Introduction to Modal Logic

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Modal Logics

:: Introduction of the nodes of truth such as necessarily $\Box$ and possibly true $\Diamond$.

syntax: $\forall \varphi \in \mathcal{L}_p \Rightarrow \Box\varphi, \Diamond\varphi \in \mathcal{L}_m$

semantics: is given by the model

- model is defined on the set of possible worlds (given by Kripke).
- model (or a knowledge base) is partitioned into several worlds and from each different information can be inferred – worlds are mutually linked by accessibility relation.
- model $\mathcal{M}_1$ is given by $\langle W, L, R\rangle$, where W is $\{w\}$, L is $w \rightarrow$ `set-of-true-formulas` and R is $R \subseteq W \times W$.
- We need to expand the logical inference $\models$ into worlds $\models_w$.

:: Application to reasoning about time, knowledge, obligation, permission, ...

Modal Logics

We do not need both $\Box$ and $\Diamond$, because

$$\Box\varphi \Leftrightarrow \neg\Diamond\neg\varphi.$$

There are two basic (**K**) axioms of modal logic:

- **distribution axiom:**

  $$\models \Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$$

  Proof: If in all accessible worlds the implication holds then provided that the if part of the implication is true in all accessible worlds then the then part needs to be also true in all accessible worlds.

- **generalization axiom**:

  $$\models \varphi \Rightarrow \Box\varphi$$

  Proof: As the this is tautology (true in all possible worlds) then if $\forall w \mathcal{M}_1 \models_w \varphi$ then obviously $\forall w' : (w, w') \in R : \mathcal{M}_1 \models_{w'} \varphi$.

Semantics of Modal Logics

s1 $\mathcal{M}_1 \models_w \varphi$ iff $\varphi \in \mathcal{L}_w$

s2 $\mathcal{M}_1 \models_w \varphi \wedge \psi$ iff $\varphi \in \mathcal{L}_w \wedge \psi \in \mathcal{L}_w$

s3 $\mathcal{M}_1 \models_w \neg\varphi$ iff $\mathcal{M}_1 \not\models_w \varphi$

s4 $\mathcal{M}_1 \models_w \Diamond\varphi$ iff $(\exists w' : R(w, w') \wedge \mathcal{M}_1 \models_{w'} \varphi)$

s5 $\mathcal{M}_1 \models_w \Box\varphi$ iff $(\forall w' : R(w, w') \Rightarrow \mathcal{M}_1 \models_{w'} \varphi)$

Accessibility Relation

- reflexive: $(\forall w : (w, w) \in R)$
- serial: $(\forall w : (\exists w' : (w, w') \in R))$
- transitive: $(\forall w_1, w_2, w_3 : (w_1, w_2) \in R \wedge (w_2, w_3) \in R \Rightarrow (w_1, w_3) \in R)$
- symmetric: $(\forall w_1, w_2, (w_1, w_2) \in R \Rightarrow (w_2, w_1) \in R)$
- euclidean: $(\forall w_1, w_2, w_3 : (w_1, w_2) \in R \wedge (w_1, w_3) \in R \Rightarrow (w_2, w_3) \in R)$

Properties of the Modal Logic

- T: $\Box\varphi \rightarrow \varphi$
- D: $\Box\varphi \rightarrow \Diamond\varphi$
- 4: $\Box\varphi \rightarrow \Box\Box\varphi$
- B: $\varphi \rightarrow \Box\Diamond\varphi$
- 5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$

Proofs

T: because $\models \varphi \Rightarrow \Box\varphi$ and due reflexivity $\forall w : (w, w) \in R$ ⊚

D: $(\mathcal{M}_1 \models_w \forall w' : (w, w') \in R : \mathcal{M}_1 \models_{w'} \varphi)$ and due to seriality $(\mathcal{M}_1 \models_w (\exists w' : (w, w') \in R))$ we can say that $\mathcal{M}_1 \models_w \exists w'' : (w, w'') \in R : \mathcal{M}_1 \models_{w'} \varphi)$ ⊚

4: provided that there is transitive relation on R we may say that $(\mathcal{M}_1 \models_w \forall w' : (w, w') \in R : \mathcal{M}_1 \models_{w'} \varphi) \Rightarrow (\mathcal{M}_1 \models_w \forall w' : (w, w') \in R : \mathcal{M}_1 \models_{w'} (\forall w'' : (w', w'') \in R : \mathcal{M}_1 \models_{w''} \varphi))$ ⊚

B: provided that there is symetric relation on R we say that $\mathcal{M}_1 \models_w \varphi \Rightarrow \forall w' : (w, w') \in R : \mathcal{M}_1 \models_{w'} \exists w'' : (w', w'') \in R : \mathcal{M}_1 \models_{w''} \varphi$ if $(\forall w, w', (w, w') \in R \Rightarrow (w', w) \in R)$ then $w = w''$ and $\mathcal{M}_1 \models_w \varphi$ ⊚

5: $(\mathcal{M}_1 \models_w \exists w' : (w, w') \in R \models_{w'} \varphi) \Rightarrow (\mathcal{M}_1 \models_w \forall w'' : (w, w'') \in R : \mathcal{M}_1 \models_{w''} \exists w' (w'', w') \in R : \mathcal{M}_1 \models_{w'} \varphi)$ due to euclidean property if $(w, w') \in R \wedge (w, w'') \in R$ then $(w', w'') \in R$ ⊚

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- 5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ due to euclidean property

Reasoning about Belief/Knowledge

:: Model logic is a classical mechanism for representing and reasoning about knowledge. The subject of study of the epistemic logic.

