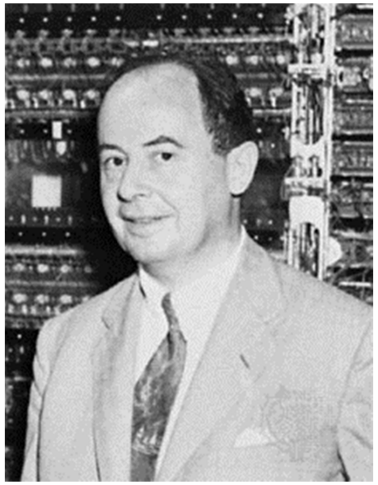
Pseudorandom numbers

John von Neumann:



Any one who considers
arithmetical methods of producing random digits
is, of course, in a state of sin.
For, as has been pointed out several times,
there is no such thing as a random number
— there are only methods to produce random numbers, and
a strict arithmetic procedure of course is not such a method.

"Various Techniques Used in Connection with Random Digits,", in *Monte Carlo Method (*A. S. Householder, G. E. Forsythe, and H. H. Germond, eds.), National Bureau of Standards Applied Mathematics Series, 12, Washington, D.C.: U.S. Government Printing Office, 1951, pp. 36–38.

Pseudorandom number generator

Random vs. pseudorandom behaviour

Random behaviour -- Typically, its outcome is unpredictable and the parameters of the generating process cannot be determined by any known method.

Examples:

Parity of number of passengers in a coach in rush hour.

Weight of a book on a shelf in grams modulo 10.

Direction of movement of a particular N_2 molecule in the air in a quiet room.

Pseudo-random -- Deterministic formula,

-- Local unpredictability, *"output looks like random*",

-- Statistical tests might reveal more or less "random behaviour"

Pseudorandom integer generator

A pseudo-random integer generator is an algorithm which produces a sequence

$$\{x_n\} = x_0, x_1, x_2, \ldots$$

of non-negative integers, which manifest pseudo-random behaviour.

Pseudorandom number generator

Pseudorandom integer generator

Two important statistical properties:

- Uniformity
- Independence

Random number in a interval $[a, b]$ must be independently drawn from a uniform distribution with probability density function:

$$f(x) = n \begin{cases} \dfrac{1}{b-a} & x \epsilon [a, b] \\ 0 & \text{elsewhere} \end{cases}$$

Good generator

- Uniform distribution over large range of values:

  Interval $[a, b]$ is long, period = $b - a,$ generates all integers in $[a, b]$.
- Speed

  Simple generation formula.

  Modulus (if possible) equal to a power of two -- fast bit operations.

Pseudorandom number generator

Random floating point number generator

Task 1: Generate (pseudo) random integer values from an interval $[a, b]$.
Task 2: Generate (pseudo) random floating point values from interval $[0,1[$.

Use the solution of Task 1 to produce the solution of Task 2.
Let $\{x_n\}$ be the sequence of values generated in Task 1.
Consider a sequence $\{y_n\} = \{x_n \,/\, (b-a-1)\}$.

Each value of $\{y_n\}$ belongs to $[0,1[$.
"Random" real numbers are thus approximated by "random" fractions.
Large length of $[a, b]$ guarantees sufficiently dense divison of $[0,1[$.

Example

$[a, b] = [0, 1024]$.

$\{x_n\} = \{712, \quad 84, \quad 233, \quad 269, \quad 810, \quad 944, \dots\}$

$$\{y_n\} = \{712/1023,\ 84/1023,\ 233/1023,\ 269/1023,\ 810/1023,\ 944/1023,\ \dots\}$$
$$= \{0.696,\ 0.082,\ 0.228,\ 0.263, 0.792,\ 0,923,\ \dots\}$$

Linear Congruential Generator

Linear Congruential generator

Linear Congruential generator produces sequence $\{x_n\}$ defined by relations

$$0 \le x_0 \le M,$$
$$x_{n+1} = (Ax_n + C) \mod M, \quad n \ge 0.$$

Modulus $M,$ seed $x_0,$ multiplier and increment A, C .

