Derivatives and Taylor polynomial

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It is given a set $X \subseteq \mathbb{R}^n$ and mappings $\mathbf{f}\colon X \to \mathbb{R}^m$. The aim is to calculate for this mapping Taylor polynomial at a given point $\mathbf{x}^*$ and then visualize mapping $\mathbf{f}$ and Taylor polynomial. The method of visualization will vary according to the dimensions of the domain and range of values, but it is always the same for display $\mathbf{f}$ and for the Taylor polynomial.

To repeat (see textbooks):

- For mapping $\mathbf{f}\colon X \to \mathbb{R}^m$ Taylor polynomial is first order in point $\mathbf{x}^*$ mapping $\mathbf{T}_1\colon X \to \mathbb{R}^m$ for given regulation
$$\mathbf{T}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x}^*) + \mathbf{f}'(\mathbf{x}^*)\,(\mathbf{x} - \mathbf{x}^*),$$
where $\mathbf{f}'(\mathbf{x}^*)$ the total derivatives (Jacobi matrix) display $\mathbf{f}$ in point $\mathbf{x}^*$.
- For function $f\colon X \to \mathbb{R}$ the Taylor polynomial of second order in point $\mathbf{x}^*$ function $T_2\colon X \to \mathbb{R}$ for given regulation
$$T_2(\mathbf{x}) = f(\mathbf{x}^*) + f'(\mathbf{x}^*)\,(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T f''(\mathbf{x}^*)\,(\mathbf{x} - \mathbf{x}^*), \tag{1}$$
where $f'(\mathbf{x}^*)$ the total derivative of the function f in point $\mathbf{x}^*$ (i.e. row vector of length n) and $f''(\mathbf{x}^*)$ the Hessian matrix function f in point $\mathbf{x}^*$ (i.e. symmetric matrix $n \times n$ second partial derivatives).

  Taylor polynomial for display $\mathbf{f}\colon X \to \mathbb{R}^m$ is mapping $\mathbf{T}_2\colon X \to \mathbb{R}^m$, whose components are functions (1) for each component of the mapping $\mathbf{f}$.

Touch it $\mathbf{T}_1$ a $\mathbf{T}_2$ strictly speaking are not polynomials, because they're not function but mapping. This imprecision in nomenclature is to be tolerated.

We prepare for the task of the following views:

1. $n = 1$, $m = 1$, $X = \langle -3, 2\rangle$, $f(x) = x^3/3 + x^2/2 - x$
2. $n = 1$, $m = 2$, $X = \langle 0, 2\pi\rangle$, $\mathbf{f}(x) = \begin{bmatrix}\cos x\\ \sin x\end{bmatrix}$
3. $n = 2$, $m = 1$, $X = \langle -2, 2\rangle \times \langle -2, 2\rangle$, $f(\mathbf{x}) = f(x_1, x_2) = 2\mathrm{e}^{-\mathbf{x}^T\mathbf{x}}$ (e represents Euler's number)
4. $n = 2$, $m = 3$, $X = \langle 0, 2\pi\rangle \times \langle 0, 2\pi\rangle$, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2) = \begin{bmatrix}(R + r\cos x_2)\cos x_1\\ (R + r\cos x_2)\sin x_1\\ r\sin x_2\end{bmatrix}$
5. $n = 2$, $m = 2$, $X = \langle -1, 1\rangle \times \langle -1, 1\rangle$, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2) = \mathbf{x}\log(1 + \mathbf{x}^T\mathbf{x})$ (log represents the natural logarithm)
6. $n = 1$, $m = 3$, X It is as described in point 2, view the show from the point of composition 2 and from the point of view 4.

Process for compilation

Download matlab functions `prikladn.m`. Number `n` corresponds to the order in the list above. Each function visualizes the mapping $\mathbf{f}$, defines at point $\mathbf{x}^*$ (designated as `xx`) and drew him like a red circle. Understand how the images are visualized. For one pair ($\mathbf{n}$, $\mathbf{m}$) may be open to more than one method of visualization - consider whether the method is the only one possible. Which images are graphs of functions and which do not?

Your task is to complete the matlab functions `T1` a `T2`, Taylor polynomials which are first and second order for mapping $\mathbf{f}$. Uncommenting the appropriate lines will be the polynomials displayed in graph.

As a model task we developed for display task number 1. For task number 6 (i.e. the composition of view 2 and 4) Develop only first-order polynomial. For calculating first total derivatives using the chain rule.

Checking the correctness of polynomials is that polynomials have at point $\mathbf{x}^*$ with mapping $\mathbf{f}$ common value and the first and second derivatives. Thus, the first order polynomial is always 'tangential' to mapping and second order polynomial approximation is quadratic display.

Required output of exercises:

- Working complemented functions in matlab's code `prikladn.m`. Instructor run the submitted functions matlab code and based on image immediately see whether everything is in order.
- Written report should include only Taylor polynomials formulas for each mapping (nothing else there may not be).