:: An agent is said to believe φ if φ is true in all the belief-accessible situations (belief alternatives). This is why we have $\Box$ to be equivalent to $(\text{Bel } A \ \varphi)$, sometimes denoted as $K_i\varphi$. Agent's belief is given by $\mathcal{B}$ – belief accessibility relation.

s1 $\mathcal{M}_4 \models_w (\text{Bel } A \ \varphi)$ iff $\forall w' \ \mathcal{B}(w, w') : \mathcal{M}_4 \models'_w \varphi$

$\mathcal{B}$ can capture reasoning about possible future events, about different knowledge of agents, etc.

Reasoning about Belief/Knowledge

Example 2.: poker card game - there the $\mathcal{B}$ is interpreted as that the agent believes in what he can see – the worlds that are accessible from his own world.

Properties of Belief

Once we are implementing an agent what do we want from functions/program that will implement its beliefs:

- to satisfy the **K** axioms
- an agent knows what it does know – positive introspection axiom – **4** axiom.
- an agent knows what it does not know – positive introspection axiom – **5** axiom.
- it beliefs are not contradictory – if it knows something it means it does not allow the negation of its being true – **D** axiom.

The $\mathcal{B}$ relation is serial, transitive and euclidean.

:: Belief is surely a **KD45** system.

Properties of Belief

:: Knowledge is more difficult – it needs to be also true – this why the knowledge accessibility relation needs to be also reflexive.

$$\models (\text{Bel } A\ \varphi) \land \varphi \Leftrightarrow (\text{Know } A\ \varphi)$$

Therefore knowledge is a **KTD45** system.

Reasoning about time

Let us have different time moments represented as possible worlds and let us define the accessibility relation with respect to the flow of time. We have $(t, t') \in R$ iff the time t' can follow time t.

In temporal logic we replace the accessibility relation by $\prec$ temporal ordering relation. Properties of the relation give us two different temporal logics.

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We have

- **linear temporal logics (LTL)** where the time is a total ordering on the time domain
- **branching time temporal logics (BTTL)** where the time is a partial ordering on the situation domain

Reasoning about time

In LTP the $\Box$ operator is replaced by G, the $\Diamond$ operator is replaced by F and Besides F we need to have $\curvearrowleft$ for until, P for past and X for next. The model is given by $\mathcal{M}_3 \equiv \langle \mathbf{T}, \prec, [\![\ ]\!] \rangle$. $[\![\ ]\!]$ gives a denotion of an atomic proposition so that $L : w \in [\![\varphi]\!]$ iff $\varphi \in L(w)$.

s1 $\mathcal{M}_3 \models_t \mathsf{P}\varphi$ iff $\exists t' : t' \prec t \wedge \mathcal{M}_2 \models_t \varphi$

s3 $\mathcal{M}_3 \models_t \mathsf{X}\varphi$ iff $\mathcal{M}_2 \models_{t+1} \varphi$

s4 $\mathcal{M}_3 \models_t \varphi \curvearrowleft \psi$ iff $(\exists t' : t \preceq t' \wedge \mathcal{M}_2 \models_t' \psi \wedge (\forall t'' : t \preceq t'' \preceq t' \Rightarrow \mathcal{M}_2 \models_t'' \varphi)$

we have

$$\mathsf{F}\varphi \Leftrightarrow \mathtt{true} \curvearrowleft \varphi,$$

$$\mathsf{G}\varphi \Leftrightarrow \neg\mathsf{F}\neg\varphi,$$

Reasoning about time

In BTTL we enhance the system with the path quantifiers A and E. This is why we talk about **satisfiability** as EFφ and about **tautology** as AGφ.

We need to introduce the concept of a path where $path(S) \rightarrow p$, so that

1. $t \in p$
2. $\forall s_1, s_2 \in p : (s_1, s_2) \in R \vee (s_2, s_1) \in R$
3. $\forall s_1 \in p : (s, s_1) \in R$

Then we may easily use the F and G operators to use on a path p so that $\mathsf{F}_p\varphi$

s1 $\mathcal{M}_3 \models_s$ AGφ iff $\forall p : p \in path(s) : \mathcal{M}_3 \models_s \mathsf{G}_p\varphi$

s3 $\mathcal{M}_3 \models_s$ EFφ iff $\exists p : p \in path(s) : \mathcal{M}_3 \models_s \mathsf{F}_p\varphi$

The model of the language M_5: $M_5 \equiv \langle \mathbf{S}, <, [\,], \mathbf{R} \rangle$

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Each action α has got a specific accessibility relation R_α that specifies the properties of the world before applying α and the properties of the resulting world.

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- we have two **achieves** type of operators:

  s1 $\mathcal{M}_2 \models_w \langle\alpha\rangle\varphi$ iff $\exists w' : R_\alpha(w, w') \wedge \mathcal{M}_2 \models_{w'} \varphi$

  s2 $\mathcal{M}_2 \models_w [\alpha]\varphi$ iff $\forall w' : R_\alpha(w, w') \wedge \mathcal{M}_2 \models_{w'} \varphi$

- From the world of regular programs we can represent the dynamics by $\alpha; \alpha'$ – followed, $\alpha|\alpha'$ – or, $\alpha*$ – repeated more than once, $\varphi?$ – test whether φ is true.

  s3 $R_{\alpha;\beta}(w, w')$ iff $\exists w'' : R_\alpha(w, w'') \wedge R_\beta(w'', w')$

  s4 $R_{\alpha|\beta}(w, w')$ iff $\exists w'' : R_\alpha(w, w') \vee R_\beta(w, w')$

  s5 $R_{\alpha*}(w, w')$ iff $\exists w_0, w_1, ..w_n : w = w_0 \wedge w' = w_n \wedge (\forall i 0 \leq i < n \Rightarrow R_\alpha(w_i, w_{i+1})$

Reasoning about action

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$\alpha; [(\neg\varphi?; \alpha)|\varphi?] *$