Combinatorial Optimization
Lab No. 3
Integer Linear Programming

Industrial Informatics Research Center

March 7, 2017

Abstract

This lab is devoted to formulating problems as ILP, namely Call center scheduling problem. Taking this problem as an example, we show how to formulate problems with absolute value functions in their objectives as ILP.

1 Call center scheduling problem

1.1 Problem Description

A call center needs to create a cyclic daily schedule (i.e. numbers of the shifts at particular hours are same on every day) of the *shifts* for its employees [1]. Let $\mathbf{d}$ be a vector of a *personnel demand* in each hour during the day, i.e. d_i is defined $\forall i = 1, 2, \ldots, 24$. The personnel demand d_i determines the minimal number of shifts (i.e. employees who will be assigned to these shifts) between $(i-1)$ and i hour, e.g. d_{10} expresses the minimal number of shifts running between 9 a.m. and 10 a.m. The objective of this problem is to obtain the cyclic daily schedule of the shifts such that the total number of the shifts used to cover the personnel demand in a day is minimized. The shift may start at arbitrary full hour and its length is set to 8 hours.

The ILP model of this problem is based on a variable $\mathbf{x}$ such that x_i represents a number of the shifts starting at hour $(i-1)$. The constraints presented in the model (1) express that demand in hour i is covered by shifts starting from hour $(i-7)$ to hour i.

$$
\begin{aligned}
& \min \sum_{i=1}^{24} x_i \\
\text{subject to} & \\
& \begin{array}{rcll}
x_{i+17} + \cdots + x_{24} + x_1 + \cdots + x_i & \geq & d_i, & \forall i = 1 \ldots 7 \\
x_{i-7} + x_{i-6} + \cdots + x_i & \geq & d_i, & \forall i = 8 \ldots 24 \\
x_i & \geq & 0 & \forall i
\end{array}
\end{aligned}
\tag{1}
$$

Lab exercise: Create the matrix-form ILP model for this problem. Namely, write the matrix $\mathbf{A}$ and the vectors $\mathbf{b}$ and $\mathbf{c}$ on the paper. The personnel demand $\mathbf{d}$ is given as follows:

$\mathbf{d}$ = (6 6 6 6 6 8 9 12 18 22 25 21 21 20 18 21 21 24 24 18 18 18 12 8)

The cyclic daily schedule of the used shifts is illustrated at the bottom of Fig. 1. You can notice the large surplus of the shifts in comparison to the personnel demand between 12 a.m. and 6 p.m. How to minimize this difference between the personnel demand and its coverage is handled in Sec. 2.

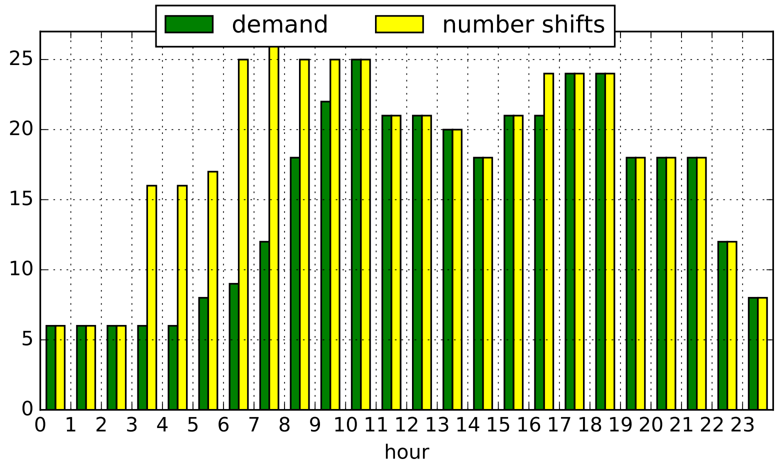

Figure 1: The coverage of the personnel demand $\mathbf{d}$

2 A homework assignment

To avoid a large surplus of shifts in comparison to the personnel demand, we modify our objective to find a daily schedule of shifts such that the difference of the personnel demand and its coverage is minimized, i.e. we try to cover the personnel demand as precisely as possible

$$\min \sum_{i=1}^{24} \left| d_i - \sum_{j=i-7}^{i} x_{((j-1) \bmod 24)+1} \right| \tag{2}$$

