Labs from Optimization
Linear programming

Ivana Duricova, 2015

Introduction

In this exercise we will try linear programming on two simple tasks. The first task we seek the optimal strategy that will give us a certain win at odds betting. In the second task we will deal with interlacing (by fitting) of linear function by set of points, which is one of the most frequent tasks in engineering practice.

Task 1: Certain win

15 of August 2009 as part of the fourth round of Champions football League took place the match between Sparta Prague (domestic) and Kladno (guests). One betting office announced on the match result following courses:

událost	1	10	0	02	2
kurz	1.27	1.02	4.70	3.09	9.00

Tabulka 1: Rates listed on the football game Sparta Prague vs Kladno.

The first line of the table 1 indicates the following events: 1..home win, 10.. home win or draw, 0..draw, 02..draw or win of guests, 2..win of guests. The second line is the rate listed on the event. Betting office allows each bettor to bet on any combination of events (eg. You can simultaneously bet on winning of Sparta and Kladno).

As part of its marketing strategy betting office offers players a one-time bonus 50% of deposited amount up to a maximum of 1000 crowns. Bonus can not be chosen directly, but it can be used for betting. This means that if, for example bettor inserts his 2000 crowns, add him betting office bonus 1000 crowns. Bettors can then wager a total of $2000 + 1000 = 3000$ crowns.

It is a lot of strategy how to bet. One option is to bet so as to maximize a certain win. Certain win is what you get regardless of the result of matches, which is (or should be) unknown. In other words, a certain win is the same as the minimum possible win. Let's take a simple example, which demonstrate a certain notion of winning. Suppose we bet the whole 3000 crowns (our 2000 + 1000 bonus) that Kladno win. In this case is certain win 0 crown, which you "get" in case Sparta win or in case there will be draw. It is obvious that to ensure a zero certain win, we need to cover some amount of each of the possible results of the match. The first thing we think of is 3000 crown uniformly to all the events, ie. $3000/5 = 600$ crowns on each at each event being written betting office. In this case, the value of certain win $600 \times (1.27 + 1.02) = 1374$ crown, we get if Sparta win (ensure that in all other cases, the win will be higher). As you can see, even this bet does not seems to be optimal because as a result we can lose $2000 - 1374 = 626$ crown.

Our goal is to find the best bet, ie. Optimum distribution of invested 2000 crown plus 1000 crown bonus for each event, so make sure the maximum win. If the job properly resolve, you will see that even guaranteeing optimum bet higher than invested win 2000 crown.

Converting verbal assignment to a problem LP Our goal is to express finding the best bet as linear programming. First, we introduce the vector variables $\boldsymbol{x} \in \mathbb{R}^5$, The coordinates will match the amount of money in the event. The particular choice of coordinates vector $\boldsymbol{x}$ is shown in table 2. When searching for the

event	1	10	0	02	2
invested money [Kč]	x_1	x_2	x_3	x_4	x_5

Tabulka 2: Vector $\boldsymbol{x} \in \mathbb{R}^5$ encodes the amount of money in each event listed by betting office.

best bet would be to pick from a set of allowable bets, which includes vectors $\boldsymbol{x}$ such that

$$\sum_{i=1}^{5} x_i = 3000\,, \qquad \text{and at the same time} \qquad x_i \geq 0,\ i = \{1,\dots,5\}. \tag{1}$$

The first equation says that the sum of bets on each event is equal to 3000 crowns, available to us and we want to wager. The set of five inequality expresses the fact that we can not put a negative amount. From the tables 1 and 2 easy to deduce what will be the value of win depending on the result of the match:

result	home win	draw	win of guests
value of win [Kč]	$1.27x_1 + 1.02x_2$	$1.02x_2 + 4.70x_3 + 3.09x_4$	$3.09x_4 + 9x_5$

The value of certain win v, i.e. the minimum win is therefore equal to

$$v = \min\{1.27x_1 + 1.02x_2, 1.02x_2 + 4.70x_3 + 3.09x_4, 3.09x_4 + 9x_5\} \tag{2}$$

Finding the best bet then can be expressed as maximizing v by all $\boldsymbol{x}$, which satisfy the constraints (1), i.e.

$$\boldsymbol{x}^* \in \operatorname*{argmax}_{\boldsymbol{x}\in\mathbb{R}^5} \min\{1.27x_1 + 1.02x_2, 1.02x_2 + 4.70x_3 + 3.09x_4, 3.09x_4 + 9x_5\} \tag{3a}$$

under conditio

$$\sum_{i=1}^{5} x_i = 3000\,, \qquad x_i \geq 0, \quad i = \{1,\dots,5\}\,. \tag{3b}$$

Optimization problem (3) requires maximizing a linear function with linear constraints. Finally the task (3) introduce an equivalent linear programming problem, the solution of which there are a large number of numerical algorithms. Equivalent to the problem of linear programming has the form

$$(\boldsymbol{x}^*, \lambda^*) \in \operatorname*{argmin}_{\substack{\boldsymbol{x}\in\mathbb{R}^5\\ \lambda\in\mathbb{R}}} -\lambda \tag{4a}$$

under condition

$$\begin{aligned} 1.27x_1 + 1.02x_2 &\geq \lambda \\ 1.02x_2 + 4.70x_3 + 3.09x_4 &\geq \lambda \\ 3.09x_4 + 9x_5 &\geq \lambda \\ \sum_{i=1}^{5} x_i &= 3000 \\ x_i &\geq 0, \quad i = \{1,\dots,5\} \end{aligned} \tag{4b}$$

Problem (4) It is equivalent in the sense that the optimal solution is equal to the optimal solution to the original problem (3).

