Cooperative Game Theory

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Introduction

- Model of **team formation**
  - entrepreneurs trying to form small companies
  - companies cooperating to handle a large contract
  - friends agreeing on a trip
- Focus of teams (coalitions) of agents
  - also called c**oalitional game theory**
- Assumes a team can (sometimes) achieve more than (the sum of) individual agents
  - Better to team up and share the utility than receive utility individually
- Called cooperative but **agents still pursue their own interests**

Example: Task Allocation

- A set of tasks needs to be performed
  - they require different expertises
  - they may be decomposed.
- Agents do not have enough resource on their own to perform a task.
- Find complementary agents to perform the tasks
  - robots have the ability to move objects in a plant, but multiple robots are required to move a heavy box.
  - transportation domain: agents are trucks, trains, airplanes, ships... a task is a good to be transported.
- **Issues:**
  - What coalition to form?
  - How to reward each each member when a task is completed?

Example: Airport Game

- A number of cities need airport capacity.
- If a new regional airport is built the cities will have to share its cost
  - which will depend on the largest aircraft that the runway can accommodate.
- Otherwise each city will have to build its own airport.
- Which cities should share an airport and how they should finance the construction?

Example: Car Pooling

- A set of twelve people all drive to work.
- They would like to form car pools.
- Some can pick up others on their way to work. Others have to go out of their way to pick up others.
- A car can only hold 5 people.
- Who should carpool together? How often should they each drive?

Example 3: Minimum Spanning Tree

- A group of customers must be connected to a critical service provided by some central facility, such as a power plant or an emergency switchboard. In order to be served, a customer must either be directly connected to the facility or be connected to some other connected customer. Let us model the customers and the facility as nodes on a graph, and the possible connections as edges with associated costs.

Example: Voting Game

- The parliament of Micronesia is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively.
- They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties.
- A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

Cooperative Game Theory

Basic Definitions

Coalitional Game

- **Transferable utility assumption**: the payoffs to a coalition may be freely redistributed among its members.
  - satisfied whenever there is a universal currency that is used for exchange in the system

Definition (Coalitional game with transferable utility)

A **coalitional game with transferable utility** is a pair $(\boldsymbol{N}, \boldsymbol{v})$ where

- N is a finite set of players (also termed **grand coalition**), indexed by i; and
- $v: 2^N \mapsto \mathbb{R}$ is a **characteristic function (**also termed **valuation function**) that associates with each coalition $S \subseteq N$ a real-valued **payoff** $\boldsymbol{v}(\boldsymbol{S})$ that the coalition's members can distribute among themselves. We assume $v(\emptyset) = 0$.

Simple Example

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
(13)	7
(23)	8
(123)	9

Payoff Division (Outcome)

Definition

An **payoff division** (also **outcome**) $\vec{x} = (x_1, \dots, x_k)$ for a coalition S in game (N, v) is a distribution of the coalition's S utility to the members of S.

Outcome Properties

Definition (Feasible outcome)

Given a coalitional game (N, v), an outcome $\vec{x}$ is **feasible** if $\sum_{i \in N} x_i \leq v(N)$

Definition (Preimputation)

Given a coalitional game (N, v), an outcome $\vec{x}$ is a **pre-imputation (efficient)** if $\sum_{i \in N} x_i = v(N)$

Definition (Imputation)

Given a coalitional game (N, v), an preimputation $\vec{x}$ is an **imputation** if $\forall i \in N, x_i \geq v(i)$

Imputations are payoff vectors that are not only efficient but **individually rational**; efficiency sometime termed **group rationality**

Example

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
(13)	7
(23)	8
(123)	9

(1)(2)(3)
$2+2+4=8$

(1)(23)
$2+8=10$

(2)(13)
$2+7=9$

(3)(12)
$4+5=9$

(123)
9

- $\vec{x} = (5,5,5)$, feasible? No
- $\vec{x} = (2,4,3)$, feasible? Yes
- $\vec{x} = (2,2,2)$, feasible? Yes, but not stable

Airport Game Example

- Airport game:
  - N is the set of cities, and v(S) is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Cooperative Games

Solution Concepts

Analyzing Coalition Games

1. **Which coalition** will form?
2. How should the coalition **divide its payoff**?

The Core

- Under what payment divisions would the agents want to form the grand coalition?
  - Sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.
- They would want to do so if and only if the payment profile is drawn from a set called the **core**.

Definition (Core)

A payoff vector x is in the **core** of a coalitional game (N, v) iff

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S)$$

- Analogue to **Strong Nash equilibrium** (allows deviations by groups of agent)

Core: Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

(1)(2)(3)
$1 + 2 + 2 = 5$

(1)(23)
$1 + 4 = 5$

(2)(13)
$2 + 3 = 5$

(3)(12)
$2 + 4 = 6$

(123)
6

- $\vec{x} = (2, 1, 2)$ in the core? No
- $\vec{x} = (2, 2, 2)$ in the core? Yes
- $\vec{x} = (1, 2, 2)$ in the core? No

Core: Existence

- Is the core always non-empty?

