ePAL - graphs

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Outline

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Example 1

Determine what is the type of graphs (directed, undirected, multigraph, simple graph, finite, infinite) in the following case:

At a numerical axis we choose all natural numbers as nodes. We connect nodes m, n with an edge if m divides n without the reminder.

Example 2

Determine what is the type of graphs (directed, undirected, multigraph, simple graph, finite, infinite) in the following case:

In the village we choose nodes as all families who own at least one car. We link with an edge two different nodes if and only if both the families own the same brand of car.

Example 3

Determine what is the type of graphs (directed, undirected, multigraph, simple graph, finite, infinite) in the following case:

Choose travel agencies in one city as nodes, and also all countries on Earth. Two nodes will be linked by an edge, if one of them is an office and the other one is a state and also the office provides a tour operator in question in this the state.

Example 4

Determine what is the type of graphs (directed, undirected, multigraph, simple graph, finite, infinite) in the following case:

Select a subset R of the set of all two elements subsets of the set K that is any non-empty finite set.

Is the set R a graph?

If so, what type?

Example 5

The table shows the real situations, which a graph can be used for a description of. Decide what type of graph, directed or undirected, is better suited to describe the facts:

	Situation	Nodes	Nodes x and y are linked by an edge, if
A	Roadmaps	settlements and cities	A straight road leads between x and y.
B	Road Network	Junctions	A road links junction x and y
C	Road network	Roads	Roads x and y have a common crossroad.
D	Molecule	Atoms	Atoms x and y are bound by a chemical bond.
E	Electrical equipment	Wires	There is a device between wires x a y.

Example 6

The table shows the real situations, which a graph can be used for a description of. Decide what type of graph, directed or undirected, is better suited to describe the facts:

	Situation	Nodes	Nodes x and y are linked by an edge, if
F	Chess	Position of chess pieces on the board	The position y is reachable by one move from the position x.
G	Office	Office worker	Officer x is a manager of officer y.
H	National economy	Products and services	x is required to produce y.
I	Job assignment	Workers and tasks	A worker x can perform a task y.

Example 7

The table shows the real situations, which a graph can be used for a description of. Decide what type of graph, directed or undirected, is better suited to describe the facts:

	Situation	Nodes	Nodes x and y are linked by an edge, if
J	Scheduling	Tasks, activities	Activity x must be completed before activity y begins.
K	Genealogy	Persons	Person x is a descendant of a person y
L	Polyhedron	Polyhedron vertices	Vertices x and y are incident to the same edge.

Example 8

The table shows the real situations, which a graph can be used for a description of. Decide what type of graph, directed or undirected, is better suited to describe the facts:

	Situation	Nodes	Nodes x and y are linked by an edge, if
M	Programming Language	Language lexical elements	The element x may follow the element y.
N	Computer program	Procedures	A subroutine y can be called by subroutine x.
O	Textbook	Chapters	To study chapter x one needs to know chapter y.

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Example 9

Must be a number of nodes in a graph of odd degree odd or even? Why?

Example 10

Must be a number of nodes in a graph of odd degree odd or even?

Is the previous statement about the sum of the degrees true in the opposite way, i.e., every even number can represent a sum of degrees of all vertices of a graph G? Or cannot any even numbers specify a sum? Which?

Example 11

Is there a graph that has fewer edges than nodes and no vertex of degree 1?

Example 12

When two participants of a conference meet for the first time, they exchange their business cards.

Is the number of cards, which changed their owners during the conference odd or even?

Is it the number of participants who distributed an odd number of cards odd or even?

Example 13

8 teams play a tournament as a round-robin system. So far, they have played a total of 9 games.

Is it possible that each team has played at most 2 games?

Is it possible that after playing 17 games each team played at most 4 games?

Example 14

Can all degrees of nodes in an ordinary graph be all different numbers?

Example 15

Given a graph with n nodes having exactly two vertices of the same degree.

May it be a degree of 0 or $n-1$?

Example 16

Prove by means as simple as possible a construction that for every $n > 2$ there is a graph with n nodes, which contains parallel edges (not loops) and whose nodes all have a different degree.

Example 17

Find a graph with 7 nodes, in which just two nodes have the same degree.

Find yet another such a graph. Are they complements to each other?

Example 18

The task whether there might be two triples at a company, one where everyone knows everyone and in the second do not know, is trivial.

Five people is enough for a positive answer. Draw the graph.

Example 19

Can a connected graph consist of a fewer number of edges than nodes?

Example 20

Can a disconnected graph have more edges than nodes?

Example 21

What is the minimum number of edges that must be removed from K_n to produce a disconnected graph?

Example 22

What is the maximum number of edges that can be removed from K_n to maintain a graph connected?

Example 23

What is the minimum number of edges that must be removed from K_n to produce a graph of k components?

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Example 24

There are three pieces of paper. We cut one of them into three pieces, one of the pieces are cut again into three pieces, etc.

