A4M33MAS - Multiagent Systems
Introduction to Auctions Theory

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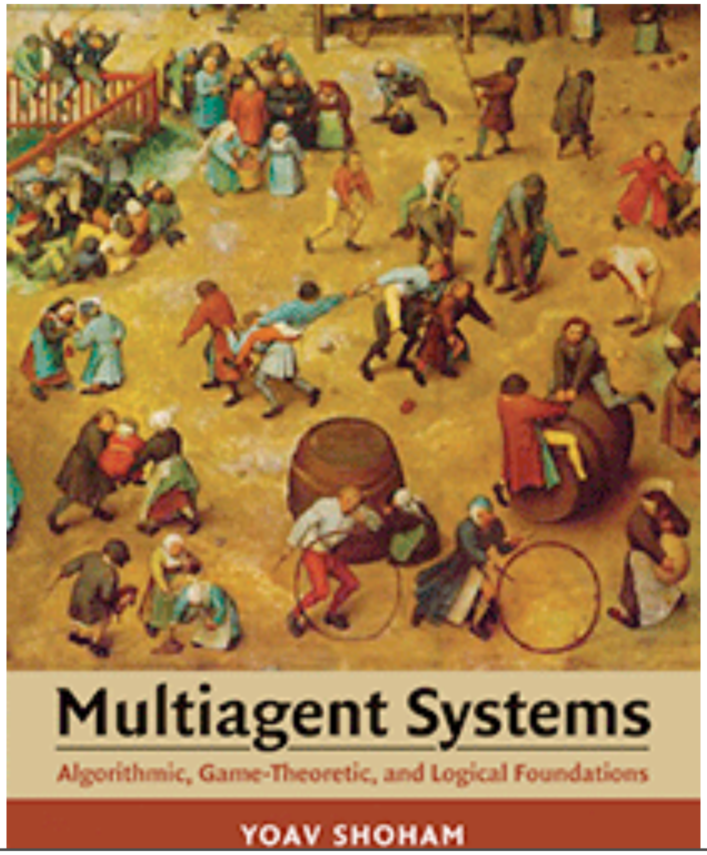
OTEVŘENÁ INFORMATIKA

In parts based on Kevin Leyton-Brown: Foundations of Multiagent Systems an introduction to algorithmic game theory, mechanism design and auctions

Game Theory

- Game theory is the study of strategic decision making, the study of mathematical models of conflict and cooperation between intelligent rational decision-makers, interactive decision theory
- Given the *rule of the game*, game theory studies strategic behaviour of the agents in the form of a mixed/pure strategy (e.g. optimality, stability)
- Given the *strategic behavior of the agents*, mechanism design (reverse game theory) studies(designs) the rule of games with respect to a specific outcome of the game



Yoav Shoham, Kevin Leyton-Brown, Multiagent Systems:
Algorithmic, Game-Theoretic, and Logical Foundations
Cambridge University Press, 2009
http://www.masfoundations.org

Game Theory

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  1.voting (social choice)

  2.auctions

Auctions

- **Auctions** are any mechanisms for allocating resources among self-interested agents: **Multiagent Resource Allocation Protocol**
  - single-good x multiunit x combinatorial

- Very widely used
  - government sale of resources
  - privatization
  - stock market
  - request for quote
  - FCC spectrum
  - real estate sales
  - eBay

Auctions and computer science

- resource allocation is a fundamental problem in CS
- increasing importance of studying distributed systems with heterogeneous agents
- markets for:
  - computational resources
  - P2P systems
  - network bandwidth
- currency needn't be real money, just something scarce
  - that said, real money trading agents are also an important motivation

Select Auctions

- English
- Japanese
- Dutch
- First-Price (Seal-bid)
- Second-Price (Vickery)
- All-Pay

Select Auctions

- English
  - auctioneer starts the bidding at *reservation price*
  - bidders then shout out ascending prices
  - once bidders stop shouting, the high bidder gets the good at that price
- Japanese
- Dutch
- First-Price
- Second-Price
- All-Pay

Select Auctions

- English
- Japanese
  - Same as an English auction except that the auctioneer calls out the prices
  - all bidders start out standing when the price reaches a level that a bidder is not willing to pay, that bidder sits down
  - once a bidder sits down, they can't get back up
  - the last person standing gets the good
- Dutch
- First-Price
- Second-Price
- All-Pay

Select Auctions

- English
- Japanese
- Dutch
  - the auctioneer starts a clock at some high value; it descends at some point, a bidder shouts *mine!* and gets the good at
  - the price shown on the clock
- First-Price
- Second-Price
- All-Pay

