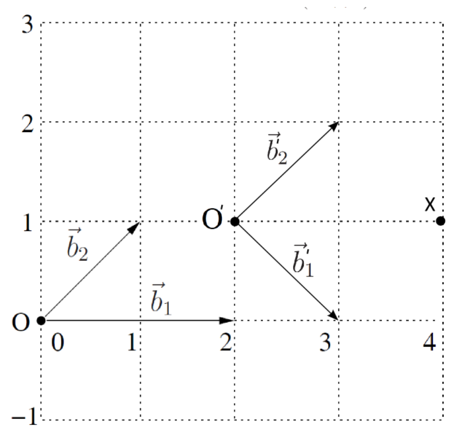
draw the coordinate system on the picture.

ii. Write the coordinates of basis vectors of $\beta'$ in basis $\beta$.

iii. Write the coordinates of point $O$ in the coordinate system $\sigma'$ and point $O'$ in the coordinate system $\sigma$.

Solution

(a) i. Ilustration of coordinates systems $(O, \beta), (O', \beta')$ and a general point $X$:




ii. Coordinates of vector $\vec{b}'_2$ in basis $\beta$:

$$\vec{x}_\beta = R\,\vec{x'}'_\beta + \vec{o'}_\beta \quad = \quad [\vec{b}'_{1_\beta}, \vec{b}'_{2_\beta}]\,\vec{x'}'_\beta + \vec{o'}_\beta \quad = \quad \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \vec{b'}_{2_{\beta_1}} \\ -1 & \vec{b'}_{2_{\beta_2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$\vec{b}'_{2_\beta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

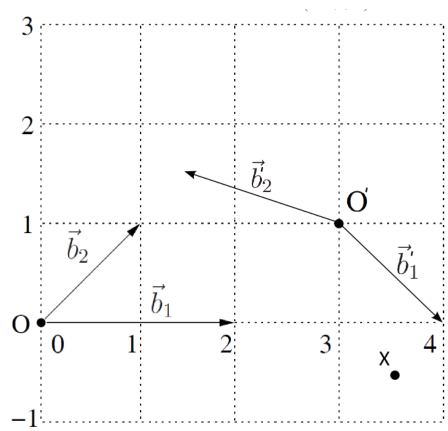
iii. To find coordinates of vector $O$ in coordinate system $\sigma'$ we need to compute vector $\vec{x'}'_\beta$ that is in opposite direction than vector to $\vec{O'}_\beta$.

$$\vec{O}_\beta = \vec{x}_\beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \vec{x}_\beta = R\,\vec{x'}'_\beta + \vec{o'}_\beta \quad \Rightarrow \quad R^{-1}\,\vec{x}_\beta = \vec{x'}'_\beta + R^{-1}\,\vec{o'}_\beta \quad \Rightarrow \quad \vec{x'}'_\beta = R^{-1}\left(\vec{x}_\beta - \vec{o'}_\beta\right)$$

$$\vec{x'}'_\beta = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1/2\\ -1\end{bmatrix} = \begin{bmatrix}-1/2\\ -3/2\end{bmatrix}$$

iv. Coordinates of basis vectors of $\beta$ in basis $\beta'$ equal $R^{-1} = [\vec{b}_{1\beta'}, \vec{b}_{2\beta'}] = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$.

(b) i. Picture with basis and $X$:




ii. It is given transformation $A = [\vec{b}_{1\beta'}, \vec{b}_{2\beta'}] = \begin{bmatrix}-1 & -2\\ -2 & -2\end{bmatrix}$ witch implies:

$$A^{-1} = [\vec{b}'_{1_\beta}, \vec{b}'_{2_\beta}] = \begin{bmatrix}1 & -1\\ -1 & 0.5\end{bmatrix}$$

iii. To find coordinates of vector $O'$ in coordinate system $\sigma$ we use $\vec{x}_\beta = R\,\vec{x'}'_\beta + \vec{o'}_\beta$

$$\vec{x}_\beta = A^{-1}\,\vec{x'}'_\beta + \vec{o'}_\beta \quad = \quad \begin{bmatrix}2\\ -1/2\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 0.5\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} + \begin{bmatrix}\vec{o}'_{\beta_1}\\ \vec{o}'_{\beta_2}\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}\vec{o}'_{\beta_1}\\ \vec{o}'_{\beta_2}\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$$

To find coordinates of vector $O$ in coordinate system $\sigma'$ we need to compute vector $\vec{x'}'_\beta$ that is in opposite direction than vector to $\vec{O'}_\beta$.

