Network structure identification

Network Application Diagnostics
B2M32DSA

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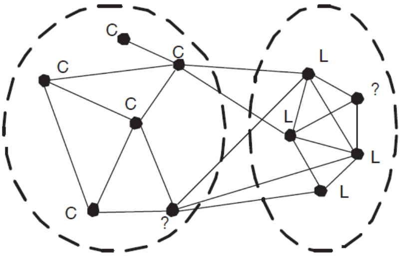
Outline

1. Node Influence
   - Node Roles
   - Hubs and Authorities
2. Network Clustering
   - Multidimensional Clustering
   - Clustering Coefficients
   - Graph Laplacian

Node Roles and Community Structure [STE07]



pre-dicting the political leaning of a person, either conservative (C) or liberal (L)

- *influence maximization* can be thought of as finding the best k people to target in order to maximize the number of people that will eventually be influenced
- *link-based classification* is the task of categorizing nodes using the node features and its link information

Community Metric - Prerequisities [STE07]

- *Assumption*: a community is defined by a clique (maximal complete subgraph) in a network.
- **rawComm** is to be an approximate measure of the number of communities to which a node is attached.
- *incomplete edge* . . . an edge that connects two nodes in different communities
- *impure edge* . . . a non-link that appears within community.
- p . . . the probability that two linked nodes are in the same community

$$p = \frac{\text{Complete node pairs}}{\text{Total linked node pairs}}$$

- q . . . the probability that two non-linked nodes are in different communities.

$$q = \frac{\text{Pure node pairs}}{\text{Total non-linked node pairs}}$$

Community Metric [STE07]

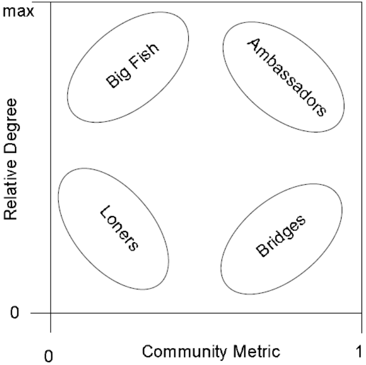
- **rawComm** metric

$$\text{rawComm} = \sum_{v \in N(u)} \tau_u(v)$$

where

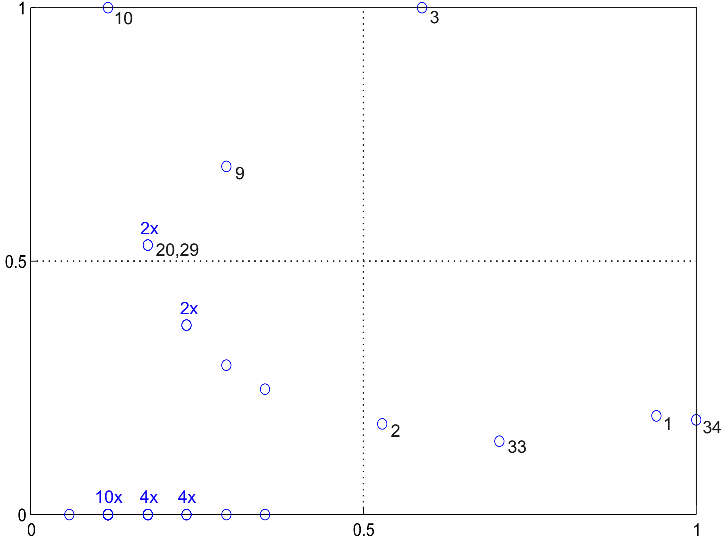
- $N(u)$ ... the neighborhood of u, that is all of the nodes that are directly linked to u
- $$\tau_u(v) = \frac{1}{1 + \sum_{v_j \in N(u)} I(v_i, v_j) \cdot p + \bar{I}(v_i, v_j) \cdot (1 - q)}$$
- $I(x, y)$ ... an indicator function that is 1 if there is a link between x and y and 0 otherwise. $\bar{I}$ is 1 if there is not a link and 0 otherwise.
- The denominator in the definition of τ is the expected number of other nodes in u's neighborhood are in a community with v_i.
  - The 1 represents the node v_i itself.
  - $I(v_i, v_j) \cdot p + \bar{I}(v_i, v_j) \cdot (1 - q)$ ... the probability of v_i and v_j being in the same community.