How many of pieces one obtains when k pieces of paper is cut (independently of size)?

Solve the task also in general for m initial pieces and cutting into n parts.

Interpret the task as a graph.

Example 25

On the table we see k sealed boxes. Whenever you open any box, it will be either empty or will contain again (probably smaller) k sealed boxes. In addition, we know that the total number of non-empty boxes is m.

When all the full boxes are open, how many sealed boxes do we see?

Example 26

A tournament in table tennis that is played in exclusionary manner, is attended by a total of 19 players and players. Six randomly selected players had to play one more game.

Draw a graph representing the progress of the tournament and determine the number of its internal nodes, i.e. the number of matches that were played during the tournament.

How many games would be played for 147 players and players?

Example 27

In a tree with n nodes we assign a random orientation to each edge.

What is the probability that the tree will be a root tree?

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Example 28

Give an example of a strongly connected graph with each edge belonging to exactly two cycles.

Example 29

Is it possible to label nodes of an acyclic graph with n nodes by numbers $1 \ldots n$ so that the source node of each edge is numbered with a number that is less than the number of target node?

In other words, can every edge lead from a "small" number to a "bigger" one?

Example 30

K_5 edges are randomly oriented.

What is the probability that it will result in an acyclic graph?

What is the probability of the same phenomenon for K_n?

Hint: *You can utilize knowledge that acyclic graphs are the ones that are can be ordered topologically.*

Example 31

Can you find a directed graph in which the sum of all output degrees differs from the sum of all input degrees?

Example 32

Is there an acyclic graph with no roots or leaves?

Example 33

Is there a directed graph with the same number of roots and leaves?

Can a graph have more roots than leaves? Or vice versa?

Example 34

Is there a directed graph with no roots and leaves?

Only with the root (roots) and without leaves?

Only the leaves (leaf) and without roots?

Example 35

Each undirected edge of a undirected circle is oriented arbitrarily.

What is the relationship between the number of roots and leaves in the resulting graph?

Example 36

Input and output degree of all vertices in a graph without loops is 1.

How does this graph look like?

Is it connected?

Example 37

Find a graph in which the input and output degree of each node is nonzero, while the graph contains a node which does not pass any cycle.

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Alg 1

Construct a directed graph of a program in a programming language so that nodes in the graph are all functions and procedures and an edge leads from node x to y if and only if the function or procedure corresponding node x calls a function or procedure corresponding to the node y.

What can be said about the program, in which the graph there are cycles or loops?

Suppose also that the body of each function and procedure must be declared before, than the function or procedure is called. Is it then necessarily that every graph belonging to any such a program is acyclic?

Alg 2

Consider the following simple algorithm:

1. As long as the graph contains at least one node with the output degree 0 (leaf) remove this node and all incident edges.
2. When the result of an empty graph then the original graph was acyclic otherwise the original graph was not acyclic

Does this algorithm always give a correct result? Why?

Alg 3

An acyclic graph has one root and one leaf. Add an edge leading from the leaf to the root.

Can you guarantee that the result will be strongly connected?

Alg 4

We are provided with a directed graph representation.

How long will it take to find the output degree of all nodes?

Alg 5

The complete binary tree with seven nodes is balanced and nodes are numbered $1 \ldots 7$ as in (binary) heap.

Write an adjacency matrix of the graph. Suppose that our tree has $2^n - 1$ nodes ($n > 0$), i.e. the depth $n - 1$ (the root has depth 0).

Describe an algorithm which will generate an adjacency matrix directly, without a real searching through the tree.

Alg 6

Square of graph G is a graph G^2, whose set of nodes coincides with the set of nodes of a graph G and a set edges is determined as follows: G^2 contains the edge (u, v) only and only if G contains both edges (u, w) and (w, v), where w is any node of the graph G. In other words, G^2 arises from G by adding edges to G between all nodes connected with a path of length 2 and remove the original edges.

Describe how you create G^2, when the graphs are represented by

1. a linked representation
2. an adjacency matrix.

Which option will be faster?

Alg 7

Modify the algorithm (or your functional program) DFS so that it lists each graph edge (each once).

Alg 8

Modify the algorithm (or your functional program) BFS so that it lists each graph edge (each once).

Alg 9

Write a functional DFS code both non-recursively and recursively.

Alg 10

Write a program that performs condensation of a directed graph.

Alg 11

Remove from the graph the maximum possible number of edges, while maintaining the condition that the condensation of the modified graph must remain the same as the original condensation graph. Write algorithm and program.

Alg 12

Describe how to find and list all paths of length 3 in the acyclic graph.

Alg 13

A directed graph is called **semiconnected** if there is a path between any two nodes u and v at least in one direction, i.e. either $u \to v$ or $v \to u$ or (possibly) both.

Draw few semiconnected graphs that are not strongly connected.

Draw an acyclic graph that is not semiconnected.

Design an algorithm verifying semiconnectivity of the tested graph. What is its complexity?

References I