Resource Allocation

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Auctions and Resource Allocation

an example of mechanism design – “reverse game theory”
the goal is to design rules of the game assuming the players are rational agents

bidding in an auction – an optimal strategy in a very large game against unknown opponents (Bayesian games)

widely used in practice (sponsored search auctions, selling advertisement slots, ...)

Facility Location

Suppose there are n agents located anywhere on the interval $[0, 1]$. We have to decide where to build an amusement park A, also anywhere on the same interval. The cost of an agent is its distance to A.

- What is the solution selected by the egalitarian collective utility function (CUF)?
- What is the solution selected by the utilitarian CUF? Design an algorithm.
- Suppose the utility function is more complex and agents have some minimal allowed distance from A (they want to be close, but not right next to A).

Divisible Goods (Cake Cutting)

Suppose there are $N = \{1, \ldots, n\}$ agents that need to divide a *cake* amongst themselves by means of a series of parallel cuts. The cake is represented by the unit interval $[0, 1]$.

A *bundle* is a finite union of subintervals of the full cake. These subintervals are not allowed to overlap (so goods cannot be shared) and we shall only be interested in complete allocations, where every piece of the cake is allocated to someone.

Divisible Goods (Cake Cutting)

Each agent $i \in N$ has got a utility function u_i (also referred to as the agents valuation or measure) mapping finite unions of subintervals of $[0, 1]$ to the reals, that satisfies the following conditions:

- Non-negativity: $u_i(B) \geq 0$ for all $B \subseteq [0, 1]$
- Normalisation: $u_i(\emptyset) = 0$ and $u_i([0, 1]) = 1$
- Additivity: $u_i(B \cup B') = u_i(B) + u_i(B')$ for disjoint $B, B' \subseteq [0, 1]$
- u_i is continuous: the Intermediate-Value Theorem applies and single points do not have any value. Specifically, if $0 < x < y \leq 1$ with $u_i([0, x]) = \alpha$ and $u_i([0, y]) = \beta$, then for every $\gamma \in [\alpha, \beta]$ there exists a $z \in [x, y]$ such that $u_i([0, z]) = \gamma$.

Divisible Goods (Cake Cutting)

- Assume there are $n = 2$ agents. What algorithm (protocol) they should use to divide the cake proportionally according to their utilities.
- Can you find an algorithm for $n = 3$ agents?

Divisible Goods (Cake Cutting)

Steinhaus Procedure (3 players)

1. Player 1 cuts the cake into three pieces (which she values equally).
2. Player 2 "passes" (if she thinks at least two of the pieces are $\geq 1/3$) or labels two of them as "bad". If player 2 passed, then players 3, 2, 1 each choose a piece (in that order) and we are done.
3. If player 2 did not pass, then player 3 can also choose between passing and labeling. If player 3 passed, then players 2, 3, 1 each choose a piece (in that order) and we are done.
4. If neither player 2 or player 3 passed, then player 1 has to take (one of) the piece(s) labeled as "bad" by both 2 and 3. The rest is reassembled and 2 and 3 play cut-and-choose.

Divisible Goods (Cake Cutting)

- Assume there are $n = 2$ agents. What algorithm (protocol) they should use to divide the cake proportionally according to their utilities.
- Can you find an algorithm for $n = 3$ agents?
- And what about n players?

Divisible Goods (Cake Cutting)

Banach-Knaster Last-Diminisher Procedure (n players)

1. Player 1 cuts off a piece (that she considers to represent $1/n$).
2. That piece is passed around the players. Each player either lets it pass (if she considers it too small) or trims it down further (to what she considers $1/n$).
3. After the piece has made the full round, the last player to cut something off (the "last diminisher") is obliged to take it.
4. The rest (including the trimmings) is then divided amongst the remaining $(n-1)$ players. Play cut-and-choose once $n = 2$.

Divisible Goods (Cake Cutting)

Envy-freeness:

- Is the allocation envy-free for the players using proposed algorithms?
- Assume there are $n = 3$ agents. What algorithm (protocol) they should use to divide the cake in an envy-free way?

Divisible Goods (Cake Cutting)

Selfridge-Conway Procedure (3 players)

1. Player 1 cuts the cake in three pieces (she considers equal).
2. Player 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). – If she passed, then let players 3, 2, 1 pick (in that order).
3. If player 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now (note: the partial allocation is envy-free).
4. Now divide the trimmings. Whoever of 2 and 3 received the untrimmed piece does the cutting. Let players choose in this order: non-cutter, player 1, cutter.

References

Lecture Notes on Fair Division by U. Endriss