Community-based Node Roles [STE07]

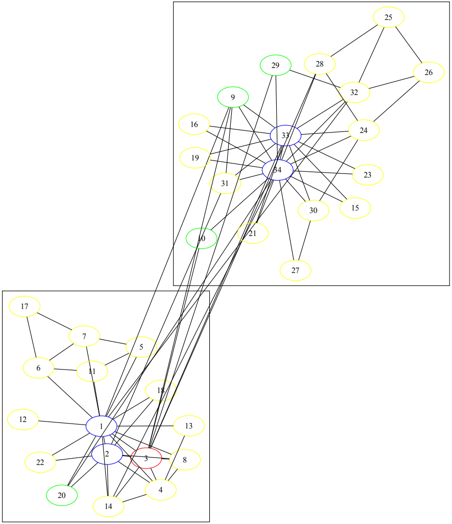


- a *authority* how much knowledge, information, etc. held by a node on a topic.
- a *hub* how well a node 'knows' where to find information on a given topic.
- an *ambassador* has links to many nodes from different communities
- a *big fish* has links only to other nodes in the same community
- a *bridge* because they serve as bridges between a small number of communities
- the *loners* . . . those with a low relative degree and low community

Community-based Node Roles - Karate Club



Community-based Node Roles - Karate Club



Hubs and Authorities I [Kle98, Kle99, New10]

- Proposed by Kleinberg in 1998 [Kle98]
- In some networks it is appropriate also to accord a vertex high centrality if it points to others with high centrality
  - papers and their reviews
  - it makes sense only in directed networks (direction)
- **Authorities** are nodes that contain useful information on a topic of interest.
- **Hubs** are nodes that tell us where the best authorities are to be found.
- The centrality algorithm is called **hyperlink-induced topic search** or **HITS**
- Each vertex i has an authority centrality x_i and a hub centrality y_i

Hubs and Authorities II [Kle98, Kle99, New10]

- The authority centrality of a vertex is defined to be proportional to the sum of the hub centralities of the vertices that point to it:

$$x_i = \alpha \sum_j A_{ij} y_j$$

  where α is a constant.
- The hub centrality of a vertex is proportional to the sum of the authority centralities of the vertices it points to:

$$y_i = \beta \sum_j A_{ji} x_j$$

  where β is a constant.
- In matrix terms

$$\mathbf{x} = \alpha \mathbf{A}\mathbf{y} \qquad \mathbf{y} = \beta \mathbf{A}\mathbf{x}$$

- Combining both and setting $\gamma = (\alpha\beta)^{-1}$

$$\mathbf{A}\mathbf{A}^T\mathbf{x} = \gamma \mathbf{x} \qquad \mathbf{A}^T\mathbf{A}\mathbf{y} = \gamma \mathbf{y}$$

Hubs and Authorities III [Kle98, Kle99, New10]

$$\mathbf{A}\mathbf{A}^T\mathbf{x} = \gamma\mathbf{x} \qquad \mathbf{A}^T\mathbf{A}\mathbf{y} = \gamma\mathbf{y}$$

- the authority and hub centralities are respectively given by eigenvectors of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ with the same eigenvalue γ.
  - The same eigenvalue γ?

$$\mathbf{A}\mathbf{A}^T\mathbf{x} = \gamma\mathbf{x} \qquad \ldots|\mathbf{A}^T\times \tag{1}$$

$$\mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{x}) = \gamma(\mathbf{A}^T\mathbf{x}) \tag{2}$$

$$\tag{3}$$

- The relation between both centralities

$$\mathbf{y} = \mathbf{A}^T\mathbf{x}$$

- $\mathbf{A}\mathbf{A}^T$ is the cocitation matrix.
- $\mathbf{A}^T\mathbf{A}$ is the bibliographic coupling matrix.
- HITS does not struggle with zero centralities outside strong components.

Data Clustering [XW05, EK10]

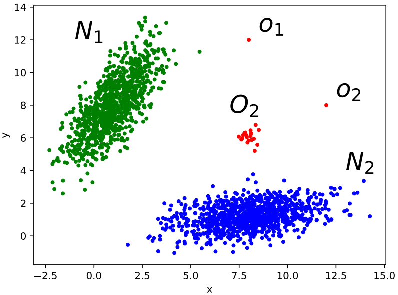
- Data are **classified** or grouped into a set of categories or clusters.
  - **supervised** ...with a teacher
    - a finite set of class/category labels/tags is provided
  - **unsupervised** ...without a teacher
    - based on similarities of objects
- A **cluster** is a collection of objects that are similar to each other using some attribute.
- A cluster of objects can be treated as a group.
- Let $P = \{p_1, \ldots, p_N\}$ be a set of N data points representing N objects.
- The goal of **clustering** (CZ shlukování) is to divide P into K groups $C_1, \ldots, C_K$ so that data belonging to a group are more similar to each other than data from different groups.
- Each C_i is called a **cluster** (CZ shluk).
- Each object p_j is described by a vector $\mathbf{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jd})^T \in \mathbb{R}^d$ and each measure x_{ji} is called to be a **feature** (attribute, dimension, or variable) (CZ příznak).