Example

$$M = 18, \ A = 7, \ C = 5.$$

$$x_0 = 4,$$
$$x_{n+1} = (7x_n + 5) \mod 18, \quad n \ge 0.$$

$$\{x_n\} = \underbrace{4, 15, 2, 1, 12, 17, 16, 9, 14, 13, 6, 11, 10, 3, 8, 7, 0, 5,}_{\text{sequence period, length = 18}} 4, 15, 2, 1, 12, 17, 16, \ldots$$

Linear Congruential Generator

Example

$$M = 15, A = 11, \ C = 6.$$

$$x_0 = 8,$$
$$x_{n+1} = (11x_n + 6) \ \text{mod} \ 15, \qquad n \geq 0.$$

$$\{x_n\} = \underbrace{8, 14, 5, 11, 2,}_{\text{sequence period, length = 5}} 8, 14, 5, 11, 2, 8, 14, \ \ldots$$

Example

$$M = 13, A = 5, \ C = 11.$$

$$x_0 = 7,$$
$$x_{n+1} = (5x_n + 11) \ \text{mod} \ 13, \qquad n \geq 0.$$

$$\{x_n\} = \underbrace{7,}_{\text{sequence period, length = 1}} 7, 7, 7, 7, \ \ldots$$

Linear Congruential Generator

Misconception

Prime numbers are "more random" than composite numbers, therefore using prime numbers in a generator improves randomness.

Counterexample: Example 3, all parameters are primes:

$$x_0 = 7, \quad x_{n+1} = (5x_n + 11) \ \mathrm{mod}\ 13.$$

Maximum period length

The lenght of period is maximum, i.e. equal to M, iff conditions 1. - 4. hold:

1. M is not a prime.
2. C and M are coprimes.
3. $A–1$ is divisible by each prime factor of M.
4. If 4 divides M then also 4 divides $A–1$.

Example

1. $M = 17, \ A = 7, \ C = 6.$ Condition 1. violated
2. $M = 18, \ A = 7, \ C = 6.$ Condition 2. violated
3. $M = 20, \ A = 17, \ C = 7.$ Condition 3. violated
4. $M = 20, \ A = 11, \ C = 7.$ Condition 4. violated
5. $M = 18, \ A = 7, \ C = 5.$ All four conditions hold

Linear Congruential Generator

Randomnes issues

Example 1: $x_0 = 4,$

$x_{n+1} = (7x_n + 5) \ \mathrm{mod} \ 18, \quad n \geq 0.$

$$\{x_n\} = \underbrace{4, 15, 2, 1, 12, 17, 16, 9, 14, 13, 6, 11, 10, 3, 8, 7, 0, 5,}_{\text{sequence period, length = 18}} 4, 15, 2, 1, 12, 17, 16, \ ...$$

$$\{x_n \ \mathrm{mod} \ 2\} = \underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,}\underbrace{0, 1,} 0, ...$$

$$\{x_n \ \mathrm{mod} \ 3\} = \underbrace{1, 0, 2,}\underbrace{1, 0, 2,}\underbrace{1, 0, 2,}\underbrace{1, 0, 2,}\underbrace{1, 0, 2,}\underbrace{1, 0 ,2,}\underbrace{1, 0, 2,}\underbrace{1, 0, 2,} 1, ...$$

$$\{x_n \ \mathrm{div} \ 4\} = \underbrace{0, 3, 0, 0, 3, 4, 4, 2, 3, 3, 1, 2, 2, 0, 2, 1, 0 ,1,} 0, 3, 0, 0, 3, 4, 4, ...$$

Trouble

Low order bits of values generated by LCG exhibit significant lack of randomness.

Remedy

Disregard the lower bits in the output (not in the generation process!).

Output the sequence $\{y_n\} = \{x_n \ \mathrm{div} \ 2^H\}$, where $\mathrm{H} \geq \log_2(\mathrm{M}/4)$.

Sequence period

Many generators produce a sequence $\{x_n\}$ defined by the general recurrence rule

$$x_{n+1} = f(x_n) \qquad n \geq 0.$$

Therefore, if $x_n = x_{n+k}$ for some $k > 0$, then also

$$x_{n+1} = x_{n+k+1},\ x_{n+2} = x_{n+k+2},\ x_{n+3} = x_{n+k+3}, \ldots$$

Sequence period

Subsequence of minimumpossible length $p > 0$, $\{x_n, x_{n+1}, x_{n+2}, \ldots\ x_{n+p-1}\}$ such that for any $n \geq 0$: $x_n = x_{n+p}$.

Random repetitions

Values $x_n,\ x_{n+1},\ x_{n+2}\ , \ldots,\ x_{n+p-1}$ are unique in some (simple) generators.

To increase the random-like behaviour of the sequence additional operations may be applied.
Typicaly, it is computing $x_n \mod W$ for some $W < \max_{n\geq0}\{x_n\}$,
often W is a power of 2 and mod is just bitwise right shift.

