Mathematical morphology

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Outline of the talk:

- Point sets. Morphological transformation.
- Erosion, dilation, properties.
- Opening, closing, hit or miss.
- Skeleton.
- Thinning, sequential thinning.
- Distance transformation.

Morphology is a general concept

In biology: the analysis of size, shape, inner structure (and relationship among them) of animals, plants and microorganisms.

In linguistics: analysis of the inner structure of word forms.

In materials science: the study of shape, size, texture and thermodynamically distinct phases of physical objects.

In signal/image processing: mathematical morphology – a theoretical model based on lattice theory used for signal/image preprocessing, segmentation, etc.

Mathematical morphology, introduction

Mathematical morphology (MM):

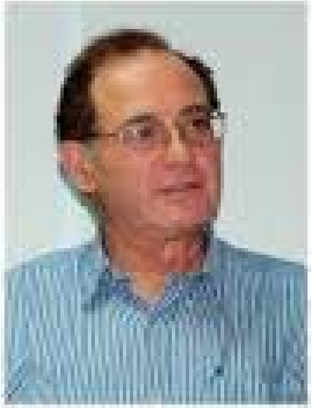
- is a theory for analysis of planar and spatial structures;
- is suitable for analyzing the shape of objects;
- is based on a set theory, integral algebra and lattice algebra;
- is successful due to a a simple mathematical formalism, which opens a path to powerful image analysis tools.

The morphological way . . .

The key idea of morphological analysis is extracting knowledge from the relation of an image and a simple, small probe (called the structuring element), which is a predefined shape. It is checked in each pixel, how does this shape matches or misses local shapes in the image.

MM founding fathers

Georges Matheron ($*$ 1930, † 2000)



Jean Serra ($*$ 1940)

- Matheron, G. Elements pour une Theorie del Milieux Poreux Masson, Paris, 1967.
- Serra, J. Image Analysis and Mathematical Morphology, Academic Press, London 1982.

Additional literature

- Course on mathematical morphology by Jean Serra: http://cmm.ensmp.fr/serra/cours/
- Jean Serra, Image analysis and mathematical morphology. Volume 2: theoretical advances, Academic Press, London, 1988
- Pierre Soille, Morphological Image Analysis: Principles and Applications, Second edition, Springer-Verlag Berlin, 2004
- Laurent Najman and Hugues Talbot (editors), Mathematical Morphology, John Wiley & Sons, Inc., London, 2010

Links with other theories and approaches

Mathematical morphology does not compete with other theories, it complements them.

- Discrete geometry (e.g., distance, skeletons).
- Graph theory (e.g., minimal spanning tree, watershed, computational geometry).
- Statistics: random models, measure theory, stereology, etc.
- Linear signal theory: replace $+$ with supremum $\wedge$.
- Scale-space: replace Gaussian smoothing with openings/closings $\Rightarrow$ granulometry.
- Level sets: dilations with PDEs, FMM is a distance function.
- *Note: there are no equivalents of Fourier, wavelet and Hough transformations. Maybe a possible research direction.*

Different mathematical structures used

Signal processing in a vector space

Vector space is a set of vectors V and set of scalars K, such that

- K is a field.
- V is a commutative group.
- Vectors can be added together and multiplied (scaled) by scalars.

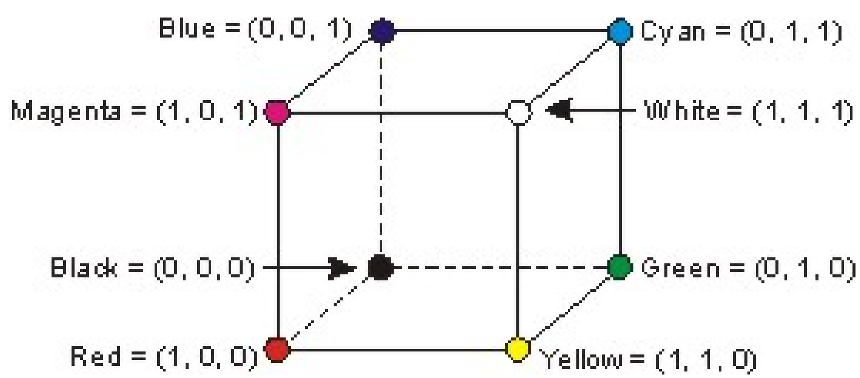
Mathematical morphology

Complete lattice $(E, \leq)$ is a set E provided with a relation $\leq$, such that

- $\forall x, y, z \in E$ holds (partial ordering)
  $x \leq x$,
  $x \leq y$, $y \leq x \Rightarrow x = y$,
  $x \leq y$, $y \leq z \Rightarrow x \leq z$.
- For any $P \subseteq E$ there exists in E (completeness)
  - A greatest lower bound $\wedge P$, called infimum.
  - A lowest upper bound $\vee P$, called supremum.

Example of Lattices

- Lattice of primary additive colors (RGB).



- Lattice of real numbers $\mathbb{R}$.
- Lattice of real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- Lattice of whole numbers (integers) $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- The Cartesian product of the natural numbers, ordered by $\leq$ so that $(a, b) \leq (c, d) \Leftrightarrow (a \leq c)$ & $(b \leq d)$.

Examples of complete lattices useful in image analysis

- Boolean lattice of sets ordered by inclusion $\Rightarrow$ binary mathematical morphology, where, e.g., the occupancy of a pixel is of interest.
- Lattice of u.s.c. functions $\Rightarrow$ gray level mathematical morphology or binary mathematical morphology in 3D binary images, where the occupancy of a voxel is of interest.

  Note: Generalization to higher dimensions is possible, e.g. for n-dimensional images, as well as for multi-valued functions, e.g. time series as in motion analysis.
- Lattice of multi-valued functions $\Rightarrow$ mathematical morphology for color images.

Comparison of basic operations

Linear signal processing

- Based on the 'superposition principle', fundamental laws are addition, multiplication and scalar product.
- Basic operations preserve addition and multiplication and commute under them.
$$\Psi\left(\sum_i \lambda_i\, f_i\right) = \sum_i \lambda_i\, \Psi(f_i)\,.$$
- The important operation is called convolution. It allows finding relation between two functions.

Mathematical morphology

- The lattice is based on ordering, supremum $\vee$ and infimum $\wedge$. Basic operations preserve supremum and infimum.
- Ordering preservation
$\{x \leq y \Rightarrow \Psi(x) \leq \Psi(y)\} \Leftrightarrow$ operation Ψ is increasing.
- Commutation under supremum
$\Psi(\vee x_i) = \vee\, \Psi(x_i) \Leftrightarrow$ dilation.
- Commutation under infimum
$\Psi(\wedge x_i) = \wedge\, \Psi(x_i) \Leftrightarrow$ erosion.

Symmetry of supremum and infimum

- Supremum and infimum play a symmetrical role in a lattice.
  - They are exchanged if we exchange ordering $x \leq y \leftrightarrow x \geq y$.
  - This leads to the concept of duality.
- Example: in a lattice of all subsets of a set $E(2^E; \leq)$,

  two operations Ψ and Ψ^* are called dual, iff

$$\Psi(X^C) = [\Psi^*(X)]^C ,$$

where $X^C = E \setminus X$ denotes a complement of X in E.

Lattice and order
Extensive, antiextensive tranform

- The operation Ψ is extensive on a lattice $(E, \leq)$ iff, for all elements in E, the transformed element is greater than or equal to the original element, i.e.

$$\Psi \text{ is extensive } \Leftrightarrow \forall\, x \in E, \quad x \leq \Psi(x)\,.$$

- The operation Ψ is anti-extensive on a lattice $(E, \leq)$ iff, for all elements in E, the transformed element is lower than or equal to the original element, i.e.

$$\Psi \text{ is antiextensive } \Leftrightarrow \forall\, x \in E, \quad \Psi(x) \leq x\,.$$

Why so abstract?

- We can define operations that work in a general way.
- The operations can be studied without specifying the definition space.
- As a result they can be applied to, e.g., discrete images, continuous images, graphs, or meshes.
- Q: How the lattice framework can be used for images?

  A: Images are often consider as a function $f : E \to T$, where E is the set of image points (pixels) and T is a set of possible pixel values.

Lattices of functions

- Let E be arbitrary set and T a closed subset of $\overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$.
- Functions $f\colon E \to T$ generate a new lattice denoted T^E (named product ordering),

$$f \leq g\,, \text{ iff } f(x) \leq g(x) \text{ for } \forall x \in E\,,$$

where supremum and infimum derive directly from those of T,

$$(\vee f_i)(x) = \vee f_i(x) \quad (\wedge f_i)(x) = \wedge f_i(x)\,.$$

- The approach extends directly to multivariate functions (e.g., color images, motion).