S	$v(S)$
(1)	0
(2)	0
(3)	0
(12)	10
(13)	10
(23)	10
(123)	10

No. Core existence guaranteed only for certain special subclasses of games.
Core is also **not unique** (there might be infinitely many payoff divisions in the core)**.**

Distributing Payments

- How should we *fairly* distribute a coalition's payoff?

S	$v(S)$
()	0
(1)	1
(2)	3
(12)	6

- If the agents form (12), how much should each get paid?

Axiomatizing Fariness

- **What is fair?**
- Axiomatic approach – a fair payoff distribution should satisfy:
  - **Symmetry:** if two agents contribute the same, they should receive the same pay-off (they are interchangeable)
  - **Dummy player:** agents that do not add value to any coalition should get what they earn on their own
  - **Additivity:** if two games are combined, the value a player gets should be the sum of the values it gets in individual games

Axiomatizing Fairness: Symmetry

- Agents i and j **are interchangeable** if they always contribute the same amount to every coalition of the other agents.
  - for all S that contains neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.
- The symmetry axiom states that such agents should receive the same payments.

Axiom (Symmetry)

For any v, if i and j are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$.

Axiomatizing Fairness: Dummy Player

- Agent i is a **dummy player** if the amount that i contributes to any coalition is exactly the amount that i is able to achieve alone.
  - for all S such that $i \notin S$: $v(S \cup \{i\}) - v(S) = v(\{i\})$.
- The dummy player axiom states that dummy players should receive a payment equal to exactly the amount that they achieve on their own.

Axiom (Dummy player)

For any v, if i is a dummy player then $\psi_i(N, v) = v(\{i\})$.

Axiomatizing fairness: Additivity

- Consider two different coalitional game theory problems, defined by two different characteristic functions v_1 and v_2, involving the same set of agents.
- The **additivity axiom** states that if we re-model the setting as a single game in which each coalition S achieves a payoff of $v_1(S) + v_2(S)$, the agents' payments in each coalition should be the sum of the payments they would have achieved for that coalition under the two separate games.

Axiom (Additivity)

For any two v_1 and v_2, we have for any player i that $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S.

Shapley Value

Theorem

Given a coalitional game (N, v), there is a **unique payoff division** $\phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms.

- This payoff division is called **Shapley value**

Lloyd F. Shapley. 1923–. Responsible for the core and Shapley value solution concepts.

Shapley Value

Definition (Shapley value)

Give a coalitional game (N, v), the **Shapley value** of player i is given by

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N\{i\}} |S|!\,(|N| - |S| - 1)!\,[v(S \cup \{i\}) - v(S)]$$

- This captures the "**average marginal contribution**" of agent i, **averaging** over all the **different sequences** according to which the grand coalition could be built up from the empty coalition.

Shapley Value: Example

If they form (12), how much should each get paid?

S	$v(S)$
()	0
(1)	1
(2)	3
(12)	6

$$\begin{aligned}\phi(1,\{1,2\}) &= \frac{1}{2}\cdot(v(1)-v()+v(21)-v(2))\\ &= \frac{1}{2}\cdot(1-0+6-3)=2\end{aligned}$$

$$\begin{aligned}\phi(2,\{1,2\}) &= \frac{1}{2}\cdot(v(12)-v(1)+v(2)-v())\\ &= \frac{1}{2}\cdot(6-1+3-0)=4\end{aligned}$$

Refining the Core

- **Problem: Core does not always exist (and is not unique)** => relax the definition of the core so that it will always exist.
- Idea: Find the solutions that **minimize the agents‘ temptation to defect**.

Excess

Definition (Excess)

The **excess of coalition** S given outcome $\vec{x}$ is

$$e(S, \vec{x}) = v(S) - \vec{x}(S),$$

where

$$\vec{x}(S) = \sum_{i \in S} \vec{x}_i$$

- The more excess S has, given $\vec{x}$, the more tempting it is of the agents in S to defect $\vec{x}$ and form S.

$$N = \{1,2,3\},\ v(\{i\}) = 0 \text{ for } i \in \{1,2,3\}$$
$$v(\{1,2\}) = 5,\ v(\{1,3\}) = 6,\ v(\{2,3\}) = 6$$
$$v(N) = 8$$

Let us consider two payoff vectors $x = \langle 3,3,2 \rangle$ and $y = \langle 2,3,3 \rangle$. Let $e(x)$ denote the sequence of **excesses** of all coalitions at x.