Combined Linear Congruential Generator

Definition Let there be r linear congruential generators defined by relations

$$0 \le y_{k,0} \le M_k$$
$$y_{k,n+1} = \left(A_k y_{k,n} + C_k\right) \bmod M_k, \quad n > 0.,$$
$$1 \le k \le r.$$

The combined linear congruential generator is a sequence $\{x_n\}$ defined by

$$x_n = (y_{1,n} - y_{2,n} + y_{3,n} - y_{4,n} + \dots (-1)^{r-1} \cdot y_{r,n}) \bmod (M_1 - 1), \quad n \ge 0.$$

Fact Maximum possible period length (not always attained!) is $(M_1 - 1)(M_2 - 1) \ldots (M_r - 1)\ /\ 2^{r-1}$.

Example 1 $\mathrm{r} = 2, \quad 1 \le y_{1,0} \le 2147483562, \quad 1 \le y_{2,0} \le 2147483398$

$$y_{1,n+1} = \left(40014 y_{1,n} + 0\right) \bmod 2147483563, \quad n \ge 0,$$
$$y_{2,n+1} = \left(40692 y_{2,n} + 0\right) \bmod 2147483399, \quad n \ge 0,$$

$$x_n = (y_{1,n} - y_{2,n}) \bmod 2147483562, \qquad n \ge 0.$$

Period length is $\frac{(M_1-1)(M_2-1)}{2} = 2305842648436451838$.

Combined Linear Congruential Generator

Example 2 $r = 3, \quad y_{1,0} = y_{2,0} = y_{3,0} = 1,$

$$y_{1,n+1} = (9y_{1,n} + 11) \bmod 16, \quad n \geq 0,$$

$$y_{2,n+1} = (7y_{2,n} + 5) \bmod 18, \quad n \geq 0,$$

$$y_{3,n+1} = (4y_{3,n} + 8) \bmod 27, \quad n \geq 0,$$

$$x_n = (y_{1,n} - y_{2,n} + y_{3,n}) \bmod 15, \quad n \geq 0.$$

$\{x_n\}$ = 1, 4, 0, 2, 7, 12, 2, 2, 6, 6, 7, 7, 5, 2, 0, 9, 1, 1, 9, 11, 7, 9, 2, 8, 9, 12, 1, 1, 14, 2, 12, 9, 7, 4, 9, 8,
1, 6, 14, 5, 9, 0, 1, 4, 8, 8, 6, 9, 4, 4, 3, 11, 4, 3, 11, 14, 9, 12, 1, 7, 11, 11, 0, 0, 1, 1, 0, 11, 10, 3, 11, 11,
3, 6, 1, 4, 11, 2, 3, 6, 10, 10, 9, 11, 7, 3, 2, 14, 3, 3, 10, 1, 8, 14, 3, 9, 10, 13, 3, 2, 1, 3, 14, 14, 12, 6, 13,
13, 5, 8, 3, 6, 10, 1, 6, 5, 10, 9, 11, 11, 9, 6, 4, 13, 5, 5, 12, 0, 10, 13, 6, 11, 13, 0, 5, 5, 3, 6, 1, 13, 11, 8,
12, 12, 4, 10, 3, 8, 13, 3, 5, 8, 12, 12, 10, 13, 8, 8, 6, 0, 7, 7, 0, 2, 13, 0, 5, 11, 0, 0, 4, 4, 5, 5, 3, 0, 13, 7,
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2, 2, 9, 12, 7, 10, 14, 5, 9, 9, 13, 13, 0, 14, 13, 9, 8, 2, 9, 9, 1, 4, 14, 2, 9, 0, 1, 4, 9, 8, 7, 9, 5, 2, 0, 12, 1,
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3, 10, 13, 9, 11, 4, 9, 11, 14, 3, 3, 1, 4, 14, 11, 9, 6, 10, 10, 3, 8, 1, 6, 11, 2, 3, 6, 10, 10, 8, 11, 6, 6, 4,
13, 6, 5, 13, 0, 11, 14, 3, 9, 13, 13, 2, 2, 3, 3, 1, 13, 12, 5, 13, 12, 5, 8, 3, 6, 13, 4, 5, 8, 12, 12, 10, 13,
9, 5, 4, 0, 5, 5, 12, 3, 10, 1, 5, 11, 12, 0, 4, 4, 3, 5, 1, 0, 14, 8, 0, 0, 7, 10, 5, 8, 12, 3, 7, 7, 12, 11, 13, 12,
11, 8, 6, 0, 7, 7, 14, 2, 12, 0, 7, 13, 0, 2, 7, 6, 5, 8, 3, 0, 13, 10, 14, 14, 6, 12, 4, 10, 0, 5, 7, 9, 14, 14, 12,
0, 10, 10, 8, 2, 9, 9, (sequence restarts:) 1, 4, 0, 2, 7, 12, 2, 2, 7, 7, 5, ...

