A4M36MAS

(notes for tutorials 5 and 6)

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- We focus on non-cooperative rational agents that aim to maximize their utility values.
- A game is defined by a mechanism that can be described using following sets (this is **not** a formal definition):
  - $N-$ players
  - $A-$ actions
  - $U-$ utility/outcome for each combination of actions
  - $S-$ states of the game
  - $K-$ knowledge of players
- We seek an optimal strategy for a given game – different knowledge of players and possibly other assumptions on the game can be incorporated using different solution concepts.
- The default solution concept is the Nash Equilibrium that describes behaviors of agents under the assumption that rationality of all agents is common knowledge.
- There are other solution concepts, such as Correlated Equilibrium or Stackelberg Equilibrium, that pose further assumptions on the game (i.e., an existence of a correlation device in the first case, or commitment to some strategy by the first player in the second case).
- We can have a different representation for games:
  - normal-form games (visualized as game matrices; typically one-shot games with simultaneous moves – prisoner's dilemma)
  - extensive-form games (visualized as game trees; typically multi-stage games with possibly imperfect information or non-determinism – chess, poker)
  - stochastic games (visualized as general graphs; possibly infinite games with rewards in any stage of the game)
- The games could be described using different representation (in principle, we can formalize extensive-form games using the (exponentially large) normal-form representation)
- Consider a two-player general-sum (or non-zero-sum) game

	X	Y	Z
A	$(1,3)$	$(6,1)$	$(4,2)$
B	$(0,4)$	$(2,9)$	$(-1,3)$
C	$(2,-1)$	$(5,2)$	$(6,3)$

- By iterative removing dominated strategies we can (in this case) find pure Nash equilibrium CZ. Pareto optimal solutions are BY and CZ, because they are not pareto-dominated by any other profile of strategies.

- Note, that we can safely remove strictly/strongly dominated strategies, because they will never be played by a rational agent (regardless on the action of the opponent, the player can always improve her strategy). This is not true for weaker notion of dominance (i.e., if we remove strategies that are weakly dominated we can possibly loose some equilibrium strategy profiles).
- Strategy can be dominated not only by a pure strategy (e.g., in the case of the example B is dominated by C), but also a mixed strategy.
- For finite games Nash equilibrium (NE) within mixed strategies always exists, which is not true only for pure strategies.
- Consider a two-player zero-sum game

	L	R
U	$(3, -3)$	$(-1, 1)$
D	$(-2, 2)$	$(1, -1)$

- This game has no pure NE, however we can find a mixed NE as follows:
  - let's assume the row player will play some mixed strategy x_1, x_2
  - now the expected utility for the column player is $\mathbb{E}[L] = -3x_1 + 2x_2$ and $\mathbb{E}[R] = x_1 - x_2$
  - the column player will play best-response to our strategy, therefore the utility value the column player will gain is equal to $max(\mathbb{E}[L], \mathbb{E}[R])$
  - since we are in the zero-sum setting, the utility value for the row player will be equal to $-max(\mathbb{E}[L], \mathbb{E}[R]) = min(\mathbb{E}[L], \mathbb{E}[R]) = min(3x_1 - 2x_2, -x_1 + x_2)$
  - let's denote this term as z, and the ultimate goal of the row player is to maximize her utility value, therefore find x_1 and x_2 such that z will be maximal
  - we can formalize that as a set of linear equations as follows:

$$\max z \tag{1}$$
$$\text{s.t. } x_1 + x_2 = 1 \tag{2}$$
$$x_1, x_2 \geq 0 \tag{3}$$
$$z \leq 3x_1 - 2x_2 \tag{4}$$
$$z \leq -x_1 + x_2 \tag{5}$$

  - by solving these equations we obtain a solution $x_1 = \frac{3}{7}$ and $x_2 = \frac{4}{7}$, for which $z = \frac{1}{7}$
  - since in zero-sum games the maxmin, minmax and NE solution concepts coincide, this is a Nash equilibrium of the game
  - general LP for computing NE is in (Shoham and Leyton-Brown: Multi-agent Systems, 2009, pp. 90-91)

$$\max \quad V^* \tag{6}$$
$$\text{s.t.} \quad \textstyle\sum_{j \in A_1} u_1(a_1^j, a_2^k) \cdot x_j \geq V^* \quad \forall k \in A_2 \tag{7}$$
$$\textstyle\sum_{j \in A_1} x_j = 1 \tag{8}$$
$$x_j \geq 0 \quad \forall j \in A_1 \tag{9}$$

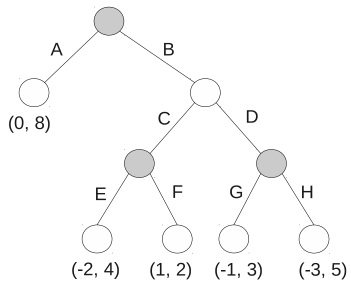
- There can be multiple Nash equilibria (even for zero-sum games). For zero-sum games, they all have the same value (termed as the *value of the game*), for general-sum games, they can have different utility values.
- from the computational perspective, the zero-sum games are the easiest class of games to solve (i.e., to find a NE) – constructing and solving a linear program can be done in polynomial time

- finding a solution to a general-sum games is computationally hard (complexity class PPAD) and finding a specific solution (e.g., a NE with maximal social welfare) is NP-complete

$$\textstyle\sum_{j\in A_1} u_1(a_1^j, a_2^k)\cdot x_j + t_2^k = V^* \quad \forall k \in A_2 \tag{10}$$
$$\textstyle\sum_{k\in A_2} u_2(a_1^j, a_2^k)\cdot y_k + t_1^j = V^* \quad \forall j \in A_1 \tag{11}$$
$$\textstyle\sum_{j\in A_1} x_j = 1 \tag{12}$$
$$\textstyle\sum_{k\in A_2} y_k = 1 \tag{13}$$
$$x_j \geq 0, \quad t_1^j \geq 0 \quad \forall j \in A_1 \tag{14}$$
$$y_k \geq 0, \quad t_2^k \geq 0 \quad \forall k \in A_2 \tag{15}$$
$$x_j \cdot t_1^j = 0 \quad \forall j \in A_1 \tag{16}$$
$$y_k \cdot t_2^k = 0 \quad \forall k \in A_2 \tag{17}$$

- different solution concepts can be simpler or harder to find (e.g., Correlated and Stackelberg equilibrium are both computationally simpler than NE – in general-sum case they are computable in polynomial time).

Consider a two-player game in the extensive form depicted in figure. Player one is playing in the dark nodes, the second player is playing in the white nodes; each of the player is maximizing its utility value.



(a) Compute a subgame-perfect pure Nash equilibrium (if one exists) and visualize the solution (i.e., the strategy profile) in the figure.

(b) Rewrite this game in the normal form (do not omit any strategies!).

(c) Write down all pure Nash equilibria (if some exist).

(d) Identify weakly dominated strategies and iteratively remove these strategies from the normal form representation of the game — remove as many strategies as possible.