$$\vec{O}_\beta = \vec{x}_\beta = \begin{bmatrix}0\\ 0\end{bmatrix}; \quad \vec{x}_\beta = R\,\vec{x'}'_\beta + \vec{o'}_\beta \quad \Rightarrow \quad R^{-1}\,\vec{x}_\beta = \vec{x'}'_\beta + R^{-1}\,\vec{o'}_\beta \quad \Rightarrow \quad \vec{x'}'_\beta = R^{-1}\left(\vec{x}_\beta - \vec{o'}_\beta\right)$$

$$\vec{x'}'_\beta = \begin{bmatrix}-1 & -2\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1\\ -1\end{bmatrix} = \begin{bmatrix}3\\ 4\end{bmatrix}$$

3. Find all solutions to system $\mathtt{A}\,\vec{x} = \vec{b}$ for

$$\mathtt{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathtt{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathtt{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

Solution

First two matrixes have $rank(A) = 2$ witch means that solution will be in form $x_0 + kv$ where $x_0$ is the one particular solution, $k \epsilon \mathbb{R}$, $\vec{v}$ generate the space of solution of the associated homogeneous system. After adding minus the third row to the first row in the first matrix, we will get two equations with three unknowns.

$$x_2 + 2x_3 = 3, \quad x_1 - x_3 = 0$$

We choose parameter $k = x_1$

$$x_1 = k, \quad x_2 = 3 - 2k, \quad x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_0 + kv = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, k \epsilon \mathbb{R}$$

We will solve second matrix in the same way.

$$2x_2 + 2x_3 = 0, \quad x_1 - x_3 = 0$$

$$x_1 = k, \quad x_2 = -k, \quad x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_0 + kv = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, k \epsilon \mathbb{R}$$

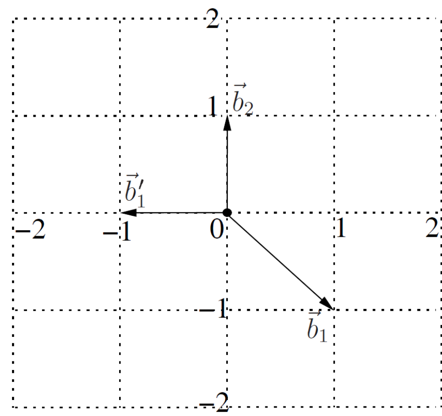
The third matrix has $rank(A) = 1$. The set of solutions is a two dimensional affine subspace represented by $x_0 + k_1 v_1 + k_2 v_2$. Vectors $v_1, v_2$ define the subspace of the associated homogeneous system, $k_1, k_2 \epsilon \mathbb{R}$ and $x_0$ is a particular solution.

$$x_1 + 2x_2 = 3$$

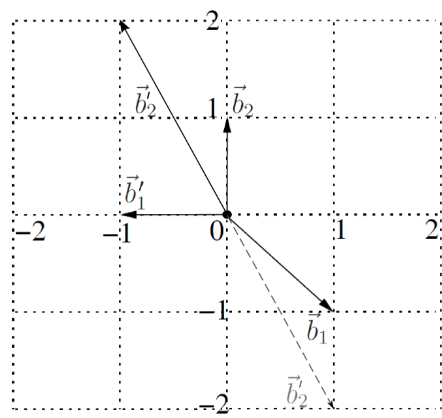
$$x_1 = k_1, \quad x_2 = 3/2 - 1/2\,k_1, \quad x_3 = k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_0 + k_1 v_1 + k_2 v_2 = \begin{bmatrix} 0 \\ 3/2 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k_1, k_2 \epsilon \mathbb{R}$$

4. Draw on the ilustration of basis vectors $\beta = (\vec{b}_1, \vec{b}_2)$ such that the transition matrix from basis $\beta$ to basis $\beta' = (\vec{b}'_1, \vec{b}'_2)$ was orthogonal. Write the matrix.