Period length is $432 < 15 \cdot 17 \cdot 26 / 2$.

Lehmer Generator

Lehmer generator produces sequence $\{x_n\}$ defined by relations

$$0 < x_0 < M, \qquad x_0 \text{ coprime to } M.$$

$$x_{n+1} = Ax_n \bmod M, \qquad n \geq 0.$$

Modulus M, seed x_0, multiplier A .

Example 1

$$x_0 = 1,$$

$$x_{n+1} = 6x_n \bmod 13.$$

$$\{x_n\} = \underbrace{1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11,}_{\text{sequence period, length = 12}} 1, 6, 10, 8, 9, 2, 12, \ldots$$

Example 2

$$x_0 = 2,$$

$$x_{n+1} = 5x_n \bmod 13.$$

$$\{x_n\} = \underbrace{2, 10, 11, 3,}_{\text{sequence period, length = 4}} 2, 10, 11, 3, 2, 10, 11, 3, \ldots$$

Lehmer Generator

$$0 < x_0 \; 1 < \; M, \; x_0 \text{ coprime to } M.$$

$$x_{n+1} = Ax_n \bmod \; M, \qquad n \geq 0.$$

Fact

The sequence period length is maximal and equal to $M{-}1$ if
M is prime and
A is a primitive root of the multiplicative group of integers modulo M,

Primitive root

G is a primitive root of the multiplicative group of integers modulo M if
$\{G, G^2, G^3, ..., G^{M-1}\} = \{1, 2, 3 \; ..., \; M{-}1\}$ (all powers are taken modulo M)

Example

$M = 13, \; G = 6,$
$\{G, G^2, G^3, ..., G^{12}\} = \{6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11, 1\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11{,}12\},$ G is a primitive root.
$M = 13, \; G = 2,$
$\{G, G^2, G^3, ..., G^{12}\} = \{2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11{,}12\}.$ G is a primitive root.
$M = 13, \; G = 5,$
$(G, G^2, G^3, ..., G^{12}) = (5, 12, 8, 1, 5, 12, 8, 1, 5, 12, 8, 1).$ G is not a primitive root.

Lehmer Generator

Finding group primitive roots

No elementary and effective method is known.
Special cases has been studied in detail.

Multiplicative group of integers modulo $M_{31} = 2^{31}–1 = 2\ 147\ 483\ 647$.

G is a primitive root iff $G \equiv 7^b \bmod M_{31}$ where b is coprime to $M_{31}–1$.
The prime factors $M_{31}–1$ are 2, 3, 7,11, 31, 151, 331.
($M_{31}–1 = 2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$)

Example

$G = 7^5 = 16807$ is a primitive root because 5 is coprime to $M_{31}–1$.

$G = 7^{1116395447} = 48271$ is a primitive root because 1116395447 is a prime, therefore it is coprime to $M_{31}–1$.

$G = 7^{1058580763} = 69621$ is a primitive root because $69621 = 19 \cdot 41 \cdot 61 \cdot 22277$ therefore 69621 is coprime to $M_{31}–1$.

Blum Blum Shub Generator

Blum Blum Shub generator produces sequence $\{x_n\}$ defined by relations

$$2 \le x_0 \le M, \quad x_0 \text{ coprime to } M.$$

$$x_{n+1} = x_n^2 \bmod M$$

Modulus M, seed x_0.

Seed x_0 coprime to M.
Modulus M is a product of two big primes P and Q.
$P \bmod 4 = \mathrm{Q} \bmod 4 = 3$,
$\gcd(\varphi(\mathrm{P}-1), \varphi(\mathrm{Q}-1))$ should be small, (cannot be 1).

Example $x_0 = 4, \quad M = 11 \cdot 47, \qquad \gcd(\varphi(10), \varphi(46)) = \gcd(4, 22) = 2,$

$x_{n+1} = x_n^2 \bmod 517.$

$$\{x_n\} = \underline{4,}\ 16, 256, 394, 136, 401, 14, 196, 158, 148, 190, 427, 345, 115, 300, 42, 213,$$
$$390, 102, 64, 477, 49, 333, 251, 444, 159, 465, 119, 202, 478, 487, 383, 378,$$
$$\underbrace{192, 157, 350, 488, 324, 25, 108, 290, 346, 289, 284,}_{\text{sequence period, length = 44}}\ \underline{4,}\ 16, 256, 394, 136, \ldots$$

Primes related notions

Prime counting function π(*n*)

Counts the number of prime numbers less than or equal to n.

