

# Advanced types

## 1 Foundations

### 1.1 Types

- Type variables

$$\frac{\Gamma \vdash \diamond \quad X \notin \text{dom}(\Gamma)}{\Gamma \cup \{X\} \vdash \diamond} \quad (1)$$

- Recursive type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \mu X.A} \quad (2)$$

$$\frac{\Gamma \vdash e : \mu X.A}{\Gamma \vdash \text{unfold } e : A[X \mapsto \mu X.A]} \quad (3)$$

$$\frac{\Gamma \vdash e : A[X \mapsto \mu X.A]}{\Gamma \vdash \text{fold } e : \mu X.A} \quad (4)$$

- Universal type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \forall X.A} \quad (5)$$

$$\frac{\Gamma \cup \{X\} \vdash e : A}{\Gamma \vdash \lambda X.e : \forall X.A} \quad (6)$$

$$\frac{\Gamma \vdash e : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash e(B) : A[X \mapsto B]} \quad (7)$$

### 1.2 Subtype Polymorphism

- Basic setup

We define a new binary relation  $<$ : on types and a new judgement:  $\Gamma \vdash A <: B$  (“ $A$  is a subtype of  $B$  in environment  $\Gamma$ ”).

$$\overline{\Gamma \vdash A <: A} \quad (8)$$

$$\frac{\Gamma \vdash A <: B \quad \Gamma \vdash B <: C}{\Gamma \vdash A <: C} \quad (9)$$

- Subsumption

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A <: B}{\Gamma \vdash e : B} \quad (10)$$

- Top type

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash Top} \quad (11)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A <: Top} \quad (12)$$

- Subtyping of functions

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B <: B'}{\Gamma \vdash A \rightarrow B <: A' \rightarrow B'} \quad (13)$$

- Subtyping of products

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' \times B' <: A \times B} \quad (14)$$

- Subtyping of unions

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' + B' <: A + B} \quad (15)$$

- Subtyping of records

$$\frac{\Gamma \vdash A'_1 <: A_1 \quad \dots \quad \Gamma \vdash A'_n <: A_n \quad \Gamma \vdash A'_{n+1} \quad \dots \quad \Gamma \vdash A'_{n+m}}{\Gamma \vdash \mathbf{Record}(l_1 : A'_1, \dots, l_{n+m} : A'_{n+m}) <: \mathbf{Record}(l_1 : A_1, \dots, l_n : A_n)} \quad (16)$$

- Bounded type variables

$$\frac{\Gamma \vdash A \quad X \notin \text{dom}(\Gamma)}{\Gamma \cup \{X <: A\} \vdash \diamond} \quad (17)$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X} \quad (18)$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X <: A} \quad (19)$$

- Subtyping of recursive types

$$\frac{\Gamma \cup \{X <: Top\} \vdash A}{\Gamma \vdash \mu X. A} \quad (20)$$

$$\frac{\Gamma \vdash \mu X. A \quad \Gamma \vdash \mu Y. B \quad \Gamma \cup \{Y <: Top, X <: Y\} \vdash A <: B}{\Gamma \vdash \mu X. A <: \mu Y. B} \quad (21)$$

- Subtyping of universal types

$$\frac{\Gamma \cup \{X <: A\} \vdash B}{\Gamma \vdash \forall X <: A. B} \quad (22)$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \cup \{X <: A'\} \vdash B <: B'}{\Gamma \vdash (\forall X <: A. B) <: (\forall X <: A'. B')} \quad (23)$$

$$\frac{\Gamma \cup \{X <: A\} \vdash e : B}{\Gamma \vdash \lambda X <: A. e : \forall X <: A. B} \quad (24)$$

$$\frac{\Gamma \vdash e : \forall X <: A. B \quad \Gamma \vdash A' <: A}{\Gamma \vdash e(A') : B[X \mapsto A']} \quad (25)$$

## 2 Tasks

1. Does  $\{A <: Top, B <: Top\} \vdash A \rightarrow B <: A \rightarrow B$  hold? Why?
2. Does  $\{A <: B, B <: C, C <: Top\} \vdash A + C <: A + B$  hold? Why?
3. Does  $\{A <: B, B <: Top\} \vdash \mathbf{Ref} A <: \mathbf{Ref} B$  hold? Why?
4. Does  $\{X <: Y, Y <: Top, A <: Top\} \vdash \mu X. (X \times Top) <: \mu Y. (Y \times A)$  hold? Why?
5. Does  $\{X <: Y, Y <: Top\} \vdash \mu X. (X \rightarrow X) <: \mu Y. (Y \rightarrow Y)$  hold? Why?
6. Does  $\{A <: B, B <: Top\} \vdash \forall X. (X \times A) <: \forall Y. (Y \times B)$  hold? Why?
7. Does  $\{A <: B, B <: Top\} \vdash \forall X. (X \times \mathbf{Ref} A) <: \forall X. (X \times \mathbf{Ref} B)$  hold? Why?
8. Implement the function  $f : (A \rightarrow B) \rightarrow (A^* \rightarrow B^*)$  in *Mathematica*.  $f(g)$  returns a function which applies  $g$  to each element of its argument (list) and returns a list of results.
9. Write the most general type of the function

$$f(a, b, c, d) = g(c, d), \quad \text{where } g = a(b)$$

10. Write the type of the function  $f$ . Use all the following types:

$$A <: Top, B <: Top, E <: Top, D <: A, C <: E$$

$$f(x, y) = SECOND(x(y))$$

11. Write the type of  $\lambda T. \lambda x : T. x$
12. Write the most general type of the function

$$f(w, x, y, z) = \text{if } w(x) = y(z) \text{ then } w(x) \text{ else } f(w, w(z), y, y(x))$$