Unfortunately, the absolute value function in Eq. (2) cannot be used in the ILP model directly. It is necessary to substitute it by an auxiliary variable z that will be bounded by the constraints representing the personnel demand coverage. We can substitute the absolute value by realizing that in general $|x| = \max\{x, -x\} = z$. The maximum function can be removed by imposing additional constraints $x \leq z, -x \leq z$ that ensure that in every optimal solution z is equal to the $\max\{x, -x\}$, and, therefore equal to $|x|$.

Since in our case we have a sum of absolute values, variable z_i has to be created for each absolute value. A general transformation is shown in Eq. (3), where $\mathbf{v}, \mathbf{z}$ are vectors of variables and $\mathbf{q}$ is a vector of constants.

$$\begin{array}{llllll} & \min \sum_i \mid q_i - v_i \mid & \longrightarrow & \min \sum_i z_i & \longrightarrow & \min \sum_i z_i \\ \text{s.t.} & & & & & \\ & & \longrightarrow & \begin{aligned} \mid v_i - q_i \mid &= z_i \\ z_i &\geq 0 \end{aligned} & \longrightarrow & \begin{aligned} q_i - v_i &\leq z_i \\ v_i - q_i &\leq z_i \\ z_i &\geq 0 \end{aligned} \end{array} \tag{3}$$

Fig. 2 shows the shifts obtained by solving the model (2) using the data from the lab exercise. You can see that the number of shifts are more close to the personnel demand coverage.

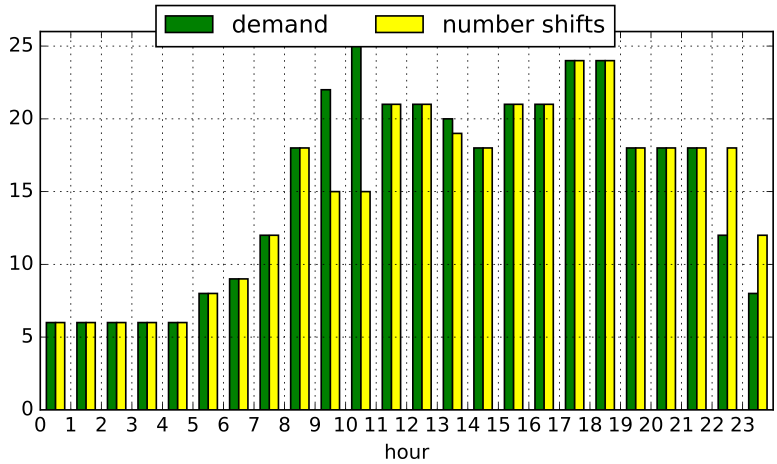

Figure 2: The coverage of the personnel demand **d** and the final daily schedule of the shifts

> **A homework assignment:** Implement a program that takes the vector of personnel demand **d** as the input and outputs the optimal allocation of personnel for each our (values of x_i variables). Given the input, construct an ILP model solving the problem described in Eq. (2). Solve the problem and output optimal values of **x** in the format specified below. Upload your source code to the CourseWare Upload System where it will be automatically evaluated.

2.1 Input/Output format

Your program will be called with two arguments: the first one is absolute path to input file and the second one is the absolute path to output file (the output file has to be created by your program).

The input file consists of a single line with 24 integers separated by space representing the vector of demands **d**.

The output consists of two lines. The first line is an integer denoting the optimal objective value. The second line are the optimal values of $x_1, x_2, \ldots, x_{24}$ variables separated by spaces.

2.1.1 Example 1

Input:

```
6 6 6 6 6 8 9 12 18 22 25 21 21 20 18 21 21 24 24 18 18 18 12 8
```

Output:

```
28
0 0 0 0 6 1 2 3 6 3 0 0 6 0 0 6 6 0 0 0 6 0 0 0
```

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall; United States Ed edition, 1993.