Input Data [Agg17]

- Record data
  - Univariate
  - Multivariate
- Attributes
  - Binary/Boolean
  - Categorical
  - Continuous
  - Hybrid
- Relations
  - Sequential
    - Temporal
  - Spatial
  - Spatio-temporal
  - **Long range correlations**
  - **Graph**
- Data Quality
  - Data Fusion
  - Data Cleansing
  - Consistency maintenance
- Processing
  - Online/Offline processing
  - Distributed processing
  - **Analysis** × **Production**
    - Feature/Property searching/selection
    - Selected features detection
- Data Volume
  - Dense/**Sparse**
  - Low/High dimensions
  - Low/**Large volumes**
  - Big data
  - Internet of Things

Simple Example - Multidimensional Space [CBK09]

- N_1 and N_2 are regions of "normal" behavior
- Points o_1 and o_2 are anomalies
- Points in region O_3 are anomalies

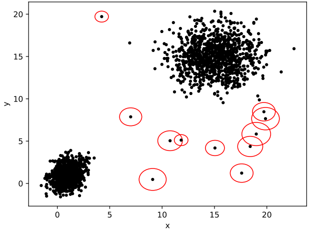


Normal behavior

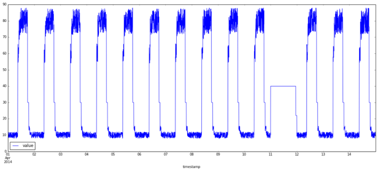
- **Normal distribution** . . . $N(\mu, \sigma)$.
  Further, it will be referred as Gaussian distribution
- **Normal behavior/pattern** . . . it is expected, not anomalous.

Anomalies? Outliers?

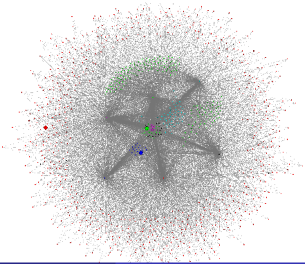
Clouds of points (multi-dimensional)



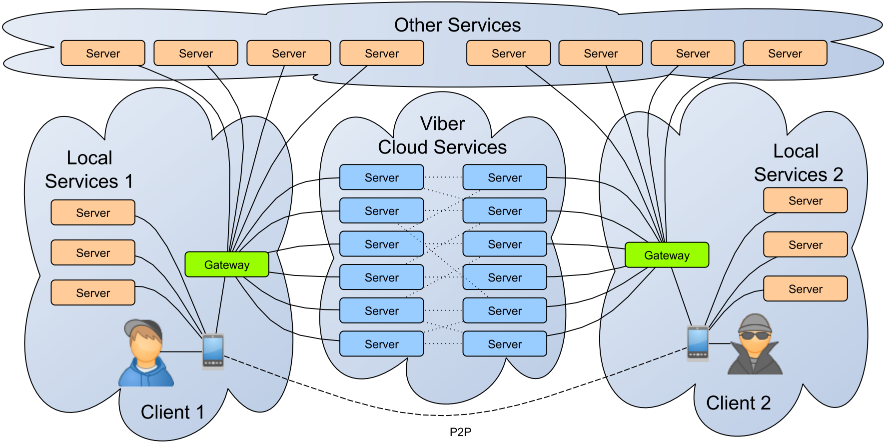
Anomaly



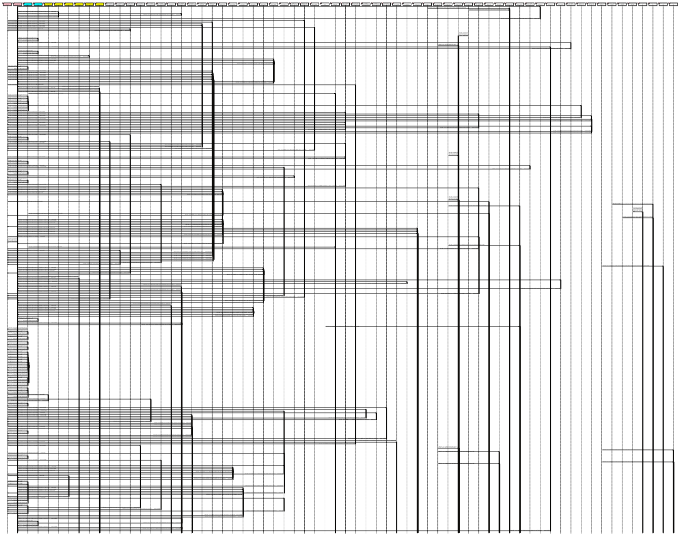
Complex Network



Exemplar (Viber) Environment [MBKK15]