A simple case, the lattice for binary images

- There are several ways how to define a lattice and induced morphological operation.
- Let us start from a simple, intuitive and practically useful case of binary images

$$f\colon E \to \{0,1\}\,,$$

$$F = \{x \in E \,|\, f(x) = 1\}\ .$$

- The lattice structure $(E, \leq)$ for binary images can be introduced as:

$$(2^E, \subseteq)\ ,$$

where $X \leq Y \Leftrightarrow X \subseteq Y$, $\wedge \leftrightarrow \cap$, $\vee \leftrightarrow \cup$.

Point sets

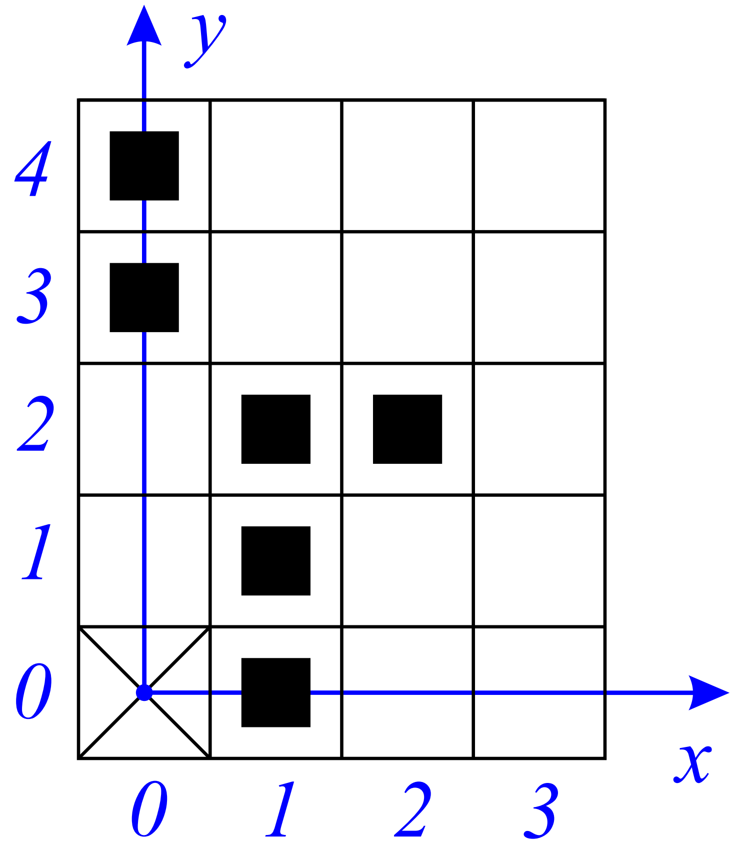
- Images can be represented by point sets of an arbitrary dimension, e.g. in a N-dimensional Euclidean space.
- 2D Euclidean space $\mathbb{E}^2$ and a system of its subsets is a natural continuous domain representing planar objects.
- A digital counterpart of the Euclidean space based on integer numbers $\mathbb{Z}$.
- Binary mathematical morphology in 2D – a set of points represented as pairs of integers ($(x, y) \in \mathbb{Z}^2$). The presence of the point informs about occupancy of a particular pixel (a location in a lattice).
- Binary mathematical morphology in 3D – a set of points represented as triplets of integers $(x, y, z) \in \mathbb{Z}^2$, where (x, y, z) are volumetric coordinates informing about occupancy of a particular voxel.
- Grayscale mathematical morphology in 2D – a set of points $(x, y, g) \in \mathbb{Z}^3$, where x, y are coordinates in a plane and g represents the gray value of a particular pixel.

Four principles of mathematical morphology

1. Compatibility with translation – The morphological operator Ψ should not depend on the translation.
2. Compatibility under the scale change – The morphological operator Ψ should not depend on the scale.
   Note: This is principle is necessarily (slightly) violated for digital images.
3. Local knowledge – The morphological operator Ψ is a local operator (see structuring elements on the next slide).
4. Semi-continuity – The morphological operator should not exhibit abrupt changes of its behavior.

We start with the binary morphology, point set

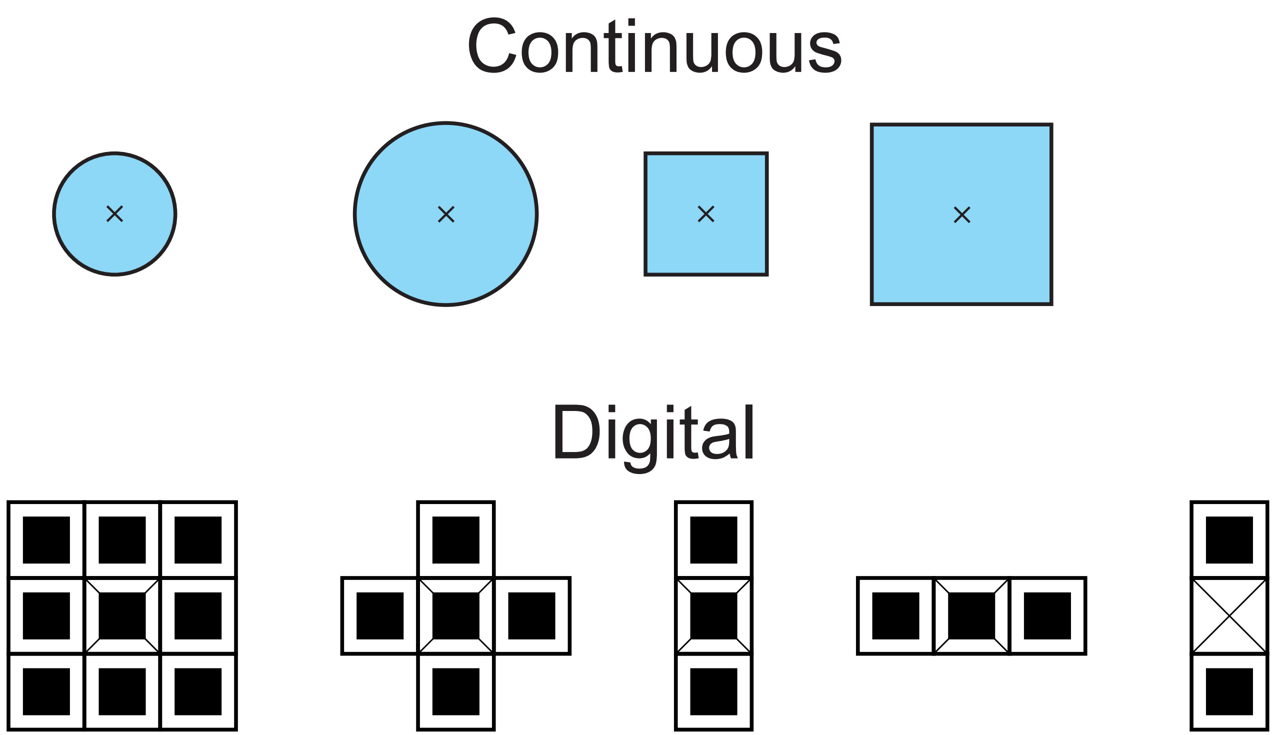
- We will constrain for a while to binary mathematical morphology.
- The example of a point set in $\mathbb{Z}^2$,



$$X = \{(1,0), (1,1), (1,2), (2,2), (0,3), (0,4)\}$$

Structuring element

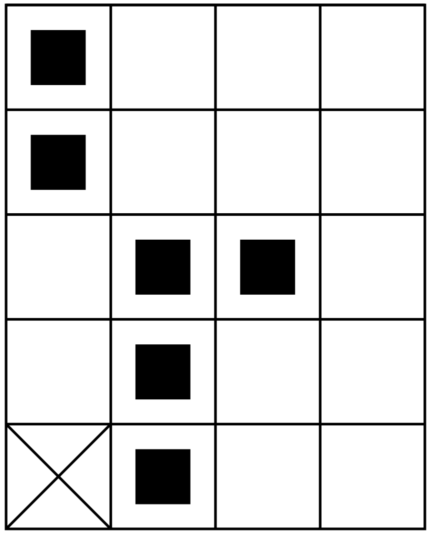
- A structuring element serves as a local probe in morphological operators.
- Structuring elements are expressed with respect to local coordinates with the origin in the representative point $\mathcal{O}$ (denoted by $\times$ in subsequent figures).
- Examples:

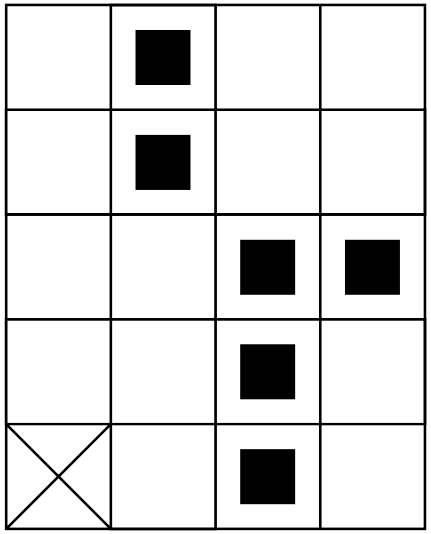


Translation of a set by a radiusvector

Translation X_h of a point set X by a radiusvector h

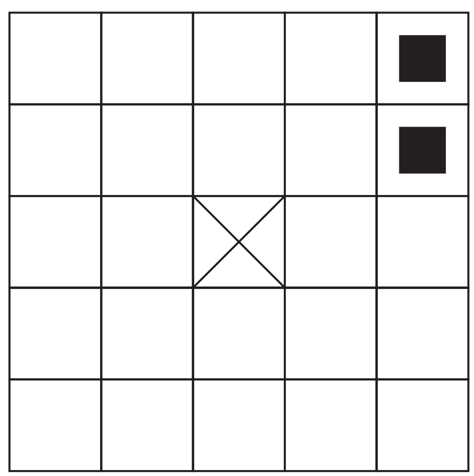
$$X_h = \left\{ p \in \mathbb{E}^2,\; p = x + h \;\text{ for some } x \in X \right\} \;.$$



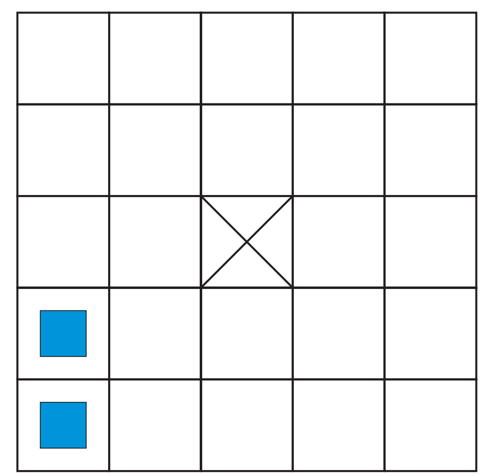


Symmetric point set

- Central symmetry is expressed with respect to a representative point $\mathcal{O}$.
- The alternative name is a transposed point set.
- Definition: $\breve{B} = \{-b : \ b \in B\}$.
- Example: $B = \{(2,1),(2,2)\}$, $\quad \breve{B} = \{(-2,-1)(-2,-2)\}$.



Original



After transposition

Binary mathematical morphology

- Deals with binary images. The domain is $\mathbb{Z}^2$. The range is $\{0, 1\}$.
- Two basic operations: dilatation and erosion, which are not invertible but dual.
- Two used formalisms for addition and subtraction
  - The ordinary addition and subtraction as taught in the elementary schools.
  - Minkowski addition, subtraction, (introduced to mathematical morphology by G. Matheron (book 1967), J. Serra (book 1982).
  - The difference between these two approaches plays a role with erosion.

Minkowski addition, subtraction

Minkowski addition (Hermann Minkowski 1864-1909, Geometry of numbers 1889)

$$X \oplus B = \bigcup_{b \in B} X_b \,.$$

Minkowski subtraction (the concept was introduced by H. Hadwiger in 1957)

$$X \ominus B = \bigcap_{b \in B} X_{-b} \,.$$

Binary dilation $\oplus$

- The dilation is a Minkowski addition, i.e. the union of translated point sets,

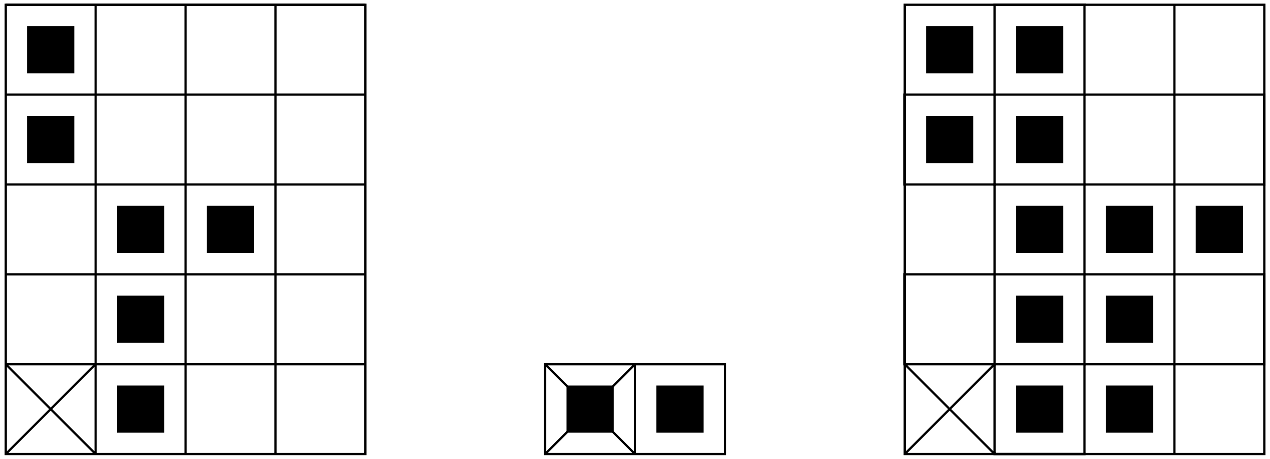
$$X \oplus B = \bigcup_{b \in B} X_b \, .$$

- Dilation operation $X \oplus B$ was denoted $\delta_B(X)$ in a functional form by J. Serra.
- The dilation operation $\oplus$ can be equivalently expressed as

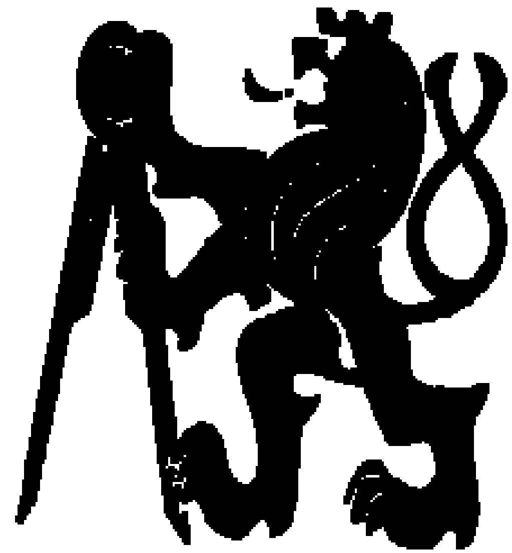
$$\delta_B(X) = X \oplus B = \left\{ p \in \mathbb{E}^2 : \; p = x + b, \; x \in X \text{ and } b \in B \right\} \, .$$

Binary dilation $\oplus$, an example

$$
\begin{aligned}
X &= \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4)\} \\
B &= \{(0,0),(1,0)\} \\
X \oplus B &= \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4), \\
& \quad (2,0),(2,1),(2,2),(3,2),(1,3),(1,4)\}
\end{aligned}
$$



Binary dilation by the isotropic structuring element 3×3



left – original, right – dilation.

Dilation is used to fill small holes and narrow bays in objects. The size of the object increases. If the size has to be kept similar then dilation is combined with erosion (to come).