Select Auctions

- English
- Japanese
- Dutch
- First-Price (Seal-bid)
  - bidders write down bids on pieces of paper
  - auctioneer awards the good to the bidder with the highest bid
  - that bidder pays the amount of his bid
- Second-Price
- All-Pay

Select Auctions

- English
- Japanese
- Dutch
- First-Price
- Second-Price (Vickery)
  - bidders write down bids on pieces of paper
  - auctioneer awards the good to the bidder with the highest bid
  - that bidder pays the amount bid by the second-highest bidder
- All-Pay

Select Auctions

- English
- Japanese
- Dutch
- First-Price
- Second-Price
- All-Pay
  - bidders write down bids on pieces of paper
  - auctioneer awards the good to the bidder with the highest bid
  - everyone pays the amount of their bid regardless of whether they win

Auctions as Structured Negotiations

- Any negotiation mechanism that is:
  - market-based (determines an exchange in terms of currency)
  - mediated (auctioneer)
  - well-specified (follows rules)
- Defined by three kinds of rules:
  - rules for bidding
  - rules for what information is revealed
  - rules for clearing

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- Defined by three kinds of rules:
  - rules for bidding
    - *who can bid, when, what is the form of a bid*
    - *restrictions on offers, as a function of:*
      - bidder's own previous bid
      - auction state (others' bids)
      - eligibility (e.g., budget constraints)
      - expiration, withdrawal, replacement
  - rules for what information is revealed
  - rules for clearing

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  - rules for what information is revealed
    - *when to reveal what information to whom*
  - rules for clearing

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- Defined by three kinds of rules:
  - rules for bidding
  - rules for what information is revealed
  - rules for clearing
    - *when to clear: at intervals, on each bid, after a period of inactivity*
    - *allocation (who gets what)*
    - *payment (who pays what)*

Intuitive comparison

	English	Dutch	Japanese	1^{st}-Price	2^{nd}-Price
Duration	#bidders, increment	starting price, clock speed	#bidders, increment	fixed	fixed
Info Revealed	2^{nd}-highest val; bounds on others	winner's bid	all val's but winner's	none	none
Jump bids	yes	n/a	no	n/a	n/a
Price Discovery	yes	no	yes	no	no

Intuitive comparison

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	English	Dutch	Japanese	1st-Price	2nd-Price
Duration	#bidders, increment	starting price, clock speed	#bidders, increment	fixed	fixed
Info Revealed	2nd-highest val; bounds on others	winner's bid	all val's but winner's	none	none
Jump bids	yes	n/a	no	n/a	n/a
Price Discovery	yes	no	yes	no	no
Regret	no	yes	no	yes	no

Second-Price (Vickery) Auction

Theorem

Truth-telling is a dominant strategy in a second-price auction.

Second-Price (Vickery) Auction

Theorem

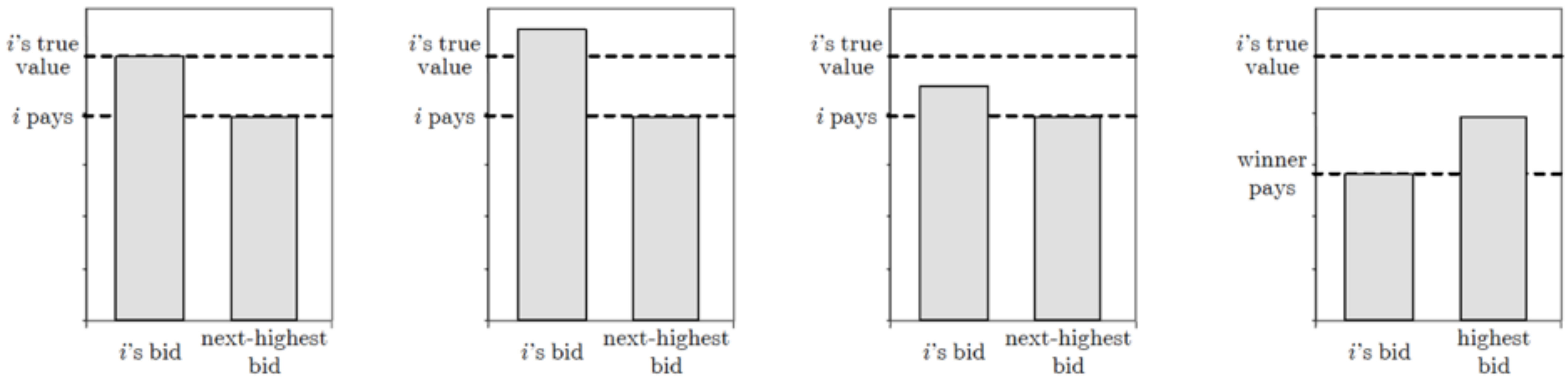
Truth-telling is a dominant strategy in a second-price auction.