Tasks which need to be solved

1. Solve the task (4) numerically using function `linprog`, which is part of the optimization toolbox in Matlab. What is the best bet to maximize certain of winning and what is its value?

   Output: i. vector (bit) and ii. scalar (minimal value of win).

2. Consider a modified task, the bitting office odds lists only home win, draw and win of guests, ie. only to events 1, 0 and 2. In addition, the bitting office sets the amount of the minimum bet 400 crown. For such modified verbal task formulate LP task, which again finds betting strategy to maximize the minimal win. Solve the task numerically.

   Output: i. LP formulation of the task (writing in Matlab code is not permitted), ii. vector (bit) and iii. scalar (minimal value of win).

Task 2: Minimax linear function introduced two interlacing points

Consider a set $\mathcal{T} = \{(x_1, y_1), \ldots, (x_m, y_m)\} \in (\mathbb{R} \times \mathbb{R})^m$. Further, consider a nonlinear function

$$f_{a,b}(x) = ax + b \,,$$

which is parameterized by numbers $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Our task is to find a linear function that best approximates a set of points $\mathcal{T}$. There are many criteria that can define a good approximation. Specific criteria for the choice depends on the particular application. We as a criterion for a good approximation we choose the maximum absolute deviation, defined as

$$\varepsilon(a, b) = \max_{i=1,\ldots,m} |f_{a,b}(x_i) - y_i| \,.$$

Our goal will be to find the parameters of the linear function for which $\varepsilon(a, b)$ minimum. In other words, we want to solve the optimization problem

$$(a^*, b^*) \in \operatorname*{argmin}_{\substack{a\in\mathbb{R}\\ b\in\mathbb{R}}} \max_{i=1,\ldots,m} |ax_i + b - y_i| \,. \tag{5}$$

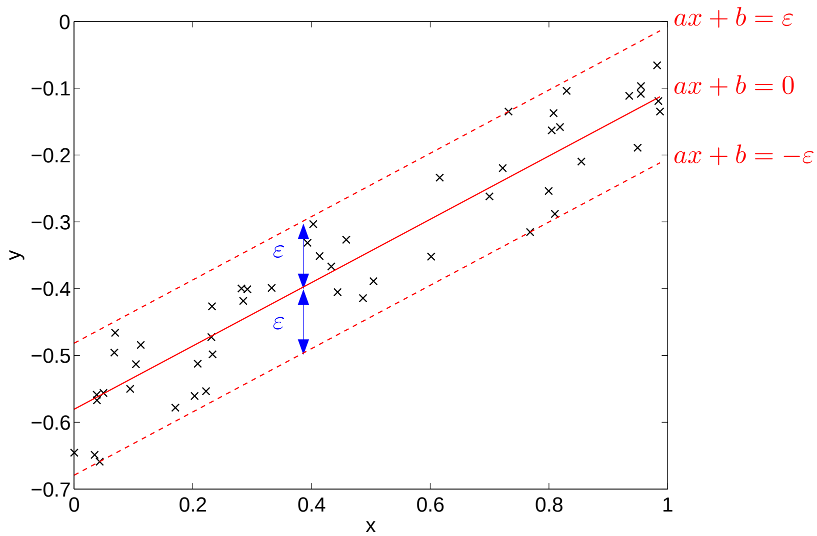
Optimization task (5) have visual graphical interpretation, which is shown in Figure 1. The task is equivalent to finding the minimum width of the belt which encloses all the points. The center line of the strip is given by $a^*x + b^* = 0$ and its width is $2\varepsilon(a^*, b^*)$.

We show that the problem (5) can be converted to linear programming. First, get rid of the absolute value of the criterion and the fact that we use the equality

$$|z| = \max\{z, -z\} \,, \qquad \forall z \in \mathbb{R} \,. \tag{6}$$

By substitution (6) with (5) we get

$$(a^*, b^*) \in \operatorname*{argmin}_{\substack{a\in\mathbb{R}\\ b\in\mathbb{R}}} \max_{i=1,\ldots,m} \max\{ax_i + b - y_i, -ax_i - b + y_i\} \,. \tag{7}$$



Obrázek 1: An illustration of the problem of leading a linear function of a set of points.

Finally we get rid of the maximum use of the very same trick that you discovered in the previous task (see. Point 1). And the problem (7) convert into an equivalent linear programming problem

$$(a^*, b^*) \in \operatorname*{argmin}_{a\in\mathbb{R}, b\in\mathbb{R}, \lambda\in\mathbb{R}} \lambda \tag{8a}$$

under condition

$$\begin{array}{rcll} ax_i + b - y_i & \leq & \lambda\,, & i = 1, \dots, m \\ -ax_i - b + y_i & \leq & \lambda\,, & i = 1, \dots, m \end{array} \tag{8b}$$

Task which need to be solved

1. Download the file `http://cmp.felk.cvut.cz/cmp/courses/OPT/cviceni/02/data1.mat`. File include the vectior `x [50 x 1]` and vector `y [50 x 1]`, which define the set of $\mathcal{T}$. In our case $m = 50$. Draw your points on a graph.

   Output: nothing.

2. For a given set of points to solve the task (8) using function `linprog`. Finding the optimal line graph to plot the points. What is the maximum absolute deviation for that line?

   Output: i. graph and ii. number.

3. Reformulate the task 2 for the case where x_i, $i = 1, \dots, m$, are vectors in $\mathbb{R}^n$. Express this role as an LP problem.

   Output: formulation of the problem (matlab's code is not permitted).