Dilation properties

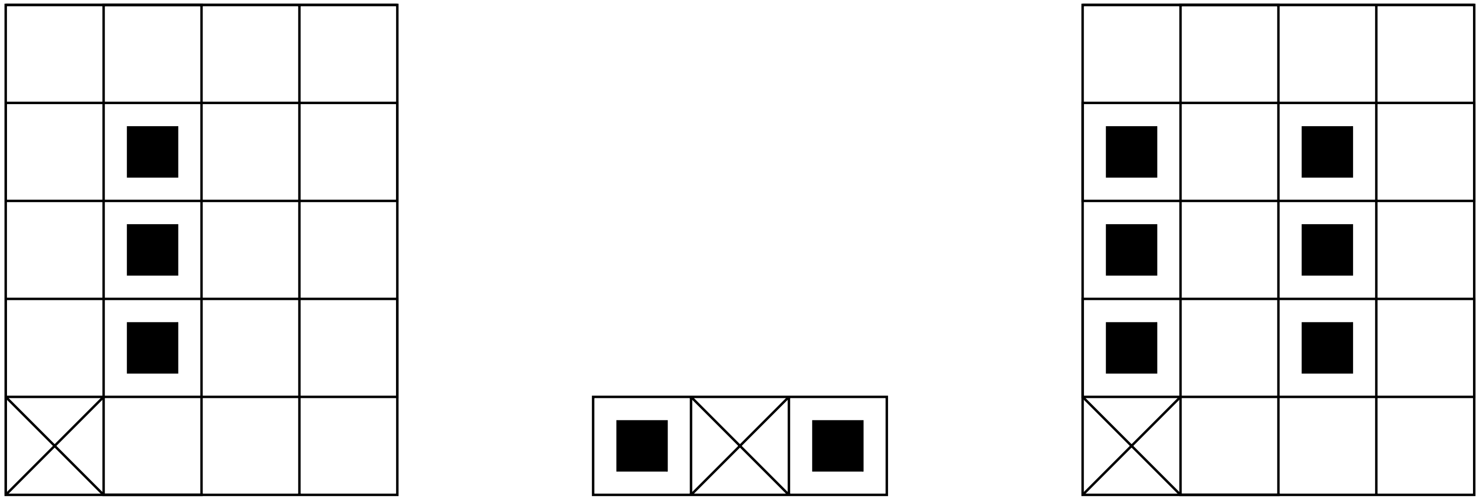
Commutative: $X \oplus B = B \oplus X$.

Associative: $X \oplus (B \oplus D) = (X \oplus B) \oplus D$.

Invariant to translation: $X_h \oplus B = (X \oplus B)_h$.

Increasing transformation: If $X \subseteq Y$ a $(0,0) \in B$ then $X \oplus B \subseteq Y \oplus B$.

Counterexample for the case of the empty representative point $(0,0) \notin B$



Binary erosion $\ominus$

- The erosion is a Minkowski subtraction, i.e. the intersection of all image X translations by the vector $-b \in B$,

$$X \ominus B = \bigcap_{b \in B} X_{-b} \,.$$

- Erosion operation $X \ominus B$ was denoted $\varepsilon_B(X)$ in a functional form by J. Serra.

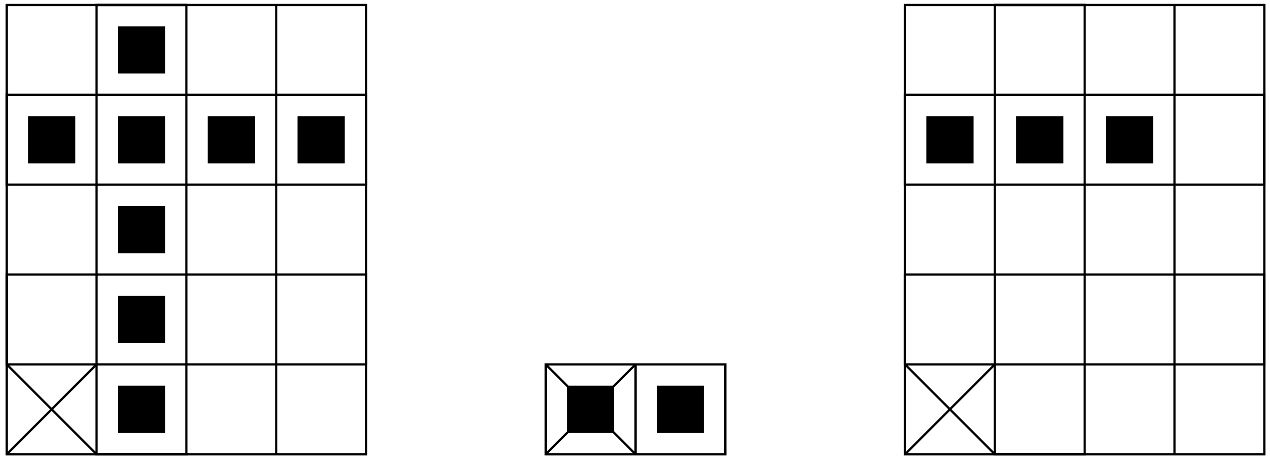
- Equivalently, it is verified for each image pixel p, whether the result fits to X for all possible $x + b$. If yes then the outcome is 1, otherwise 0.

$$\varepsilon_B(X) = X \ominus B = \{p \in \mathbb{E}^2 : \; p = x + b \in X \text{ for all } b \in B\} \,.$$

- Dilation and erosion are dual morphological operations.

Binary erosion $\ominus$, an example

$$
\begin{aligned}
X &= \{(1,0),(1,1),(1,2),(0,3),(1,3),\\
&\quad (2,3),(3,3),(1,4)\}\\
B &= \{(0,0),(1,0)\}\\
X \ominus B &= \{(0,3),(1,3),(2,3)\}
\end{aligned}
$$



Binary erosion with an isotropic structuring element $3\times\ 3$

left – original,   right – erosion.

Objects smaller than the structuring element vanish (e.g. lines of lenght one 1).

Erosion is used for simplifying structure (decomposition of an object into simpler parts).

Contour by binary erosion

Contour of a binary region ∂X (boundary mathematically) has width one.

$$\partial X = X \setminus (X \ominus B) \, .$$

left – original X, right boundary (contour) ∂X.

Erosion properties

Antiextensive: If $(0,0) \in B$ then $X \ominus B \subseteq X$.

Invariant with respect to translation: $X_h \ominus B = (X \ominus B)_h$,
$X \ominus B_h = (X \ominus B)_{-h}$.

Preserves inclusion: If $X \subseteq Y$ then $X \ominus B \subseteq Y \ominus B$.

Duality of erosion and dilation: $(X \ominus Y)^C = X^C \oplus \breve{Y}$.

Combination of erosion and intersection:
$(X \cap Y) \ominus B = (X \ominus B) \cap (Y \ominus B)$,
$B \ominus (X \cap Y) \supseteq (B \ominus X) \cup (B \ominus Y)$.

Dilation and erosion properties (2)

The order of dilation and erosion can be interchaged:

$(X \cap Y) \oplus B = B \oplus (X \cap Y) \subseteq (X \oplus B) \cap (Y \oplus B)$.

Dilation of the intersection of two sets (images) is contained in the intersection of their dilations.

A possible interchange of erosion and set intersection (e.g. enables decomposition of more complex structural elements into simpler ones):

$$
\begin{aligned}
B \oplus (X \cup Y) = (X \cup Y) \oplus B &= (X \oplus B) \cup (Y \oplus B)\,, \\
(X \cup Y) \ominus B &\supseteq (X \ominus B) \cup (Y \ominus B)\,, \\
B \ominus (X \cup Y) &= (X \ominus B) \cap (Y \ominus B)\,.
\end{aligned}
$$

Dilation and erosion properties (3)

Sequential dilation (resp. erosion) of the image X by the structural element B followed by the structural element D is equivalent to dilation (resp. erosion) of the image X by $B \oplus D$

$$
\begin{aligned}
(X \oplus B) \oplus D &= X \oplus (B \oplus D)\,, \\
(X \ominus B) \ominus D &= X \ominus (B \oplus D)\,.
\end{aligned}
$$

Transformation hit or miss $\otimes$

- Uses a composite structuring element $B = (B_1, B_2)$, $B_1 \cap B_2 = \emptyset$.

$$X \otimes B = \{x : \; B_1 \subset X \text{ and } B_2 \subset X^c\} \, .$$

- Indicates the match between the composite structuring element and the part of the image. B_1 checks the objects and B_2 the background.
- Transformation $\otimes$ can be expressed by erosions and dilations

$$X \otimes B = (X \ominus B_1) \cap (X^c \ominus B_2) = (X \ominus B_1) \setminus (X \oplus \breve{B}_2) \, .$$