Solution




We start from equation $\vec{x_\beta} = R\vec{x_{\beta'}}$, where R represent transformation from $\beta'$ to $\beta$. To assure orthogonal transformation from $\beta$ to $\beta'$ we need $R^{-1}$ which is $\lambda R^T$, $\lambda \epsilon \mathbb{R}$ for orthogonal matrices with orthogonal columns of equal size. The solution is choose unknown parameters $R^T$.

$$R = [\vec{b}'_{1_\beta}, \vec{b}'_{2_\beta}] = \begin{bmatrix} -1 & a \\ -1 & b \end{bmatrix}, \quad R^{-1} = \lambda R^T = \lambda \begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}$$

Parameters can be

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors of following matrices and draw them.

$$\mathtt{R} = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{R} = \begin{bmatrix} -6/10 & 0 & -8/10 \\ 0 & -1 & 0 \\ -8/10 & 0 & 6/10 \end{bmatrix}$$

Solution

We obtain eigenvalues from $det(\mathtt{R} - \lambda\mathtt{I}) = 0$ and eigenvectors from $(\mathtt{R} - \lambda\mathtt{I})\, v = 0$.

$$det(\mathtt{R}{-}\lambda\mathtt{I}) = \begin{bmatrix} 3/5-\lambda & -4/5 & 0 \\ 4/5 & 3/5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1{-}\lambda)((3/5{-}\lambda)(3/5{-}\lambda){+}16/25) = (1{-}\lambda)(\lambda^2{-}6/5\lambda{+}1) = 0$$

$$\lambda_{2,3} = \frac{6/5 \pm \sqrt{-64/25}}{2} \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 0.6 + 0.8i, \quad \lambda_3 = 0.6 - 0.8i$$

$$\lambda_1 = 1, \quad (\mathtt{R} - \lambda\mathtt{I})\, v = \begin{bmatrix} -0.4 & -0.8 & 0 \\ 0.8 & -0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1_1} \\ v_{1_2} \\ v_{1_3} \end{bmatrix} = 0$$

$$\Rightarrow \quad v_{1_1} = v_{1_2} = 0, \quad a = v_{1_3}, \quad a\epsilon\mathbb{R}$$

$$\lambda_2 = 0.6 + 0.8i, \quad (\mathtt{R} - \lambda\mathtt{I})\, v = \begin{bmatrix} -0.8i & -0.8 & 0 \\ 0.8 & -0.8i & 0 \\ 0 & 0 & 0.4-0.8i \end{bmatrix} \begin{bmatrix} v_{2_1} \\ v_{2_2} \\ v_{2_3} \end{bmatrix} = 0$$

$$-0.8iv_{2_1} - 0.8v_{2_2} = 0, \quad 0.8v_{2_1} - 0.8iv_{2_2} = 0, \quad v_{2_1} = iv_{2_2}$$

$$\Rightarrow \quad a + ai = v_{2_1}, \quad a - ai = v_{2_2}, \quad v_{2_3} = 0, \quad a\epsilon\mathbb{R}$$

$$\lambda_3 = 0.6 - 0.8i, \quad (\mathtt{R} - \lambda\mathtt{I})\, v = \begin{bmatrix} 0.8i & -0.8 & 0 \\ 0.8 & 0.8i & 0 \\ 0 & 0 & 0.4+0.8i \end{bmatrix} \begin{bmatrix} v_{3_1} \\ v_{3_2} \\ v_{3_3} \end{bmatrix} = 0$$

$$0.8iv_{3_1} - 0.8v_{3_2} = 0, \quad 0.8v_{3_1} + 0.8iv_{3_2} = 0, \quad v_{3_1} = -iv_{3_2}$$

$$\Rightarrow \quad a - ai = v_{3_1}, \quad a + ai = v_{3_2}, \quad v_{3_3} = 0, \quad a\epsilon\mathbb{R}$$

Second matrix we solve the same way.

$$det(\mathtt{R}{-}\lambda\mathtt{I}) = \begin{bmatrix} -6/10-\lambda & 0 & -8/10 \\ 0 & -1-\lambda & 0 \\ -8/10 & 0 & 6/10-\lambda \end{bmatrix} = (-1{-}\lambda)((-0.6{-}\lambda)(-0.6{-}\lambda){-}0.64) = (-1{-}\lambda)(\lambda^2{-}1) = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = -1$$

$$\lambda_1 = 1, \quad (\mathtt{R} - \lambda\mathtt{I})\, v = \begin{bmatrix} -1.6 & 0 & -0.8 \\ 0 & -2 & 0 \\ -0.8 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} v_{1_1} \\ v_{1_2} \\ v_{1_3} \end{bmatrix} = 0$$

$$-1.6v_{1_1} - 0.8v_{1_3} = 0, \quad -2v_{1_2} = 0, \quad -0.8v_{1_1} - 0.4v_{1_3} = 0$$