Example

π(*10*) = 4. Primes less than or equal to 10: 2, 3, 5, 7.

π(*37*) = 12. Primes less than or equal to 37: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37.

π(*100*) = 25. Primes less than or equal to 100: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Estimate

$$\frac{n}{\ln n} < \pi(n) < 1.25506\frac{n}{\ln n} \text{ for } n > 16.$$

Example

$$\frac{100}{\ln 100} < \pi(100) < 1.25506\frac{100}{\ln 100} \qquad \frac{10^6}{\ln 10^6} < \pi(10^6) < 1.25506\frac{10^6}{\ln 10^6}$$

$$21.715 < \pi(100) = 25 < 27.253 \qquad 72382.4 < \pi(10^6) = 78498 < 90844.3$$

Limit behaviour

$$\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\ln n}} = 1$$

Primes related notions

Euler's totient function $\varphi(n)$

Counts the positive integers less than or equal to n that are relatively prime to n.

Example

$n = 21$, $\varphi(20) = 8$.
coprimes to 21, smaller than 21: 1, 2, 4, 5, 8, 10, 11 13, 16, 17, 19, 20.
$n = 24$, $\varphi(24) = 8$.
coprimes to 24, smaller than 24: 1, 5, 7, 11, 13, 17, 19, 23.

Mersenne prime M_n

Mersenne prime M_n is a prime in the form $2^n–1$, for some $n > 1$.