$x = \langle 3,3,2 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},x)$
$\{1\}$	-3
$\{2\}$	-3
$\{3\}$	-2
$\{1,2\}$	-1
$\{1,3\}$	1
$\{2,3\}$	1
$\{1,2,3\}$	0

$y = \langle 2,3,3 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},y)$
$\{1\}$	-2
$\{2\}$	-3
$\{3\}$	-3
$\{1,2\}$	0
$\{1,3\}$	1
$\{2,3\}$	0
$\{1,2,3\}$	0

Which payoff should we prefer? x or y? Let us write the excess in the decreasing order (from the greatest excess to the smallest)

$\langle 1,1,0,-1,-2,-3,-3 \rangle$ $\langle 1,0,0,0,-2,-3,-3 \rangle$

Nucleolus

Definition (nucleolus)

The *nucleolus* is the set

$$\{\vec{u} \mid \theta(\vec{u}) \not\succ \theta(\vec{v}) \text{ for all } \vec{v}, \text{ given that } \vec{u} \text{ and } \vec{v} \text{ are feasible.}\}$$

where,

$$\theta(\vec{u}) = \langle e(S_1^{\vec{u}}, \vec{u}), e(S_2^{\vec{u}}, \vec{u}), \ldots, e(S_{2^{|A|}}^{\vec{u}}, \vec{u})\rangle,$$

where $e(S_i^{\vec{u}}, \vec{u}) \geq e(S_j^{\vec{u}}, \vec{u})$ for all $i < j$.
$\succ$ is a lexicographical ordering over all subsets S given some $\vec{u}$.
$\theta(\vec{u}) \succ \theta(\vec{v})$ is true when there is some number $q \in 1 \ldots 2^{|A|}$ such for all $p < q$ we have that $e(S_p^{\vec{u}}, \vec{u}) = e(S_p^{\vec{v}}, \vec{v})$ and $e(S_q^{\vec{u}}, \vec{u}) > e(S_q^{\vec{v}}, \vec{v})$ where the S_i have been sorted as per θ.

Nucleolus: Lexicographical Order

- For example, if we had the lists:

$$\{(2,2,2),(2,1,0),(3,2,2),(2,1,1)\}$$

- They would be ordered as

$$\{(3,2,2),(2,2,2);\ (2,1,1);\ (2,1,0)\}$$

Nucleolus

- Nucleolus **always exists** and **is unique.**
- If the core is non-empty, the nucleolus **is in the core**.
- Captures idea of minimizing temptation, somewhat.
  - However, notice that the lexicographic order it defines only cares about the first coalition that has a higher excess, it does not care about the ones after that.
- In fact, **minimizes the greatest temptation.**

Classes of Coalition Games

- Superadditive game
- Additive game
- Constant-sum game
- Convex game
- Simple game

Superaditive Games

Definition (Superadditive game)

A game $G = (N, v)$ is superadditive if for all $S, T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

- Superadditivity is justified when coalitions can always **work without interfering** with one another
  - the value of two coalitions will be no less than the sum of their individual values.
  - implies that the grand coalition **has the highest total payoff**
- All our examples are superadditive.

Convex Games

- An important subclass of superadditive games

Definition (Convex game)

A game $G = (N, v)$ is convex if for all $S, T \subset N$,
$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

- Convexity is a stronger condition than superadditivity.
  - However, convex games are not too rare in practice (e.g., the airport game is convex)
- Convex games have a number of useful properties
  - the core is always non-empty
  - Shapley value is in the core

Simple Games

- A coalitional game is **simple** if the value of any coalition is either 0 (losing) or 1 (winning)
- Simple games model **yes/no** voting systems
- $Y = \langle Ag, W \rangle$, where $W \subseteq \mathbf{2}^{Ag}$ is the set of winning coalitions
- If $C \in W$, C would be able to determine the outcome, 'yes' or 'no'
- Important conditions:
  - **Non-triviality:** $\emptyset \subset W \subset \mathbf{2}^{Ag}$
  - **Monotonicity:** if $C_1 \subseteq C_2$ and $C_1 \in W$ then $C_2 \in W$
  - **Zero-sum:** if $C \in W$ then $Ag \setminus C \notin W$
  - **Empty coalition loses:** $\emptyset \notin W$
  - **Grand coalition wins:** $Ag \in W$
- Naive representation is exponential in the number of agents

Relation of Game Clases

$$\text{Superadditive} \supset \text{Convex} \supset \text{Additive}$$

$$\text{Constant sum} \supset \text{Additive}$$

$$\text{Constant sum} \supset \text{Proper simple}$$

$$\text{Simple} \supset \text{Proper simple}$$

Computational Aspects

Finding Optimal Coalition Structure

- Assuming utilitarian solution, i.e., maximizing the total payoff
- Trivial if superadditive → grand coalition
- Otherwise: search for the best coalition structure