Proof.

Assume that the other bidders bid in some arbitrary way. We must show that i's best response is always to bid truthfully. We'll break the proof into two cases:

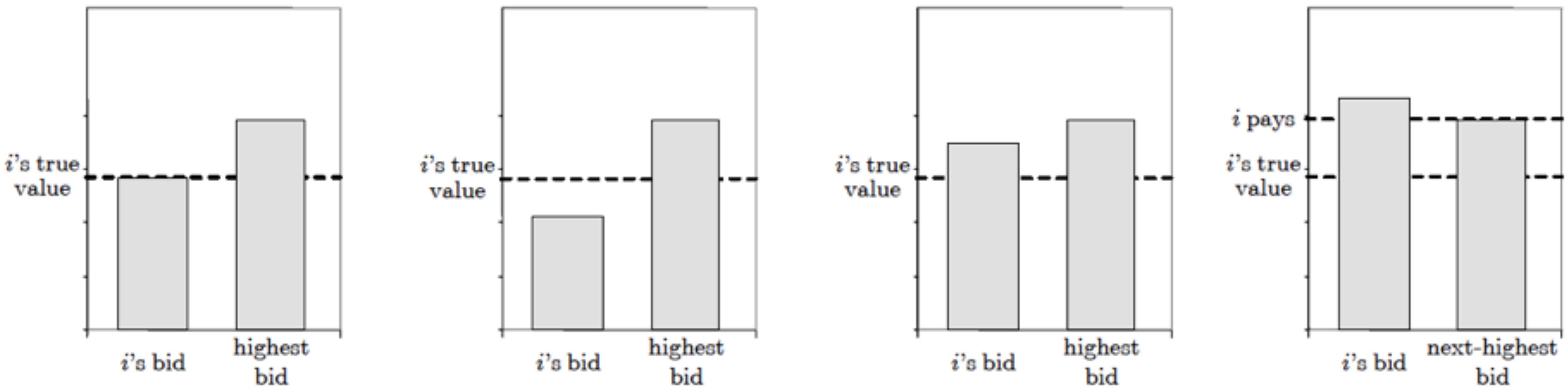
1. Bidding honestly, i would win the auction
2. Bidding honestly, i would lose the auction

Second-Price (Vickery) Auction



- Bidding honestly, i is the winner
- If i bids higher, he will still win and still pay the same amount
- If i bids lower, he will either still win and still pay the same amount. . . or lose and get utility of zero.

Second-Price (Vickery) Auction



- Bidding honestly, i is not the winner
- If i bids lower, he will still lose and still pay nothing
- If i bids higher, he will either still lose and still pay nothing. . . or win and pay more than his valuation.

English and Japanese Auctions

- A much more complicated strategy space
  - extensive form game
  - bidders are able to condition their bids on information revealed by others
  - in the case of English auctions, the ability to place jump bids
- intuitively, though, the revealed information doesn't make any diff erence in the IPV setting.

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Theorem

Under the independent private values model (IPV), it is a dominant strategy for bidders to bid up to (and not beyond) their valuations in both Japanese and English auctions.

Dutch and First-price Auction

- There is no dominant strategy. The best strategy is to bid a bit less that than private value
  - but how much it depend bidders atitude to risk:
    - *risk seekers would bid substantially less and thus would he for higher payoff, while risk averse would bid high by which they lower payoff but increase likelyhood of winning*

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Theorem

First-Price and Dutch auctions are strategically equivalent.

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  - but how much it depend bidders atitude to risk:
    - *risk seekers would bid substantially less and thus would he for higher payoff, while risk averse would bid high by which they lower payoff but increase likelyhood of winning*

Proposition 11.1.2 *In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from the interval* $[0, 1]$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ *is a Bayes–Nash equilibrium strategy profile.*

Dutch and First-price Auction

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  - but how much it depend bidders atitude to risk:
    - *risk seekers would bid substantially less and thus would he for higher payoff, while risk averse would bid high by which they lower payoff but increase likelyhood of winning*

Theorem 11.1.3 *In a first-price sealed-bid auction with n risk-neutral agents whose valuations are independently drawn from a uniform distribution on the same bounded interval of the real numbers, the unique symmetric equilibrium is given by the strategy profile* $(\frac{n-1}{n}v_1, \ldots, \frac{n-1}{n}v_n)$.