Example: detection of convex corners

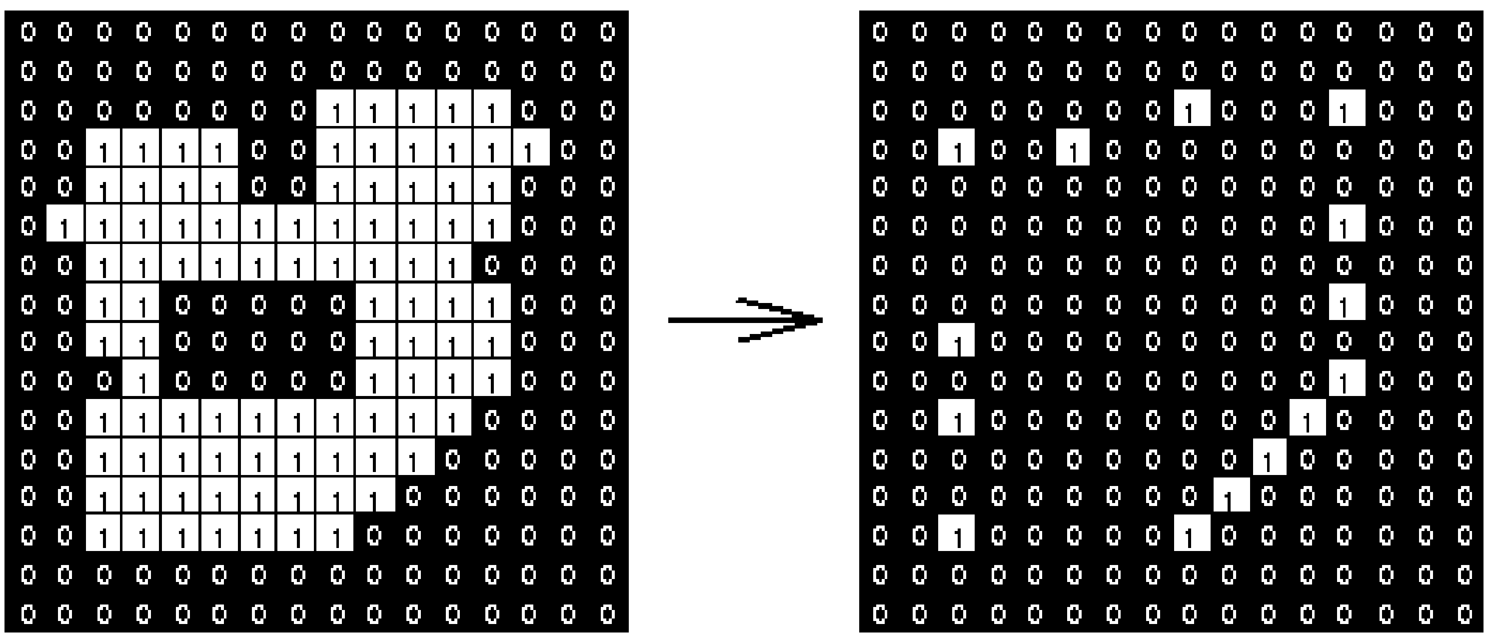
	1	
0	1	1
0	0	

	1	
1	1	0
	0	0

	0	0
1	1	0
	1	

0	0	
0	1	1
	1	

Masks detecting four possible configuration of convex corners using hit or miss transformaton.



The result of corner detection.

Morphological filtering

- In ‘classical’ signal/image processing, the term filter denotes an arbitrary processing procedure having a signal/image both as an input and an output.
- A filter has a precise meaning in mathematical morphology, i.e.

  Operation Ψ is a morphological filter $\Leftrightarrow$ Ψ is increasing and idempotent.

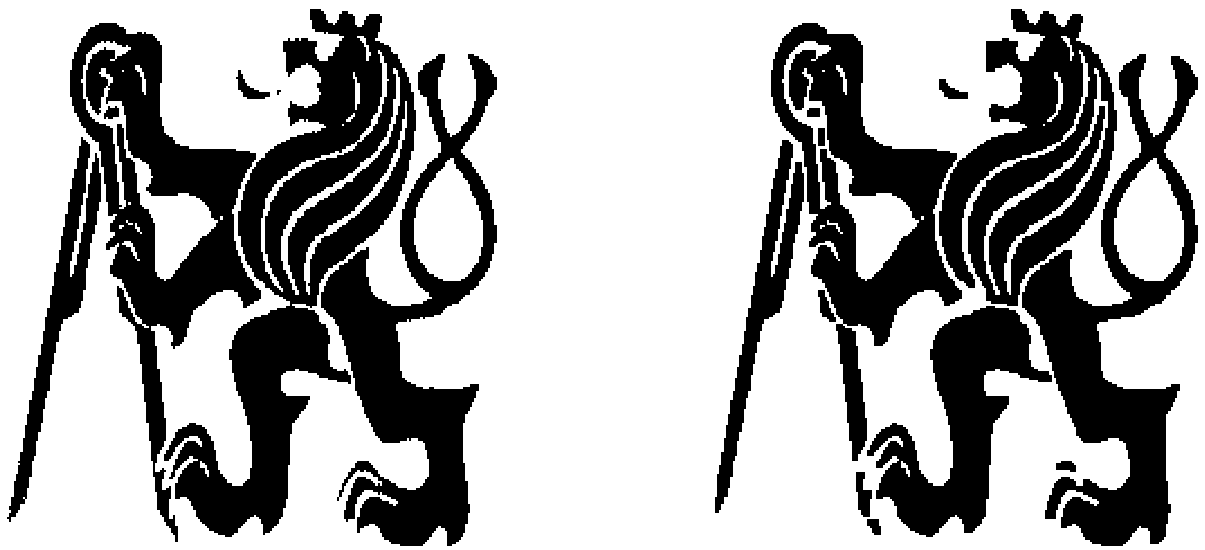
- In words: morphological filters preserve ordering relation and converge in one iteration.
- Two most important operations are openings and closings in this context.
- Openings are anti-extensive morphological filters.
- Closings are extensive morphological filters.

Binary opening $\circ$

Erosion followed by dilation.

$$X \circ B = (X \ominus B) \oplus B$$

If the image X remains unchanged after opening by the structuring element B, it is open with respect to B.

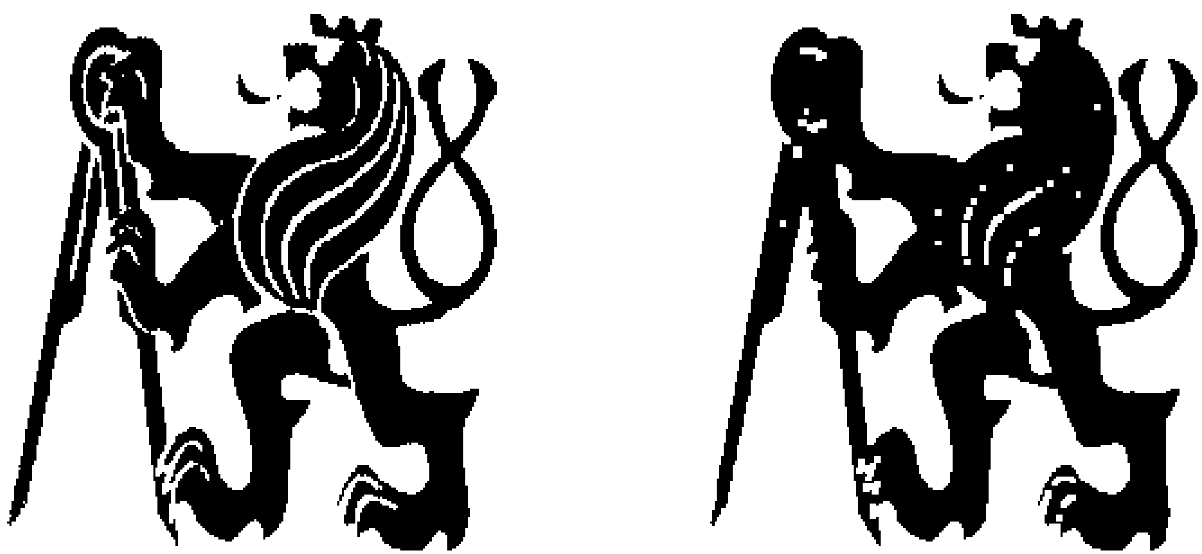


Binary closing $\bullet$

Dilation followed by erosion.

$$X \bullet B = (X \oplus B) \ominus B$$

If the image X remains unchanged after closing with the structural element B, it is closed with respect to B.