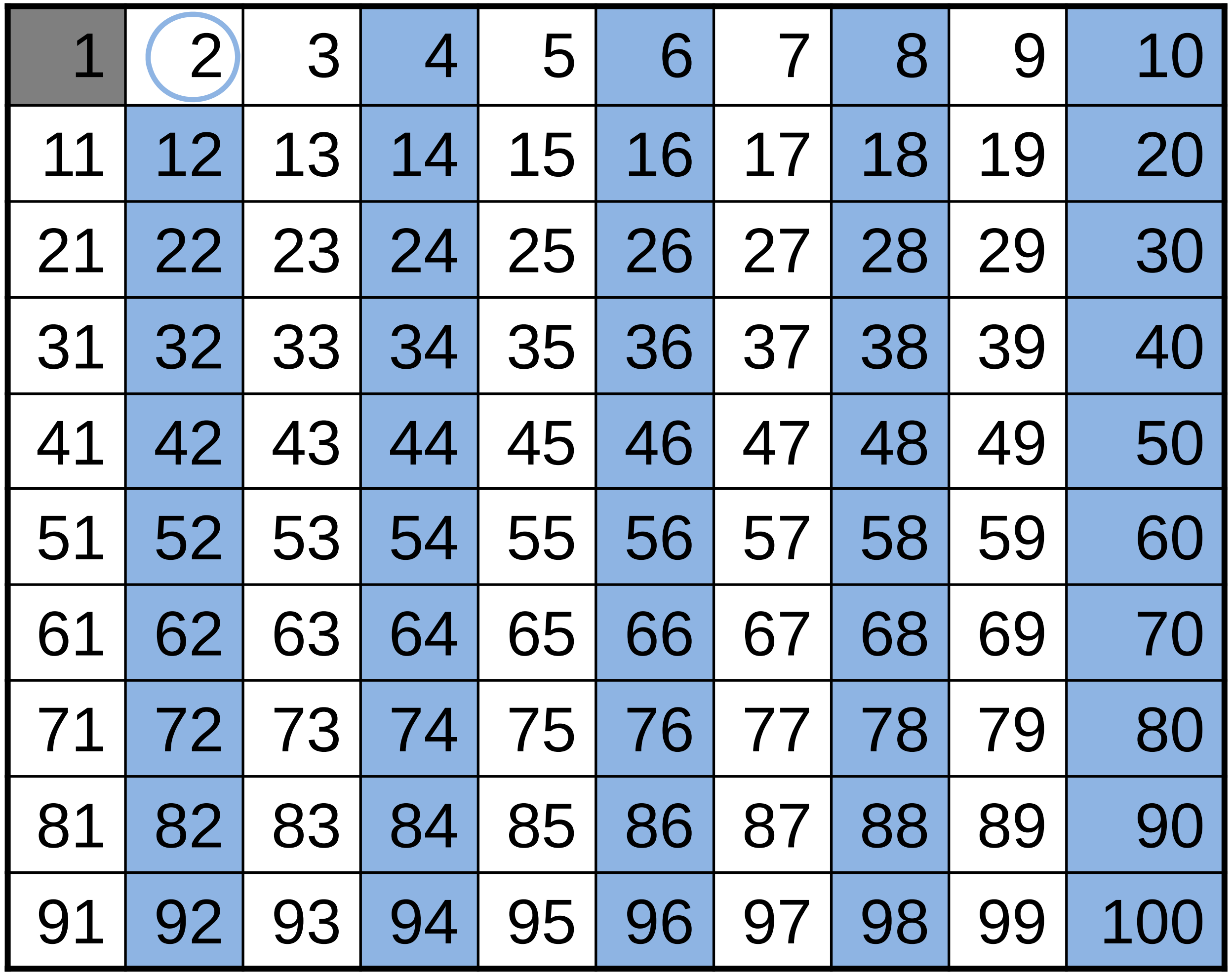
Example

$n = 3$, $M_3 = 2^3–1 = 7$,
$n = 7$, $M_7 = 2^7–1 = 127$,
$n = 31$, $M_{31} = 2^{31}–1 = 2147483647$.