Example Capture Characteristics - Message Sequences [MBKK15]

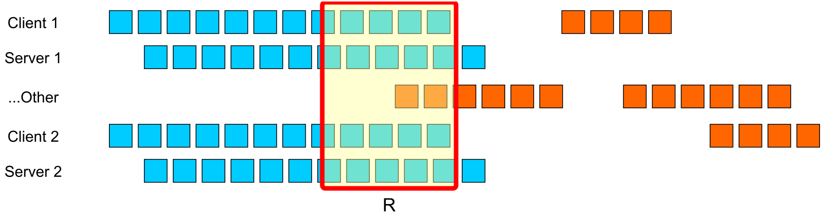


- 138882 PCAP blocks
- 1788 transport sessions
- 2 clients
- 22 viber.com servers
- 150 peers of 2 clients
- 5660 possible concurrent sessions
- How to analyze?

Concurrent Communication Detection [MBKK15]

Selection of IP nodes

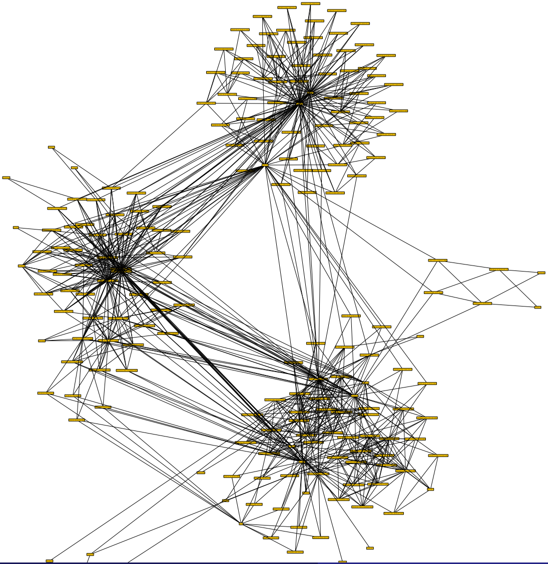
- *viber.com* servers → viber clients → other Viber servers
- Classified based on entropy based characteristics of TCP/IP distributions



$$s(a,b) = \frac{\sum_{\forall i,j: t_a[i]-t_b[j]<R} R/(t_a[i]-t_b[j])}{\sum_{\forall i,j: t_a[i]-t_b[j]<R} 1}$$

In our experiments: $R = 50ms, \quad s(a,b) > 0.001$

UDP Packet Sequence Concurrency as a Complex Network [MBKK15]



- Captures with two clients
- **Communities** of concurrent sessions
- Some clusters related to only one client
- Interesting clusters consist of nodes of **both** clients

Hierarchical or Partitional Clustering [XW05, EK10, eHS09]

- **Partitional clustering** (CZ rozkladové shlukování)
  - the objects are divided into non-overlapping, unnested, clusters
  - Given a set of input patterns $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$
  - It attempts to seek a K-partition of $\mathbf{X}$, $C = \{C_1, \ldots, C_K\}$, $(K \leq N)$ so that
    1. $C_i \neq \emptyset, i = 1, \ldots, K$;
    2. $\bigcup_{i=1}^{K} C_i = \mathbf{X}$;
    3. $C_i \cap C_j = \emptyset; i, j = 1, \ldots, K$ and $i \neq j$.
- **Hierarchical clustering** (CZ hierarchické shlukování)
  - the clusters are nested and can be displayed as a tree
  - It attempts to construct a tree-like nested structure partition of $\mathbf{X}$, $H = \{H_1, \ldots, H_Q\}$, $(Q \leq N)$ so that

$$C_i \in H_m, C_j \in H_\ell, \text{ and } m > \ell \implies C_i \subset C_j \text{ or } C_i \cap C_j = \emptyset \quad (4)$$