Finding Optimal Coalition Structure

(1)(2)(3)(4)

(12)(3)(4) (13)(2)(4) (14)(2)(3) (23)(1)(4) (24)(1)(3) (34)(1)(2)

(1)(234) (2)(134) (3)(124) (4)(123) (12)(34) (14)(23) (13)(24)

(1234)

Number of combination (just) at level 2 exponential $2^N - 1$

➔ use approximation algorithm and/or exploit the properties of the game (class)

Compact Representations

- A naive representation of a coalition game is infeasible (**exponential** in the number of agents):
  - e.g. for three agents $\{1, 2, 3\}$:

    $(1) = 5$ $\quad$ $(1, 3) = 10$

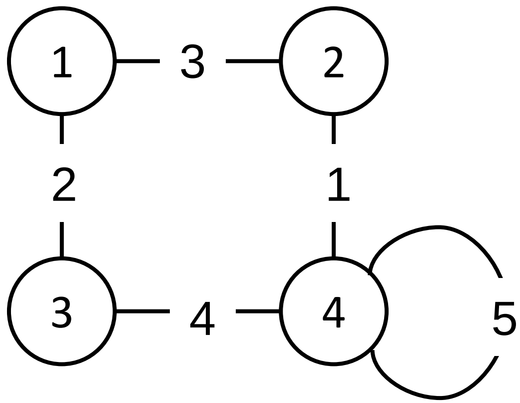
    $(2) = 5$ $\quad$ $(2, 3) = 20$

    $(3) = 5$ $\quad$ $(1, 2, 3) = 25$

    $(1, 2) = 10$
- We need a **succinct** representations
- **Modular representations** exploit Shapley's axioms directly
- Basic idea: divide the game into smaller games and exploit additivity axiom

Weighted Graph Games

- Define a characteristic function by an undirected weighted graph. Value of a coalition $S \subseteq N$: $v(S) = \sum_{\{i,j\} \subseteq S} w_{i,j}$



$$v(\{1,2,3\}) = 3 + 2 = 5$$
$$v(\{4\}) = 5$$
$$v(\{2,4\}) = 1 + 5 = 6$$
$$v(\{1,3\}) = 2$$

- **Not a complete** representation (not all characteristic functions can be represented)
- But **easy to compute** the Shapley value for a given agent in polynomial time: $sh_i = \frac{1}{2} \sum_{j \neq i} w_{i,j}$
- Checking emptiness of the core is NP-complete, and membership to the core is co-NP-complete

Marginal Contribution Nets

- Represent characteristic function as rules: pattern $\longrightarrow$ value
  - the pattern is a conjunction of agents, e.g. $1 \wedge 3$
  - $1 \wedge 3$ would apply to $\{1, 3\}$ and $\{1, 3, 5\}$, but not to $\{1\}$ or $\{8, 12\}$
  - $C \vDash \varphi$, means the rule $\varphi \longrightarrow x$ applies to coalition C
  - $rs_C = \{\varphi \longrightarrow x \in rs | C \vDash \varphi\}$ are the rules that apply to coalition C
- $\nu_{rs}(C) = \sum_{\varphi \longrightarrow x \in rs_C} x$
- Example:
  - $rs_1 = \{a \wedge b \longrightarrow 5,\ b \longrightarrow 2\}$
  - $\nu_{rs_1}(\{a\}) = 0$, $\nu_{rs_1}(\{b\}) = 2$ and $\nu_{rs_1}(\{a, b\}) = 7$
- Extension: allow negation in rules, e.g. $b \wedge \neg c \longrightarrow -2$
- Shapley value can be computed in polynomial time
- Complete representation, but not necessarily succinct

Weighted Voting Games

- For each agent $i \in Ag$ define a weight w_i and an overall **quota** q
- A coalition is winning if the sum of their weights exceeds the quota:
$$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$
- Example: **Simple majority voting**, $w_i = 1$ and $q = \frac{\lceil |Ag|+1 \rceil}{2}$
- Succinct (but incomplete) representation: $\langle q; w_1, \ldots, w_n \rangle$
- Extension: ***k*-weighted voting games** are a complete representation
  - overall game = "conjunction" k of k different weighted voting games
  - Winning coalition is the one that wins in all component games
  - **Game dimension**: k is at most exponential in the number of players
  - Checking whether a k-weighted voting game is minimal is NP-complete

Conclusions

- Cooperative game theory models **teams of selfish agents**
- It studies which / how coalitions form and how they should distribute its payoff to its members
- Core / Nucleus address the issue of coalition stability
- Shapley value represents a fair distribution of payments
- In practice, compact representation of coalition games are required required
- Reading:
  - [Shoham]: 12.1-12.2
  - [Vidal]: Chapter 4