Auctions Comparison

- from the perspective of the revenue

Risk-neutral, IPV	Jap	=	Eng	=	2nd	=	1st	=	Dutch
Risk-averse, IPV		=		=		<		=	
Risk-seeking, IPV		=		=		>		=	

Collusion of Bidders

- Cooperation between the bidders aimed at providing the same result while lowering the expected payments (and revenue).
- Good auction for collusion:
  - English
    - *no special protocol required*
      if an agent breaks the collusion, it can be corrected
- In other auctions:
  - risk of collusion being evaded
  - cartel (bidding ring) run by trusted agent, who is not interested in bidding

Collusion of Bidders

- Collusion protocol for Vickery auction:
  1. Each agent in the cartel submits a bid to the ring center.
  2. The cartel identifies the max bid that he received: v_1^r and the second: v_2^r
  3. Cartel submits v_1^r in the main auction and drops the other bids.
  4. If cartel wins in the main auction at v_2^r, the cartel awards the good to the v_1^r bidder and requires that him to pay $\max(v_2, v_2^r)$.
  5. The ring center gives every agent who participated in the bidding ring a payment of k, regardless of the amount of that agent's bid and regardless of whether or not the cartel's bid won the good in the main auction

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How big k is supposed to be?

for $k = 0$, the auction works like Vickery as nobody is intencentivized to joint the cartel, for large k nobody is interested in organizing the cartel

Collusion of Bidders

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  5. The ring center gives every agent who participated in the bidding ring a payment of k, regardless of the amount of that agent's bid and regardless of whether or not the cartel's bid won the good in the main auction

How big k is supposed to be?

$$k = \frac{\texttt{expected}(v_2 - v_2^r)}{n}$$

Collusion of Bidders

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- Collusion protocol for First Price auction:
  1. Each agent in the cartel submits a bid to the ring center.
  2. The cartel identifies the max bid that he received: v_1^r and bidder must pay this price in full.
  3. The ring center bids in the main auction at 0. Note that the bidding ring always wins in the main auction as there are no other bidders.
  4. The ring center gives the good to the bidder who placed the winning bid in the preauction.
  5. The ring center pays every bidder other than the winner $\dfrac{1}{n-1}v_1^r$

Multiunit Auctions

- There are multiple units available for bidding:
  - each bidder provides a independent private valuation bid for single unit
  - or each bidder can bid an arbitrary number of units.
- What the bidder shall pay (provided that the winners are chosen):
  - discriminatory pricing rule (pay-your-bid scheme)
  - uniform pricing rule (highest among loosing or lowest among winning)
- Proposed bids are (i) all-or-nothing or (ii) divisible

Multiunit Auctions

Definition 11.2.2 (Winner determination problem (WDP)) *The* winner determination problem *(WDP) for a general multiunit auction, where m denotes the total number of units available and $\hat{v}_i(k)$ denotes bidder i's declared valuation for being awarded k units, is to find the social-welfare-maximizing allocation of goods to agents. This problem can be expressed as the following integer program.*

$$\text{maximize} \quad \sum_{i \in N} \sum_{1 \le k \le m} \hat{v}_i(k) x_{k,i} \tag{11.11}$$

$$\text{subject to} \quad \sum_{i \in N} \sum_{1 \le k \le m} k \cdot x_{k,i} \le m \tag{11.12}$$

$$\sum_{1 \le k \le m} x_{k,i} \le 1 \qquad \forall i \in N \tag{11.13}$$

$$x_{k,i} = \{0, 1\} \qquad \forall 1 \le k \le m, i \in N \tag{11.14}$$

Multiunit Auctions

- **Additive valuation.** The bidder's valuation of a set is directly proportional to the number of goods in the set, so that $v_i(S) = c|S|$ for some constant c.
- **Single item valuation.** The bidder desires any single item, and only a single item, so that $v_i(S) = c$ for some constant c for all $S \neq \emptyset$.
- **Fixed budget valuation.** Similar to the additive valuation, but the bidder has a maximum budget of B, so that $v_i(S) = \min(c|S|, B)$.
- **Majority valuation.** The bidder values equally any majority of the goods, so that

$$v_i(S) = \begin{cases} 1 & \text{if } |S| \geq m/2; \\ 0 & \text{otherwise.} \end{cases}$$

Multiunit Auctions

- **General symmetric valuation.** Let $p_1, p_2, \ldots, p_m$ be arbitrary nonnegative prices, so that p_j specifies how much the bidder is willing to pay of the j^{th} item won. Then

$$v_i(S) = \sum_{j=1}^{|S|} p_j$$

- **Downward sloping valuation.** A downward sloping valuation is a symmetric valuation in which $p_1 \geq p_2 \geq \cdots \geq p_m$.