$$\text{for all } i, j \neq i, m, l = 1, \ldots, Q \quad (5)$$

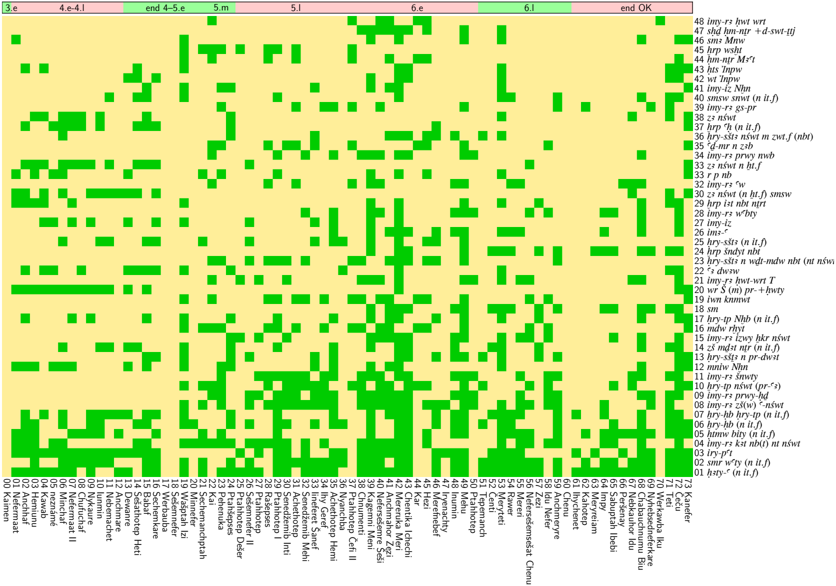
  - The tree-like partition can visualizes as a **dendrogram**

Input Data - The Old Kingdom of Egypt [MD15]

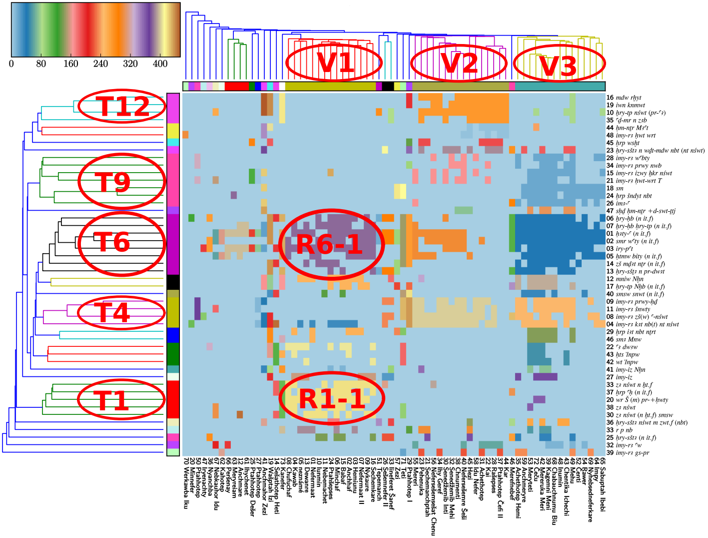


- Continuous . . . tomb dimensions
- Categorical . . . titles
- Binary, boolean . . . titles
- Multivariate . . . people, titles, tombs
- Temporal . . . dynasties, king reigns
- Spatio-temporal . . . location of tombs in time

Titles of Viziers [DMBC17]



Titles of Viziers - Jaccard, Single Linkage Clustering [DMBC17, JD88]



Clustering [XW05, EK10]

- The steps of **cluster analysis**:
  1. **Feature selection and extraction** selects distinguishing features from a set of candidates.
  2. **Clustering algorithm design or selection** in which a proximity measure, a criterion function and an algorithm is determined.
  3. **Cluster validation** is performed to provide the users with a degree of confidence that the clustering results make sense.
  4. **Results interpretation** in which experts in the relevant fields interpret the data partition.
- The assigned **membership** of the nodes in the resulting clusters:
  - **Disjoint clusters** . . . each node is a member of exactly one cluster.
  - **Overlapping clusters** . . . a node may be a member of more than one cluster.
  - **Fuzzy clustering** methods assign a membership weight between 0 and 1 to each node such that
    1 means absolute membership,
    0 means a non-member.

K-Means Clustering [For65, Mac67, Har75, HW79, Llo06, XW05, EK10]

- Proposed by Lloyd in 1957 and published in 1982 [Llo06] and by Forgy in 1965 [For65]
- The term "k-means" was first used by MacQueen in 1967 [Mac67]
- A partitional clusterig
- Given a set of observation $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$
- The aim to minimize the within-cluster sum of squares (WCSSE)

$$WCSSE = \underset{C}{\operatorname{argmin}} \sum_{i=1}^{K} |C_i| Var(C_i) = \underset{C}{\operatorname{argmin}} \sum_{i=1}^{K} \sum_{\mathbf{x} \in C_i} \parallel \mathbf{x} - \boldsymbol{\mu}_i \parallel^2$$

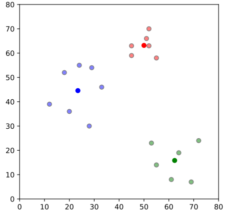
- The mean $\boldsymbol{\mu}_i$ of the data points within the cluster C_i:

$$\boldsymbol{\mu}_i = \frac{1}{n_i} \sum_{\mathbf{x} \in C_i} \mathbf{x}$$

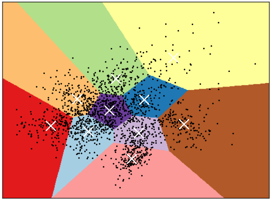
K-Means Algorithm [HW79, EK10]

