Social choice

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Social choice introduction

- Designer perspective
- How to evaluate the preferences of a population to reflect their wishes?

Formalism

- $\mathcal{N} = \{1, ..., n\}$ is a set of agents
- $\mathcal{U}$ set of choices (outcomes)
- Complete transitive preference relation $\succeq_i$ for every agent i
- $\mathcal{R}(\mathcal{U})$ set of all preference relations over $\mathcal{U}$. We call $\mathcal{R}(\mathcal{U})^n$ a preference profile.
- Social choice function (SCF) is a function $f : \mathcal{R}(\mathcal{U})^n \times \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{U})$ and $f(R, A) \subseteq A$. It takes a preference profile and some subset of choices and chooses a subset of choices.
- Social welfare function (SWF) is a function $f : \mathcal{R}(\mathcal{U})^n \to \mathcal{R}(\mathcal{U})$. It takes a preference profile and chooses one preference relation.

Voting rule

- Special case of social choice function
- Voting rule is a function $f : \mathcal{R}(\mathcal{U})^n \to \mathcal{F}(\mathcal{U})$
- We say that that a voting rule is resolute if $|f(\mathcal{R})| = 1$

Scoring rules

- A scoring vector $s = (s_1, ..., s_{|U|})$, $s_1 \geq ... \geq s_{|U|}$ and $s_1 > s_{|U|}$, describes the number of points an alternative should obtain based on its position in the preference relation of every agent
- The outcomes with the highest cumulative score are chosen
- Plurality rule
  - Only the most preferred option gets 1 point
- Borda's rule
  - An alternative get k points if agent prefers it to k other alternatives
- Example

Condorcet winner

- An alternative is a Condorcet winner if, when compared with every other candidate, is preferred by more voters.
- It is unique, but does not always exists (example)
- Does Borda's or Plurality guarantee to find condorcet winner if one exists? (example)

Condorcet extension

- A voting rule that selects the condorcet extension whenever it exists
- Copeland's rule
  - An alternative gets point for every pairwise majority win, and a value in $(0, 1)$ for every majority tie.
- Maximin rule
  - Evaluate every alternative by its worst pairwise defeat count by another alternative; the winners are those who lose by the lowest margin in their worst pairwise defeats. (If there are any alternatives that have no pairwise defeats, then they win.)

Condorcet's paradox

- There exist scenarios, where no matter what you choose the majority will be unhappy (the condorcet winner does not exist).

Other rules

- Single transferable vote: looks for the alternatives that are ranked in first place the least often, removes them from all voters ballots, and repeats. The alternatives removed in the last round win.
- Pairwise elimination: pairwise comparison of alternatives, the winning alternative continues to another comparison (sensitive to the order, dominated outcomes can be the result)

Example

- Plurality rule: Only the most preferred option gets 1 point
- Borda's rule: An alternative get k points if agent prefers it to k other alternatives
- Copeland's rule: An alternative gets point for every pairwise majority win, and a value in $(0, 1)$ for every majority tie.
- Maximin rule: Evaluate every alternative by its worst pairwise defeat count by another alternative; the winners are those who lose by the lowest margin in their worst pairwise defeats.
- Single transferable vote: looks for the alternatives that are ranked in first place the least often, removes them from all voters ballots, and repeats. The alternatives removed in the last round wins.
- Pairwise elimination: pairwise comparison of alternatives, the winning alternative continues to another comparison

Manipulation

- There are situations, where the voters might be incentivized to report different preferences then their true ones (example)
- Why to avoid it?
  - Fairness issue (agents stating their true preference are punished)
  - Resources wasted on determining the manipulation strategy

The Gibbard-Satterthwaite Impossibility

- We assume f to be resolute (it always return one outcome), furthermore i knows the preferences of other agents
- We say that f is manipulable by voter i if there exist preference profiles $\mathcal{R}$ and $\mathcal{R}'$ such that $\mathcal{R}_j = \mathcal{R}'_j$ for all $j \neq i$ and $f(\mathcal{R}') \succ_i f(\mathcal{R})$ (i has incentive to vote differently than according to his true preference). A voting rule is strategyproof if it is not manipulable.
- f is non-dictatorial if there is no voter i such that for all preference profiles $\mathcal{R}$, $a \in f(\mathcal{R})$ a is voter i's most preferred alternative
- f is non-imposing if it can return as an outcome every alternative from $\mathcal{U}$.

The Gibbard-Satterthwaite Impossibility

- Every non-imposing, strategyproof, resolute voting rule is dictatorial when $|U| \geq 3$.
- Very negative result
- In practice we meet mostly non-imposing, non-dictatorial and resolute voting rules, therefore manipulable
- Workarounds are restricted domains (single peaked preferences)

Computational hardness of manipulation

- Manipulation is possible, but how difficult is it to manipulate?
- Even though there are voting rules where manipulation is NP-hard (Single transferable vote), there has been a recent negative result (Isaksson et al., 2010): Essentially, for every efficiently computable, neutral voting rule, a manipulable preference profile with a corresponding manipulation can easily be found

Arrow's theorem

- Social welfare function (SWF) is a function $f : \mathcal{R}(\mathcal{U})^n \to \mathcal{R}(\mathcal{U})$. It takes a preference profile and chooses one preference relation.
- f is Pareto optimal if $a \succ_i b$ for all $i \in N$ implies that $a \succ_f b$
- f is independent of irrelevant alternatives if for preference profiles $\mathcal{R}$, $\mathcal{R}'$ where the pairwise comparison of alternatives a and b are identical the ranking according to f must be also identical (ordering between two alternatives depends only on relative ordering given by agents)
- f is nondictatorial if there is no agent i such that for all preference profiles $\mathcal{R}$ and alternatives a, b, $a \succ_i b$ implies $a \succ_f b$

Arrow's theorem

- There exists no social welfare function that simultaneously satisfies IIA, Pareto optimality, and non-dictatorship whenever $\mathcal{U} \geq 3$.

Arrow's theorem

- Social choice function (SCF) is a function $f : \mathcal{R}(\mathcal{U})^n \times \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{U})$ and $f(R, A) \subseteq A$.
- f is Pareto optimal if $a \notin f(R, A)$ if there exists some $b \in A$ such that $b \succ_i a$ for all $i \in N$
- f is nondictatorial if there is no agent i such that for all preference profiles $\mathcal{R}$ and alternative a, $a \succ_i b$ for all $b \in A \backslash a$ implies $a \in f(R, A)$
- f is IRR if for preference profiles $\mathcal{R}$, $\mathcal{R}'$ where the pairwise comparison of alternatives a and b are identical the ranking according to f must be also identical (ordering between two alternatives depends only on relative ordering given by agents) and $f(R, A) = f(R', A)$
- f satisfies weak axiom of revealed preferences iff for all feasible sets A and B and preference profiles R if $B \subseteq A$ and $f(R, A) \cap B \neq \emptyset$ then $f(R, A) \cap B = f(R, B)$ (correlation of choices in sets and subsets)

Arrow's theorem

- There exists no social choice function that simultaneously satisfies IIA, Pareto optimality, non-dictatorship, and WARP whenever $U \geq 3$