1 Lecture

1.1 Foundations

Disjoint union $\bigsqcup_{i=1,\ldots,n} A_i$ of sets $A_1, \ldots, A_n$ is defined as

$$\bigsqcup_{i=1,\ldots,n} A_i = \bigcup_{i=1,\ldots,n} \{(x, i) \colon x \in A_i\}\,. \tag{1}$$

1.2 Types

- Type variables

$$\frac{\Gamma \vdash \diamond \quad X \notin dom(\Gamma)}{\Gamma \cup \{X\} \vdash \diamond} \tag{2}$$

- Recursive type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \mu X.A} \tag{3}$$

$$\frac{\Gamma \vdash e : \mu X.A}{\Gamma \vdash \texttt{unfold e} : A[X \mapsto \mu X.A]} \tag{4}$$

$$\frac{\Gamma \vdash e : A[X \mapsto \mu X.A]}{\Gamma \vdash \texttt{fold e} : \mu X.A} \tag{5}$$

- Universal type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \forall X.A} \tag{6}$$

$$\frac{\Gamma \cup \{X\} \vdash e : A}{\Gamma \vdash \lambda X.e : \forall X.A} \tag{7}$$

$$\frac{\Gamma \vdash e : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash e(B) : A[X \mapsto B]} \tag{8}$$

1.3 Subtype Polymorphism

- Basic setup

  We define a new binary relation $<:$ on types and a new judgement: $\Gamma \vdash A <: B$ ("A is a subtype of B in environment Γ").

$$\frac{}{\Gamma \vdash A <: A} \tag{9}$$

$$\frac{\Gamma \vdash A <: B \quad \Gamma \vdash B <: C}{\Gamma \vdash A <: C} \tag{10}$$

- Subsumption

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A <: B}{\Gamma \vdash e : B} \tag{11}$$

- Top type

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash Top} \tag{12}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A <: Top} \tag{13}$$

- Subtyping of functions

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B <: B'}{\Gamma \vdash A \to B <: A' \to B'} \tag{14}$$

- Subtyping of products

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' \times B' <: A \times B} \tag{15}$$

- Subtyping of unions

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' + B' <: A + B} \tag{16}$$

- Subtyping of records

$$\frac{\Gamma \vdash A'_1 <: A_1 \quad \ldots \quad \Gamma \vdash A'_n <: A_n \quad \Gamma \vdash A'_{n+1} \quad \ldots \quad \Gamma \vdash A'_{n+m}}{\Gamma \vdash \texttt{Record}(l_1 : A'_1, \ldots, l_{n+m} : A'_{n+m}) <: \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)} \tag{17}$$

- Bounded type variables

$$\frac{\Gamma \vdash A \quad X \notin dom(\Gamma)}{\Gamma \cup \{X <: A\} \vdash \diamond} \tag{18}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X} \tag{19}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X <: A} \tag{20}$$

- Subtyping of recursive types

$$\frac{\Gamma \cup \{X <: Top\} \vdash A}{\Gamma \vdash \mu X.A} \tag{21}$$

$$\frac{\Gamma \vdash \mu X.A \quad \Gamma \vdash \mu Y.B \quad \Gamma \cup \{Y <: Top, X <: Y\} \vdash A <: B}{\Gamma \vdash \mu X.A <: \mu Y.B} \tag{22}$$

- Subtyping of universal types

$$\frac{\Gamma \cup \{X <: A\} \vdash B}{\Gamma \vdash \forall X <: A.B} \tag{23}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \cup \{X <: A'\} \vdash B <: B'}{\Gamma \vdash (\forall X <: A.B) <: (\forall X <: A'.B')} \tag{24}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash e : B}{\Gamma \vdash \lambda X <: A.e : \forall X <: A.B} \tag{25}$$

$$\frac{\Gamma \vdash e : \forall X <: A.B \quad \Gamma \vdash A' <: A}{\Gamma \vdash e(A') : B[X \mapsto A']} \tag{26}$$

2 Seminar

1. Does $\{A <: Top, B <: Top\} \vdash A \to B <: A \to B$ hold? Why?
2. Does $\{A <: B, B <: C, C <: Top\} \vdash A + C <: A + B$ hold? Why?
3. Does $\{A <: B, B <: Top\} \vdash \texttt{Ref}\ A <: \texttt{Ref}\ B$ hold? Why?
4. Does $\{X <: Y, Y <: Top, A <: Top\} \vdash \mu X.(X \times Top) <: \mu Y.(Y \times A)$ hold? Why?
5. Does $\{X <: Y, Y <: Top\} \vdash \mu X.(X \to X) <: \mu Y.(Y \to Y)$ hold? Why?
6. Does $\{A <: B, B <: Top\} \vdash \forall X.(X \times A) <: \forall Y.(Y \times B)$ hold? Why?
7. Does $\{A <: B, B <: Top\} \vdash \forall X.(X \times \texttt{Ref}\ A) <: \forall X.(X \times \texttt{Ref}\ B)$ hold? Why?

3 Homework

Your task is to implement the missing methods in the library of types.