K-means Algorithm Basic Structure

1: **Input:** $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$
2: **Input:** K ▷ a given number of clusters
3: **Output:** K clusters
4: **select** K points as the initial centroids
5: **repeat**
6: **assign** each data point to its closest centroid and form clusters
7: **compute** the centroid for each cluster
8: **until** centroids do not change significantly



K-means clustering on the digits dataset (PCA-reduced data)
Centroids are marked with white cross



[PVG$^+$11]

Summary of Approaches [New06, Weh13, CRTV07, HK13]

- The **density** of graph is the proportion of present lines to the maximum possible number of lines.
- **Clustering coefficient** is a measure of the degree to which nodes in a graph tend to cluster together

Global clustering coefficient [HK13]

the ratio of the total number of triangles to the total number of connected triplets.

$$C_g = \frac{2\sum_{i=1}^{N} \ell_i}{\sum_{i=1}^{N} d_i(d_i - 1)}$$

- **Modularity** . . . is - up to a normalization constant - the number of edges within communities c minus those for a **null model**
  - *"A good division of a network into communities is not merely one in which there are few edges between communities; it is one in which there are fewer than expected edges between communities".*

Clustering, Triplets and Triangles [CRTV07, HK13]

Clustering coefficient

- a measure of the degree to which nodes in a graph tend to cluster together
- a group of nodes with a relatively high density of ties
- ℓ_i . . . the number of edges between neighbors of v_i

A triplet

- **open triangle** . . . three nodes connected by two undirected ties
- of nodes (v_j, v_i, v_k) is called **connected** if v_j is connected to v_i, v_i is connected to v_k, and $j < k$. Formally, if $A_{ji} = 1, A_{ik} = 1$, and $j < k$.

A triangle

- **closed triangle** . . . a triplet connected by three undirected ties
- is a connected triplet (v_j, v_i, v_k) in which v_j and v_k are connected. Formally, if $A_{jk} = 1$.

Numbers of Triplets and Triangles [CRTV07, HK13]

For a specific node v_i

- a triplet . . . if $j < k$ and $A_{ji}A_{ik} = 1$
- a triangle . . . if $j < k$ and $A_{ji}A_{ik}A_{jk} = 1$
- a number of connected triplets

$$N_3(i) = \sum_{j<k} A_{ji}A_{ik} = d_i(d_i - 1)/2$$

  - a number of choices how edges incident to v_i can be combined.
- a number triangles

$$N_{\triangle}(i) = \sum_{j<k} A_{ji}A_{ik}A_{jk}$$

  - the number of edges between neighbors of v_i

Clustering coefficient [CRTV07, HK13]

Transitivity, Transitivity Ratio [CRTV07]

$$C = \frac{3N_\triangle}{N_3}$$

Clustering Coefficient, Local Clustering Coefficient [CRTV07, HK13]

$$C_i = \frac{N_\triangle(i)}{N_3(i)} = \frac{2\ell_i}{d_i(d_i - 1)}$$

Network average clustering coefficient [HK13]

$$C_\ell = 1/n \sum_{i=1}^{N} C_i$$

Global clustering coefficient [HK13]

the ratio of the total number of triangles to the total number of connected triplets.

$$C_g = \frac{2\sum_{i=1}^{N} \ell_i}{\sum_{i=1}^{N} d_i(d_i - 1)}$$

Diffusion Equation [Cra75]

- The **diffusion equation** is a partial differential equation.
- In physics, it describes the behavior of the *collective motion* of micro-*particles* in a material resulting from the random movement of each micro-particle.

$$\frac{\partial \phi(\mathbf{r},t)}{\partial t} = \nabla \cdot [D(\phi,\mathbf{r})\ \nabla \phi(\mathbf{r},t)]$$

- where $\phi(\mathbf{r},t)$ is the density of the diffusing material
- at location $\mathbf{r}$ and time t and
- $D(\phi,\mathbf{r})$ is the collective diffusion coefficient for density ϕ at location $\mathbf{r}$; and
- ∇ represent the vector differential operator del (nabla).
- If D is constant, then the equation reduces to the linear differential equation (the **heat equation**), ∇^2 is the Laplacian operator:

$$\frac{\partial \phi(\mathbf{r},t)}{\partial t} = D\nabla^2 \phi(\mathbf{r},t) \qquad \Delta = \nabla^2 = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$$

Diffusion on Networks [New10]

- **Diffusion process** as a simple model of spread across a network
  - the spread of an idea,
  - the spread of a disease/virus
- An amount ψ_i of some commodity or substance at vertex i.
- The commodity flows from vertex i to an adjacent one i at a rate $C(\psi_j - \psi_i)$
- where C is a constant called the **diffusion constant**.
- The rate at which ψ_i is changing is given by

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t} = C \sum_j A_{ij}(\psi_j - \psi_i)$$

- The equation works for both undirected and directed networks.
- Let us assume an undirected and simple network.

