Filtering and convolution

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

Filtering (1)

kernel

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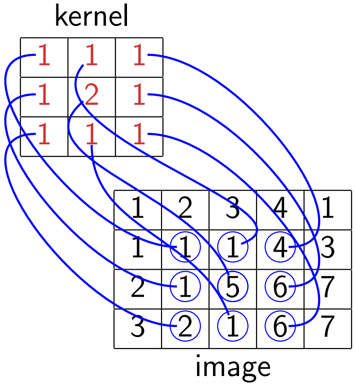
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

Filtering (1)



Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 +$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
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3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 + 1 \cdot 1 +$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 +$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 +$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 5 +$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$
\begin{aligned}
g_{\circ} \;=\;& 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + \\
& 1 \cdot 1 + 2 \cdot 5 + 1 \cdot 1 +
\end{aligned}
$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$
\begin{aligned}
g_{\circ} \;=\;& 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + \\
& 1 \cdot 1 + 2 \cdot 5 + 1 \cdot 1 + \\
& 1 \cdot 4 +
\end{aligned}
$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$
\begin{aligned}
g_{\circ} \quad = \quad & 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + \\
& 1 \cdot 1 + 2 \cdot 5 + 1 \cdot 1 + \\
& 1 \cdot 4 + 1 \cdot 6 +
\end{aligned}
$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image

$$g_{\circ} = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 5 + 1 \cdot 1 + 1 \cdot 4 + 1 \cdot 6 + 1 \cdot 6$$

Filtering (1)

kernel

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	(5)	6	7
3	2	1	6	7

image

$$\begin{aligned} g_{\circ} &= 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + \\ &\quad 1 \cdot 1 + 2 \cdot 5 + 1 \cdot 1 + \\ &\quad 1 \cdot 4 + 1 \cdot 6 + 1 \cdot 6 \\ &= 32 \end{aligned}$$

Filtering (2)

Assume matlab indexing of kernel and image, then

kernel h

1	1	1
1	2	1
1	1	1

$$g_{3,3} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i+2,j+2} \, f_{3+i,3+j} \, ,$$

.

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image f

Filtering (2)

kernel h

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image f

Assume matlab indexing of kernel and image, then

$$g_{3,3} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i+2,j+2} \, f_{3+i,3+j} \, ,$$

$$g_{u,v} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i+2,j+2} \, f_{u+i,v+j} \, .$$

Filtering (2)

kernel h

1	1	1
1	2	1
1	1	1

1	2	3	4	1
1	1	1	4	3
2	1	5	6	7
3	2	1	6	7

image f

Assume matlab indexing of kernel and image, then

$$g_{3,3} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i+2,j+2} \, f_{3+i,3+j} \, ,$$

$$g_{u,v} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i+2,j+2} \, f_{u+i,v+j} \, .$$

Index the kernel h differently such that $(0,0)$ is at its center, then

$$g_{u,v} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i,j} \, f_{u+i,v+j} \, .$$

Filtering, relationship with convolution (1)

$$g_{u,v} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i,j} \, f_{u+i,v+j} \, .$$

- h: filter kernel, centered at origin
- f: image
- g: filtering result

Filtering, relationship with convolution (1)

$$g_{u,v} = \sum_{i=-1}^{1} \sum_{j=-1}^{1} h_{i,j}\, f_{u+i,v+j}\,.$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_{i,j}\, f_{u+i,v+j}\,.$$

- h: filter kernel, centered at origin, padded by zeros where necessary
- f: image, padded by zeros where necessary
- g: filtering result

Filtering, relationship with convolution (2)

$$g_{u,v} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_{i,j}\, f_{u+i,v+j}$$

This is almost a convolution ...

Filtering, relationship with convolution (2)

$$g_{u,v} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_{i,j}\, f_{u+i,v+j}$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{h}_{i,j}\, f_{u-i,v-j}$$

This is almost a convolution . . . and is a convolution when the kernel (or the image) is flipped in both axes; $\hat{h}_{i,j} = h_{-i,-j}$.

Convolution's properties are well understood $\Rightarrow$ benefits for analysis and practice.

Convolution - interesting properties

- Commutative: $f * h = h * f$, that is,

$$g_{u,v} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{h}_{i,j} \, f_{u-i,v-j} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{h}_{u-i,v-j} \, f_{i,j}$$

Convolution - interesting properties

- Commutative: $f * h = h * f$, that is,

$$g_{u,v} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{h}_{i,j} \, f_{u-i,v-j} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \hat{h}_{u-i,v-j} \, f_{i,j}$$

- Associative: $f * (h * w) = (f * h) * w$.