Properties of opening, closing

Opening and closing are dual morphological transformations

$$(X \bullet B)^C = X^C \circ \breve{B}$$

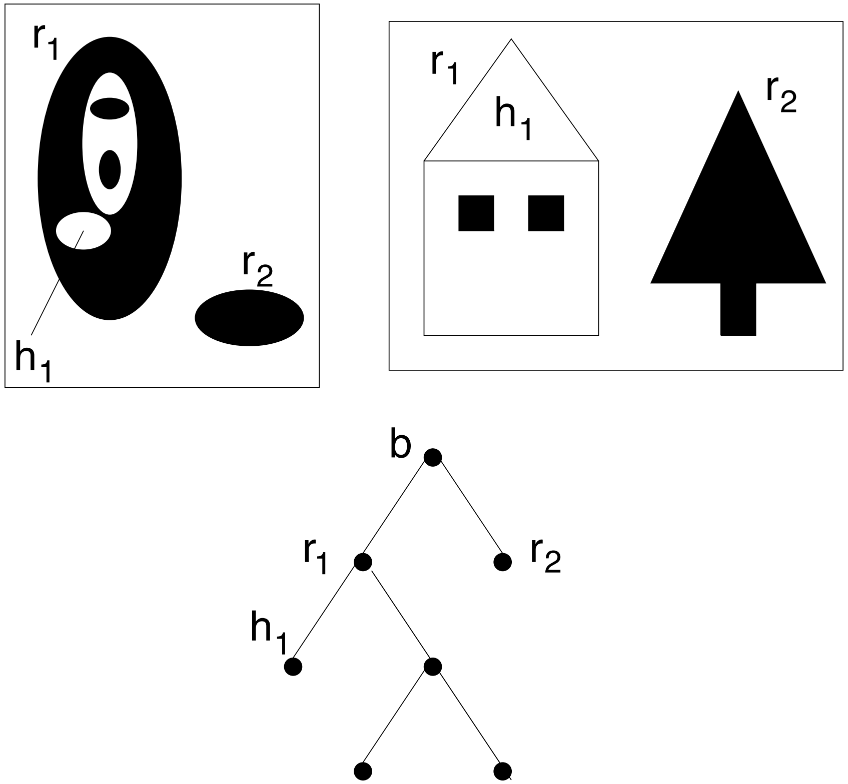
Idempotence – is an important property in mathematics. Here: after one opening, resp. closing, the set is open, resp. closed. Subsequent use of these transformation does not change anything.

$$X \circ B = (X \circ B) \circ B$$

$$X \bullet B = (X \bullet B) \bullet B$$

Homotopic transformation

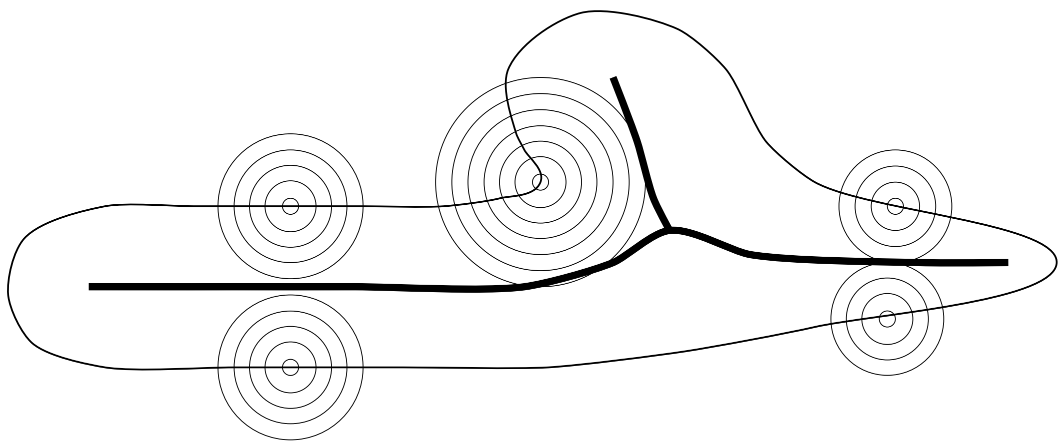
- It is based on the connectedness relation between points, regions, which are expressed by a homotopic tree.
- The homotopic transformation does not change the homotopic tree.



Example: The same homotopic tree corresponds to two different images.

Skeleton

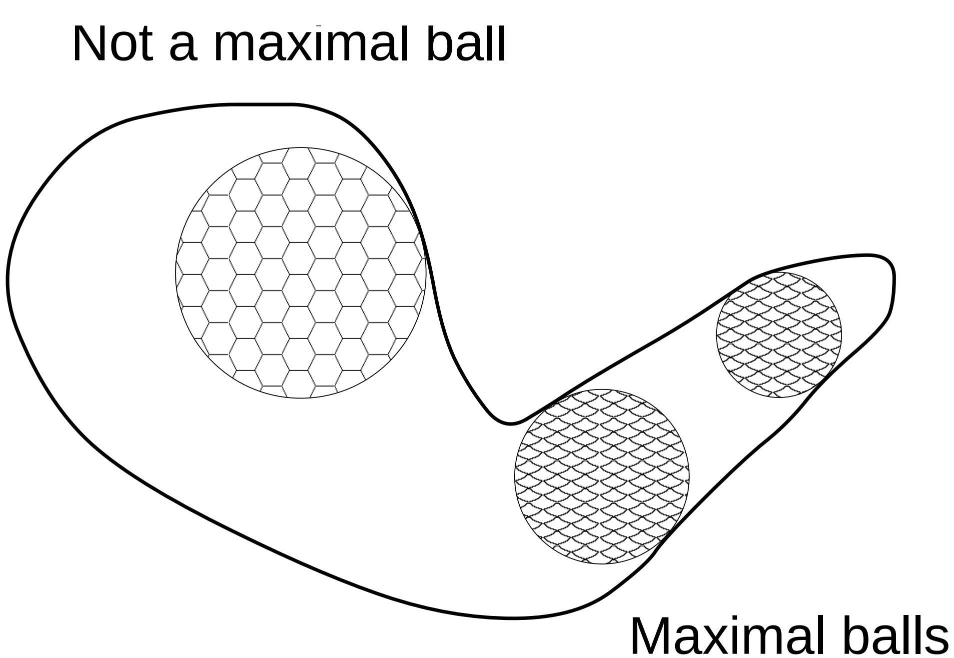
- It is natural to represent elongated objects by a skeleton.
- H. Blum proposed in 1967 a "Medial axis transformation". A locus of points equidistant from contour (the analogy: grass fire).



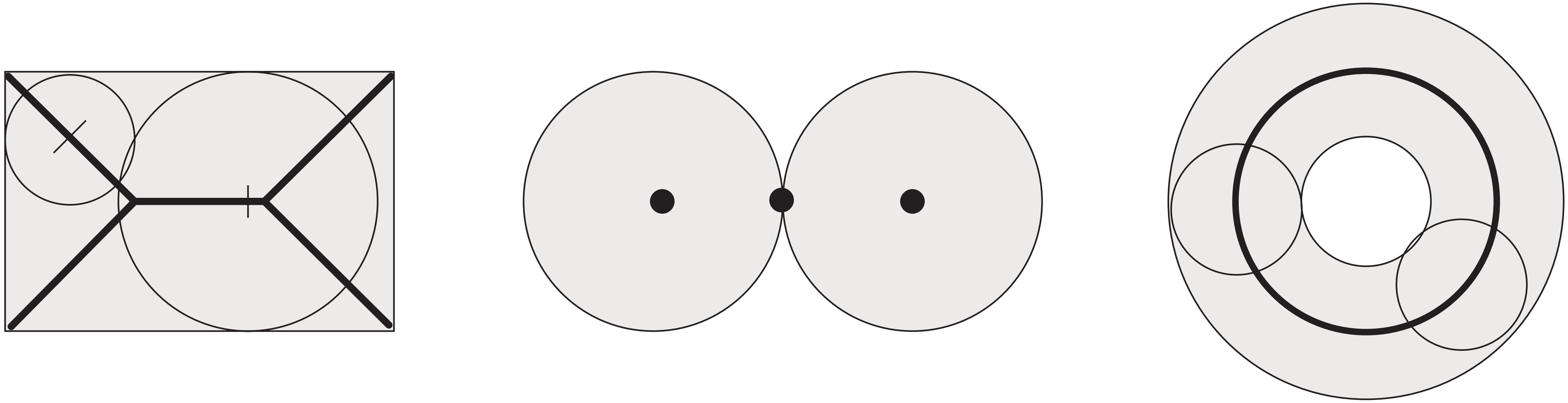
- A formal definition of the skeleton is based on the concept of the maximal circle (the ball in 3D).