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

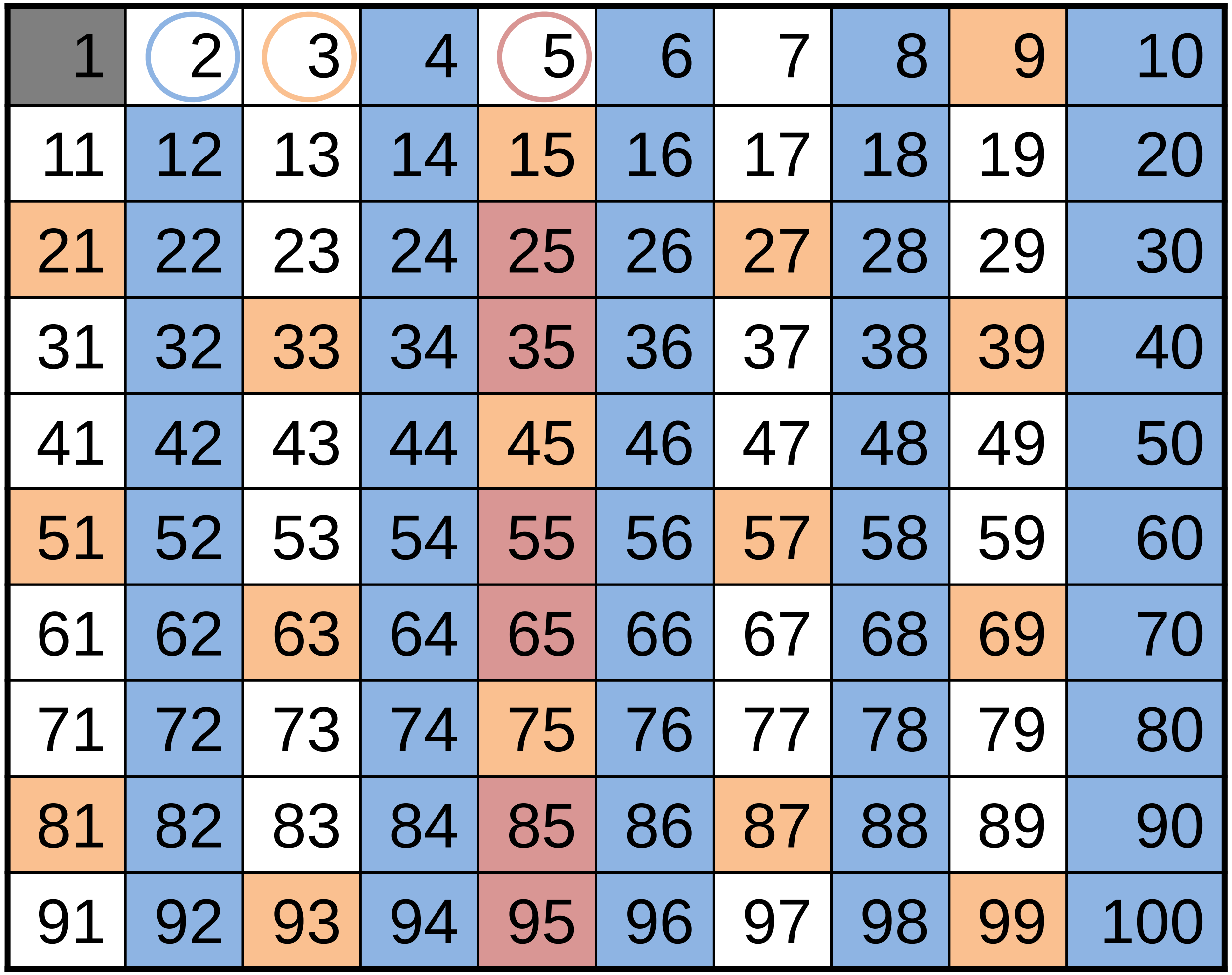


1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

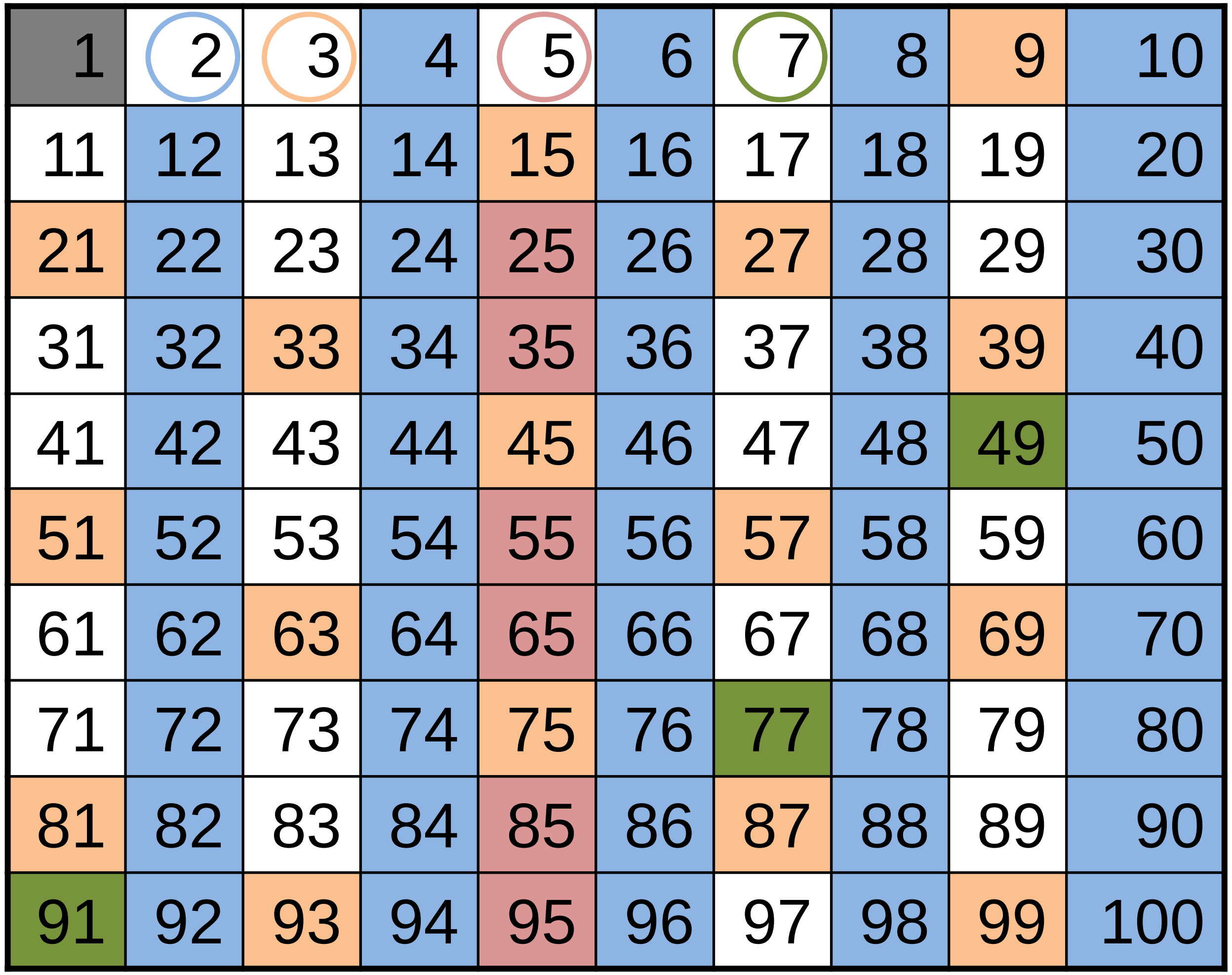
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

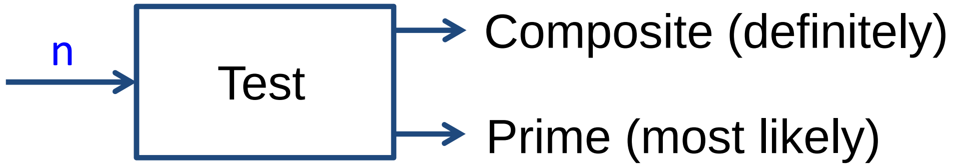
Algorithm

```
EratosthenesSieve ($n$)
    Let $A$ be an array of Boolean values, indexed by integers 2 to $n$, initially all set to **true**
    for $i$ = 2 to $\sqrt{n}$
        if $A[i]$ = **true** then
            for $j = i^2, i^2+i, i^2+2i, i^2+3i$, ..., not exceeding $n$
                $A[i]$ := **false**
    end
    output all $i$ such that $A[i]$ is **true**
end
```

Time complexity: $\mathrm{O}(n \log \log n)$.