Diffusion Matrix Form Derivation [New10]

- Network diffusion equation

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t} = C \sum_j A_{ij}(\psi_j - \psi_i)$$

- Splitting, rewriting, merging

$$\begin{aligned}\frac{\mathrm{d}\psi_i}{\mathrm{d}t} &= C \sum_j A_{ij}\psi_j - C\psi_i \sum_j A_{ij} \\ &= C \sum_j A_{ij}\psi_j - C\psi_i k_i \\ &= C \sum_j (A_{ij} - \delta_{ij} k_i)\psi_j\end{aligned}$$

- where $k_i = \sum_j A_{ij}$ is the degree of vertex i and
- δ_{ij} is the Kronecker delta.

Diffusion Matrix Form [New10]

- Network diffusion equation

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t} = C \sum_j (A_{ij} - \delta_{ij} k_i) \psi_j$$

- In matrix form

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t} = C(\mathbf{A} - \mathbf{D})\boldsymbol{\psi}$$

- where $\boldsymbol{\psi}$ is the vector whose components are amounts ψ_i,
- $\mathbf{A}$ is the adjacency matrix, and
- $\mathbf{D}$ is the diagonal matrix with the vertex degrees along the diagonal

$$\mathbf{D} = \begin{pmatrix} k_1 & 0 & 0 & \cdots \\ 0 & k_2 & 0 & \cdots \\ 0 & 0 & k_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Graph Laplacian [New10, EK10]

- Network diffusion equation

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t} = C(\mathbf{A} - \mathbf{D})\boldsymbol{\psi}$$

- In matrix form
- where $\boldsymbol{\psi}$ is the vector whose components are amounts ψ_i,
- $\mathbf{A}$ is the adjacency matrix, and
- The **graph Laplacian** is the real symmetric matrix

$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

$$L_{ij} = \begin{cases} k_i & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and there is an edge between vertices } j \text{ and } i, \\ 0 & \text{otherwise} \end{cases}$$

$$L_{ij} = \delta_{ij} k_i - A_{ij}$$

Diffusion Equation Solution [New10]

- Network diffusion equation

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t} = C(\mathbf{A} - \mathbf{D})\boldsymbol{\psi} \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}t} = -C\mathbf{L}\boldsymbol{\psi}$$

- Assuming the vector $\boldsymbol{\psi}$ as a linear combination of the Laplacian eigenvectors $\mathbf{v_i}$, i.e. $\mathbf{L}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$(real,sym $\mathbf{L}$)

$$\boldsymbol{\psi}(t) = \sum_i a_i(t)\mathbf{v}_i$$

- with the coefficients $a_i(t)$ varying over time.
- By the substitution into the diffusion equation and the dot product with $\mathbf{v}_j$

$$\sum_i \left(\frac{\mathrm{d}a_i}{\mathrm{d}t} + C\lambda_i a_i\right)\mathbf{v_i} = 0 \quad \xRightarrow{\cdot \mathbf{v}_j} \quad \frac{\mathrm{d}a_i}{\mathrm{d}t} + C\lambda_i a_i = 0$$

- The solution

$$a_i(t) = a_i(0)\mathrm{e}^{-C\lambda_i t}$$

Eigenvalues of the Graph Laplacian [New10]

- Let $\mathbf{B}$ be the edge incidence matrix (i.e. edges $\times$ vertices)
  - If $i \neq j$ then $\sum_k B_{ki} B_{kj}$ is -1 if there is an edge between vertices i and j, and zero otherwise.
  - If $i = j$ then $\sum_k B_{ki}^2$ has a term $+1$ for every edge connected to vertex i so that the sum is equal to the degree k_i of vertex i .

$$L_{ij} = \sum_k B_{ki} B_{kj} \qquad \Rightarrow \qquad \mathbf{L} = \mathbf{B}^T \mathbf{B}$$

- Let $\mathbf{v}_i$ be an eigenvector of $\mathbf{L}$ with eigenvalue λ_i, i.e. $\mathbf{L}\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

$$\mathbf{v}_i^T \mathbf{B}^T \mathbf{B} \mathbf{v}_i = \mathbf{v}_i^T \mathbf{L} \mathbf{v}_i = \lambda_i \mathbf{v}_i^T \mathbf{v}_i = \lambda_i \qquad \text{as } \mathbf{v}_i^T \mathbf{v}_i = 1$$