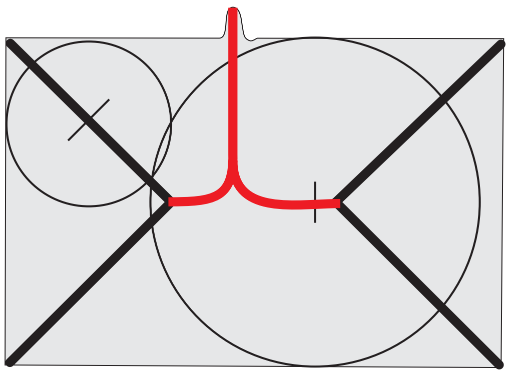
Skeleton by maximal balls

- A circle $B(p, r)$ with the center p and the radius r, where $r \geq 0$ is a set of points, for which the distance $d \leq r$.
- The maximal circle B inscribed into the set X touches its boundary ∂X in two or more points.
- The skeleton is a union of maximal circles centers.



Skeleton example, continuous case



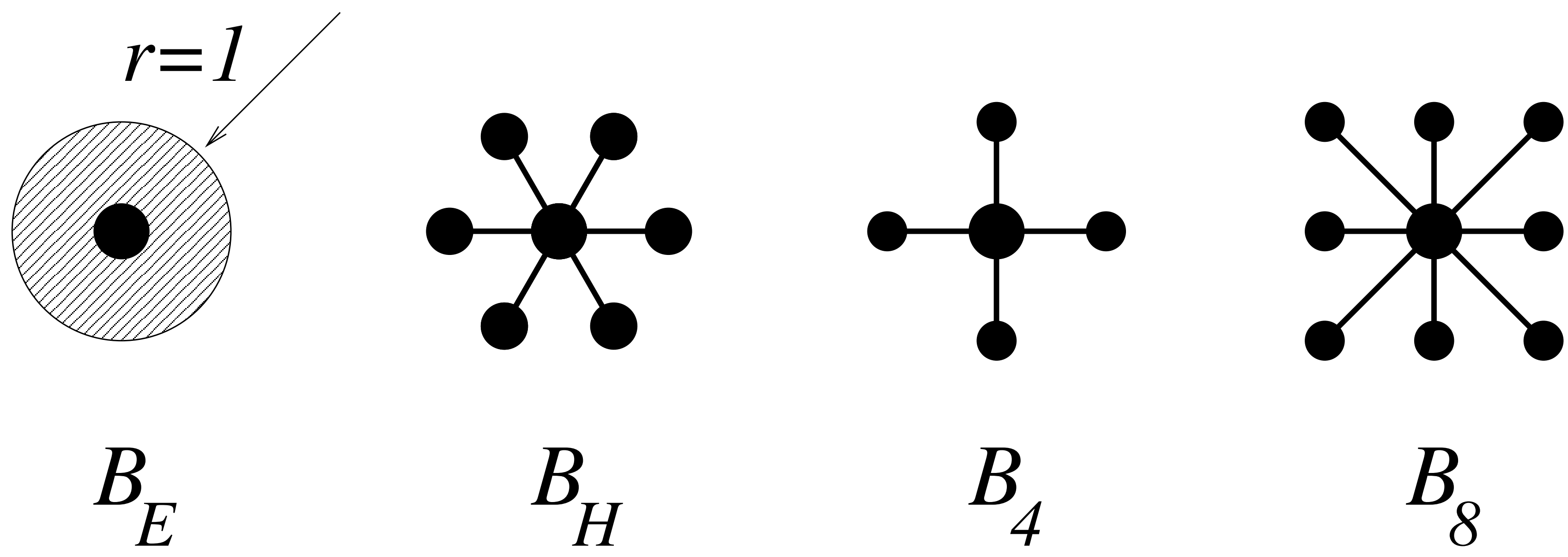


Troubles with noise.

Discrete circles with the radius 1

Circles look differently in the discrete lattice due to different ways how the distance is defined.

Examples:



Taxonomy of binary skeletonization algorithms

Inscription of circles based on previsous definition is almost unused in practice. The computational complexity is high. The connectivity is being lost. The skeleton width > 1.

Sequential thinning. The region is eroded by a suitable structuring element, which ensures that the connectivity is not broken. The oldest approach uses structuring elements from the Golay alphabet (1969) providing a homotopic thinning.

Via distance transformation. Fast and most often used.

In the corner representation. Regions are first compressed losslessly (providing corners). The skeleton is obtained by inscribing maximal rectangles directly in the compressed data (Schlesinger M.I., 1986).

Thinning and thickening

- Let X be an image and $B = (B_1, B_2)$ be a composite structuring element introduced for the hit or miss transformation.
- Thinning $X \oslash B = X \setminus (X \otimes B)$.

  When thinning, a part of the boundary of the object given by the structuring element B is subtracted from it by the set difference operation.
- Thickening $X \odot B = X \cup (X^c \otimes B)$.

  When thickening, a part of the boundary given by the structuring element B of the background is added to the object.
- Thinning and thickening are dual transformations

  $(X \odot B)^c = X^c \oslash B \,, \quad B = (B_2, B_1)$.

Sequential thinning and thickening

- Let $\{B_{(1)}, B_{(2)}, B_{(3)}, \ldots, B_{(n)}\}$ be a sequence of composite structuring elements $B_{(i)} = (B_{i_1}, B_{i_2})$.
- Sequential thinning can then be expressed as a sequence of n structuring elements for square rasters (e.g. eight elements of the size 3×3, as will be shown later with Golay alphabet).

$$X \oslash \{B_{(i)}\} = (((X \oslash B_{(1)}) \oslash B_{(2)}) \ldots \oslash B_{(n)}) \, .$$

- Sequential thickening (analogically)

$$X \odot \{B_{(i)}\} = (((X \odot B_{(1)}) \odot B_{(2)}) \ldots \odot B_{(n)}) \, .$$

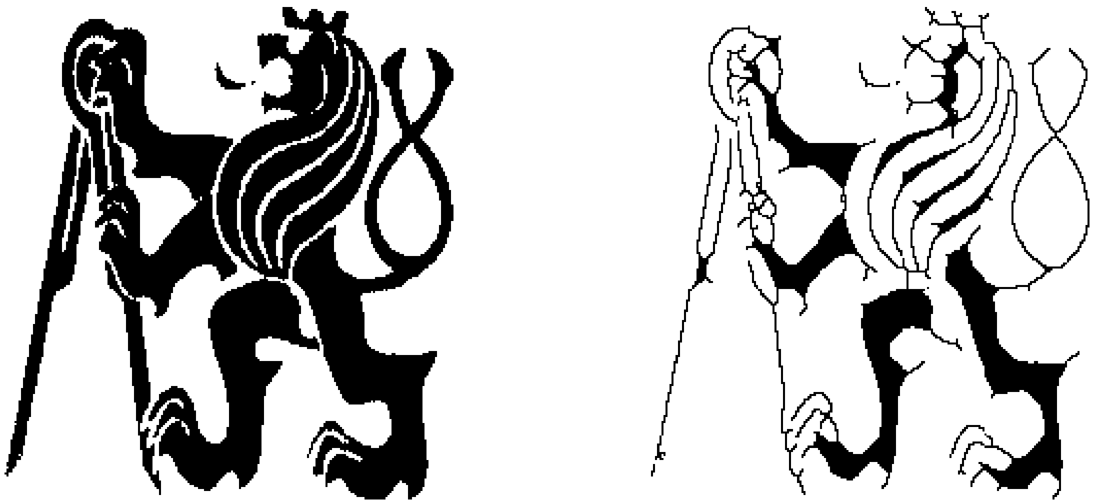
Useful sequences from the Golay's alphabet

- Several sequences of structuring elements $\{B_{(i)}\}$ are useful in practice.
- Let us show two of them from the Golay alphabet (1969) for octagonal lattice. Structuring elements are displayed for the first two rotations, from which the other rotations can be derived.
- A concise representation of the composite structuring element in one matrix: the value 1 checks if the corresponding point (image pixel) is a subset of B_1 (objects). Simultaneously, the value 0 checks if the point is a subset of B_2 (background). Finally, the value $*$ means that this element is not used in the matching process.
- Thinning and thickening sequential by Golay alphabet elements is idempotent.