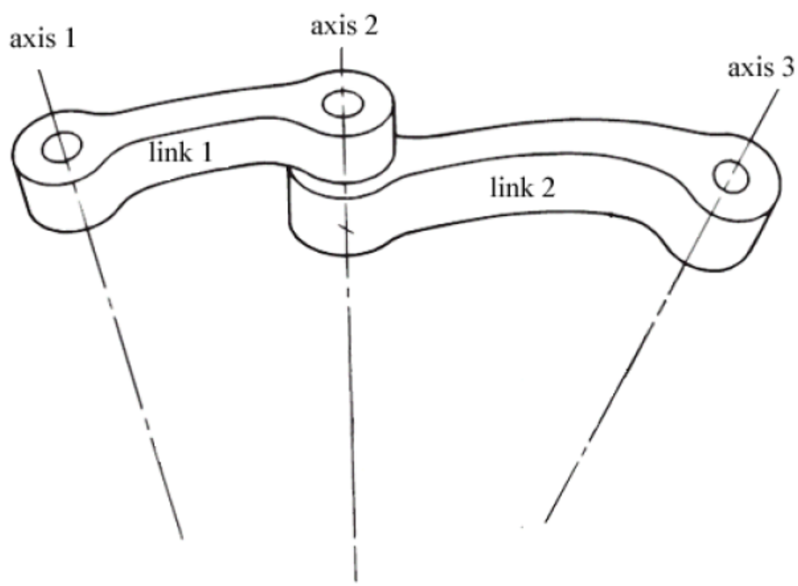
$$\Rightarrow \quad v_{1_1} = -0.5a, \quad v_{1_2} = 0, \quad v_{1_3} = a, \quad a \epsilon \mathbb{R}$$

$$\lambda_{2,3} = -1, \quad (\mathtt{R} - \lambda\mathtt{I})\, v = \begin{bmatrix} 0.4 & 0 & -0.8 \\ 0 & 0 & 0 \\ -0.8 & 0 & 1.6 \end{bmatrix} \begin{bmatrix} v_{2,3_1} \\ v_{2,3_2} \\ v_{2,3_3} \end{bmatrix} = 0$$

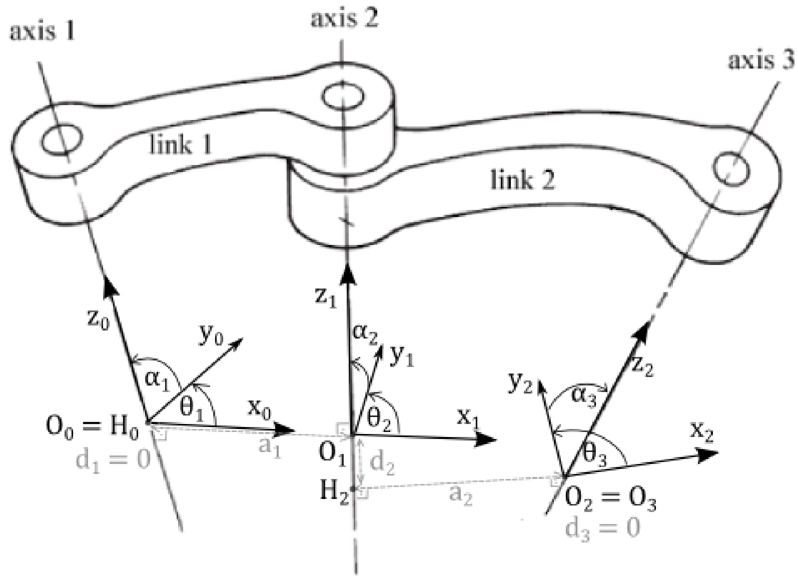
$$0.4v_{2,3_1} - 0.8v_{2,3_3} = 0, \quad -0.8v_{2,3_1} + 1.6v_{2,3_3} = 0$$

$$\Rightarrow \quad v_{2,3_1} = 2a, \quad v_{2,3_2} = b, \quad v_{2,3_3} = a, \quad a, b \,\epsilon\, \mathbb{R}$$

6. Let us have a manipulator with three axis of motion that do not intersect, as shown in the picture below. On the picture
   (a) draw the coordinate systems of objects according to Denavit-Hartenberg notation;
   (b) draw all parameters along with their orientation which are needed in the Denavit-Hartenberg notation.



Solution
Denavit-Hartenberg parameter $d_2$ is not in the picture because $d_2 = 0$.




7. Write some matrix with eigenvalues $1, 2, -1, -3$.

Solution
We can use the formula $trace(A) = \Sigma_i \lambda_i$ for $A \,\epsilon\, \mathbb{R}^{NxN}$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$