Combinatorial Auctions

- Agents bid for combination of different ammounts of different objects, the result of combinatorial auction is an assignment.

Imagine that each of the objects in X has an associated price; the price vector is $p = (p_1, \ldots, p_n)$, where p_j is the price of object j. Given an assignment $S \subseteq M$ and a price vector p, define the "utility" from an assignment j to agent i as $u(i,j) = v(i,j) - p_j$. An assignment and a set of prices are in *competitive equilibrium* when each agent is assigned the object that maximizes his utility given the current prices. More formally, we have the following.

Definition 2.3.4 (Competitive equilibrium) *A feasible assignment S and a price vector p are in* competitive equilibrium *when for every pairing $(i,j) \in S$ it is the case that $\forall k,\ u(i,j) \geq u(i,k)$.*

Theorem 2.3.5 *If a feasible assignment S and a price vector p satisfy the competitive equilibrium condition then S is an optimal assignment. Furthermore, for any optimal solution S, there exists a price vector p such that p and S satisfy the competitive equilibrium condition.*

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Naive Auction Algorithm

$S \leftarrow \emptyset$
forall $j \in X$ **do**
  $p_j \leftarrow 0$
repeat
  // Bidding Step:
  let $i \in N$ be an unassigned agent
  // Find an object $j \in X$ that offers i maximal value at current prices:
  $j \in \arg\max_{k|(i,k) \in M}(v(i,k) - p_k)$
  // Compute i's bid increment for j:
  $b_i \leftarrow (v(i,j) - p_j) - \max_{k|(i,k) \in M; k \neq j}(v(i,k) - p_k)$
  // which is the difference between the value to i of the best and second-best objects at current prices (note that i's bid will be the current price plus this bid increment).
  // Assignment Step:
  add the pair (i, j) to the assignment S
  if *there is another pair* (i', j) **then**
    remove it from the assignment S
  increase the price p_j by the increment b_i
until S *is feasible*   // that is, it contains an assignment for all $i \in N$

Naive Auction Algorithm

An example of an assignment problem is the following (in this example, $X = \{x_1, x_2, x_3\}$ and $N = \{1, 2, 3\}$).

$\mathbf{i}$	$\mathbf{v(i, x_1)}$	$\mathbf{v(i, x_2)}$	$\mathbf{v(i, x_3)}$
1	2	4	0
2	1	5	0
3	1	3	2

round	$\mathbf{p_1}$	$\mathbf{p_2}$	$\mathbf{p_3}$	**bidder**	**preferred object**	**bid incr.**	**current assignment**
0	0	0	0	1	x_2	2	$(1, x_2)$
1	0	2	0	2	x_2	2	$(2, x_2)$
2	0	4	0	3	x_3	1	$(2, x_2), (3, x_3)$
3	0	4	1	1	x_1	2	$(2, x_2), (3, x_3), (1, x_1)$

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Theorem 2.3.6 *The naive algorithm terminates only at a competitive equilibrium.*

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1	0	0	0	2	x_2	0	$(1, x_1), (2, x_2)$
2	0	0	0	3	x_1	0	$(3, x_1), (2, x_2)$
3	0	0	0	1	x_2	0	$(3, x_1), (1, x_2)$
4	0	0	0	2	x_1	0	$(2, x_1), (1, x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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1	ϵ	2ϵ	0	2	x_2	2ϵ	$(1, x_1), (2, x_2)$
2	3ϵ	2ϵ	0	3	x_1	2ϵ	$(3, x_1), (2, x_2)$
3	3ϵ	4ϵ	0	1	x_2	2ϵ	$(3, x_1), (1, x_2)$
4	5ϵ	4ϵ	0	2	x_1	2ϵ	$(2, x_1), (1, x_2)$

Naive Auction Algorithm

Definition 2.3.7 (ϵ-competitive equilibrium) *S and p satisfy ϵ-competitive equilibrium when for each $i \in N$, if there exists a pair $(i, j) \in S$ then $\forall k,\ u(i, j) + \epsilon \geq u(i, k)$.*

Theorem 2.3.8 *A feasible assignment S with n goods that forms an ϵ-competitive equilibrium with some price vector is within $n\epsilon$ of optimal.*

Corollary 2.3.9 *Consider a feasible assignment problem with an integer valuation function $v : M \mapsto \mathbb{Z}$. If $\epsilon < \frac{1}{n}$ then any feasible assignment found by the terminating auction algorithm will be optimal.*