- It is just an inner product of a real vector $(\mathbf{B}\mathbf{v}_i)$ with itself.
  - $\implies$ It is a sum of squares $\implies$ the sum $\geq 0 \implies \lambda_i \geq 0$.
  - The solution of the diff. equation contains only decaying exponentials.
  - The solution tends to an equilibrium value as $t \to \infty$.
- $\mathbf{L} \cdot \mathbf{1} = \mathbf{0} \Leftrightarrow \sum_j L_{ij} \times 1 = \sum_j (\delta_{ij} k_i - A_{ij}) = k_i - \sum_j A_{ij} = k_i - k_i = 0$
  - $\mathbf{1}$ is always an eigenvector of $\mathbf{L}$ with the smallest eigenvalue $\lambda_1 = 0$
  - $\implies$ $\mathbf{L}$ is singular, the Laplacian has no inverse.

Eigenvalues of the Graph Laplacian [New10]

- Let $\mathbf{B}$ be the edge incidence matrix (i.e. edges $\times$ vertices)
  - If $i \neq j$ then $\sum_k B_{ki} B_{kj}$ is -1 if there is an edge between vertices i and j, and zero otherwise.
  - If $i = j$ then $\sum_k B_{ki}^2$ has a term $+1$ for every edge connected to vertex i so that the sum is equal to the degree k_i of vertex i .

$$L_{ij} = \sum_k B_{ki} B_{kj} \qquad \Rightarrow \qquad \mathbf{L} = \mathbf{B}^T \mathbf{B}$$

- Let $\mathbf{v}_i$ be an eigenvector of $\mathbf{L}$ with eigenvalue λ_i, i.e. $\mathbf{L}\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

$$\mathbf{v}_i^T \mathbf{B}^T \mathbf{B} \mathbf{v}_i = \mathbf{v}_i^T \mathbf{L} \mathbf{v}_i = \lambda_i \mathbf{v}_i^T \mathbf{v}_i = \lambda_i \qquad \text{as } \mathbf{v}_i^T \mathbf{v}_i = 1$$

- It is just an inner product of a real vector $(\mathbf{B}\mathbf{v}_i)$ with itself.
  - $\implies$ It is a sum of squares $\implies$ the sum ≥ 0 $\implies$ $\lambda_i \geq 0$.
  - The solution of the diff. equation contains only decaying exponentials.
  - The solution tends to an equilibrium value as $t \to \infty$.
- $\mathbf{L} \cdot \mathbf{1} = \mathbf{0} \Leftrightarrow \sum_j L_{ij} \times 1 = \sum_j (\delta_{ij} k_i - A_{ij}) = k_i - \sum_j A_{ij} = k_i - k_i = 0$
  - $\mathbf{1}$ is always an eigenvector of $\mathbf{L}$ with the smallest eigenvalue $\lambda_1 = 0$
  - $\implies$ $\mathbf{L}$ is singular, the Laplacian has no inverse.

Algebraic Connectivity [New10]

$$\mathbf{L} = \left(\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{L_1}} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{L_2}} & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\ddots$}} \\ 0 & 0 & 0 & 0 & & \end{array}\right)$$

$$\mathbf{v} = (\underbrace{1, 1, 1, \ldots}_{n_1 \text{ ones}}, \underbrace{0, 0, 0, \ldots}_{\text{zeros}})$$

- Suppose we have a network that is divided up into c different components of sizes $n_1, n_2, \ldots, n_c$.
- $\Longrightarrow$ at least c eigenvectors with eigenvalue zero
- The number of zero eigevalues is always exactly equal to the number of components.
- The second eigenvalue λ_2 is non-zero if and only if the network is connected, it is called the **algebraic connectivity**.

Summary

- Node roles
- Hubs and Authorities (HITS)
- Data Clustering
  - Introduction and examples
  - K-means clustering
- Clustering,, triplets, and triangles
- Diffusion Equation
  - Graph Laplacian

Competencies

- What are the basic roles of nodes?
- How is it possible to assess a role of a given nodes?
- Provide definitions of authorities and hubs.
- How are the hub and authority centralities defined?
- What is the goal of clustering?
- What are the two fundamental approaches to data clustering?
- What are the typical steps of a cluster analysis?
- What ate the basic forms of node memberships in clusters?
- Describe k-means clustering.
- Define a triplet and triangle.
- Describe a diffusion equation.
- What is the graph Laplacian?
- Name basic properties of the graph Laplacian eigenvalues?

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