Randomized primality tests

General scheme



n

Test

Composite (definitely)

Prime (most likely)

Fermat (little) theorem

If p is prime and $0 < a < p$, then $a^{p-1} \equiv 1 \pmod{p}$.

Fermat primality test

```
FermatTest (n, k)
    for i = 1 to k
        $a$ = random integer in [2, n-2]
        if $a^{n-1} \not\equiv 1 \pmod{n}$ then return Composite
    end
    return Prime
end
```

Flaw: There are infinitely many composite numbers for which the test always fails. (Carmichael numbers: 561, 1105, 1729, 2465, ...)

Randomized primality tests

Miller-Rabin primality test

Lemma: If p is prime and $x^2 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

⇨ Let $n > 2$ be prime, $n-1 = 2^r \cdot d$ where d is odd, $1 < a < n-1$.
Then either $a^d \equiv 1 \pmod{n}$ or $a^{2^s \cdot d} \equiv -1 \pmod{n}$ for some $0 \le s \le r-1$.

```
MillerRabinTest ($n$, $k$)
    compute $r$, $d$ such that $d$ is odd and $2^r \cdot d = n-1$
    for $i = 1$ to $k$   // WitnessLoop
        $a$ = random integer in $[2, n-2]$
        $x = a^d \bmod n$
        if $x = 1$ or $x = n-1$ then goto EndOfLoop
        for $j = 1$ to $r-1$
            $x = x^2 \bmod n$
            if $x = 1$ then return Composite
            if $x = n-1$ then goto EndOfLoop
        end
        return Composite
        EndOfLoop:
    end
    return Prime
end
```

Examples:

$n = 1105 = 2^4 \cdot 69 + 1$
$a = 389$
$x_0 = 1039$
$x_1 = 1041$
$x_2 = 781$
$x_3 = 1$ -> Composite

$n = 1105 = 2^4 \cdot 69 + 1$
$a = 390$
$x_0 = 539$
$x_1 = 1011$
$x_2 = 1101$
$x_3 = 16$
-> Composite

$n = 13 = 2^2 \cdot 3 + 1$
$a = 7$
$x_0 = 5$
$x_1 = 12 \equiv -1 \pmod{13}$
WitnessLoop passes

Randomized primality tests

Miller-Rabin primality test

- Time complexity: $\mathrm{O}(k \log^3 n)$.
- If n is composite then the test declares n prime with a probability at most 4^{-k}.
- A deterministic variant exists, however it relies on unproven generalized Riemann hypothesis.

AKS primality test

- First known deterministic polynomial-time primality test.
- Agrawal, Kayal, Saxena, 2002 - Gödel Prize in 2006.
- Time complexity: $\mathrm{O}(\log^3 n)$.
- If n is composite then the test declares n prime with a probability at most 4^{-k}.
- The algorithm is of immense theoretical importance, but not used in practice.

Integer factorization

Difficulty of the problem

- No efficient algorithm is known.
- The presumed difficulty is at the heart of widely used algorithms in cryptography (RSA).

Pollard's rho algorithm

- Effective for a composite number having a small prime factor.

```
PollardRho (n)
    x = y = 2; d = 1
    while d = 1
        x = g(x); y = g(g(y))
        d = GCD(|x-y|, n)
    end
    if d = n return Failure
    else return d
end
```

g(x) .. a suitable polynomial function
For example, $g(x) = x^2 - 1$

GCD .. the greatest common divisor

Pollard’s rho algorithm – analysis

- Assume $n = pq$.
- Values of x and y form two sequences $\{x_k\}$ and $\{y_k\}$, respectively, where y_k=x_{2k} for each k. Both sequences enter a cycle. This implies there is t such that y_t=x_t.
- Sequences $\{x_k \bmod p\}$ and $\{y_k \bmod p\}$ typically enter a cycle of shorter length. If, for some $\mathrm{s} < t$, $x_s \equiv y_s \pmod p$, then p divides $|x_s$-$y_s|$ and the algorithm halts.

- The expected number of iterations is O($\sqrt{p}$)=O($n^{1/4}$) .

References

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