Thinning by the element L, homotopic approximation of the skeleton of the width 1

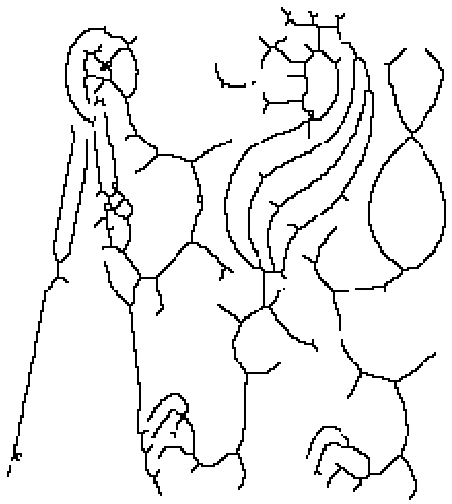
$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ * & 1 & * \\ 1 & 1 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} * & 0 & 0 \\ 1 & 1 & 0 \\ * & 1 & * \end{bmatrix} \ldots$$

5 iterations



Thinning by the element L (2)

Thinning until the idempotence is reached..

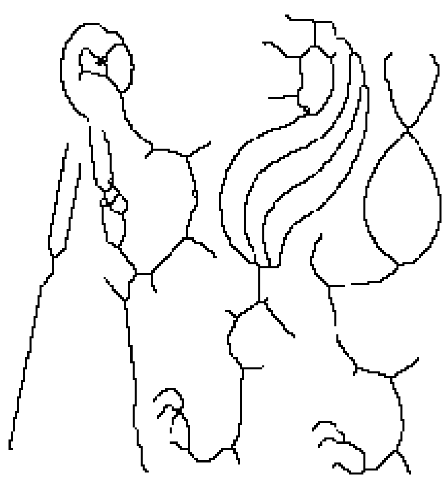


Cutting from free ends by the element E

$$E_1 = \begin{bmatrix} * & 1 & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dots$$

If thinning by element E is performed until the image does not change, then only closed contours remain.

5 iterations

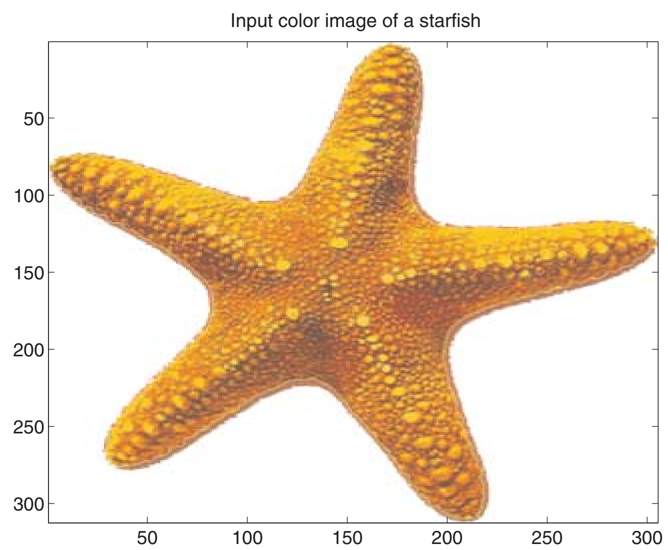


Motivation for sequential morphology
Distance transformation

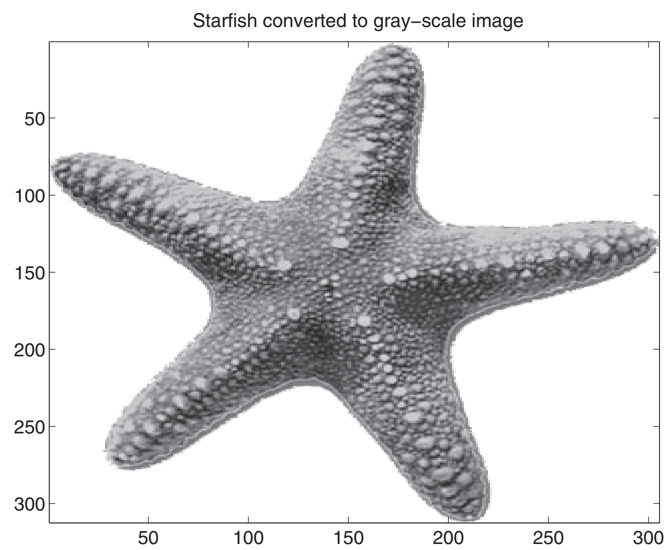
- It has not matter so far, in which order the morphological operation have been used in different location in the image. Operation could have been used in a random order, line after line, in parallel, . . .
- A more specific approach, in which the order of operator positions in the image is appropriately prescribed, can yield a substantial calculation speed up. The outcome of the operator will depend not only on the input image and the operator, but also on partial results of applying the operator previous locations in the image.
- By doing this, the useful global information can be accumulated, and the operator can explore previous results. The benefit is in gaining the speed and simplification of algorithms.
- Morphological operators based on the effective algorithm calculating the distance transformation (dealt with earlier) constitute an important example of this approach, e.g. in calculating skeletons in binary morphology.

Distance transformation
Starfish example, input image

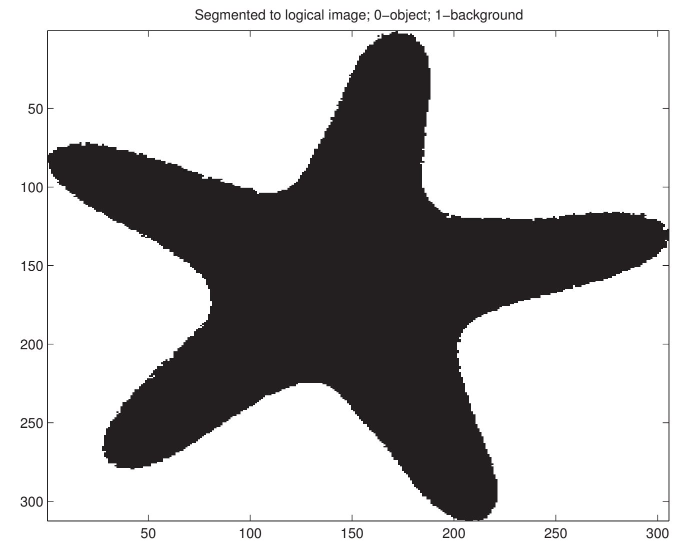
The distance transformation was explained in the lecture Digital image, basic concepts.



color

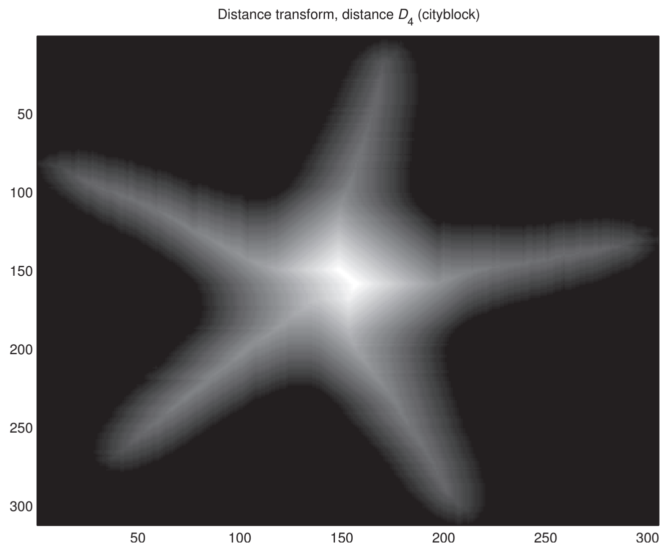


grayscale

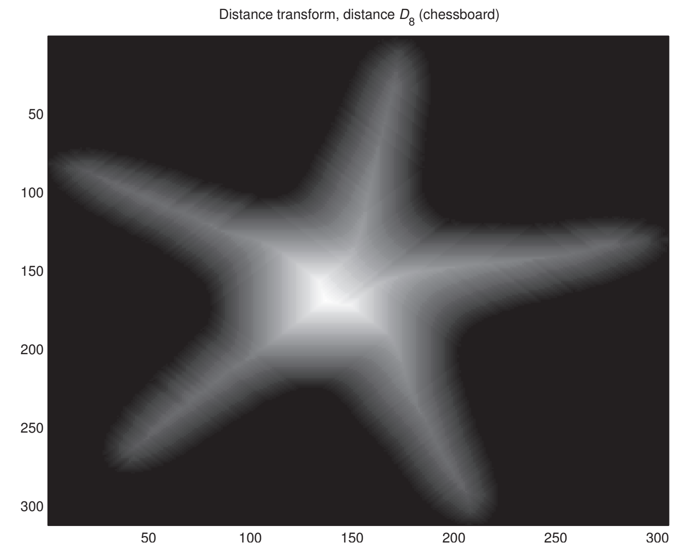


binary

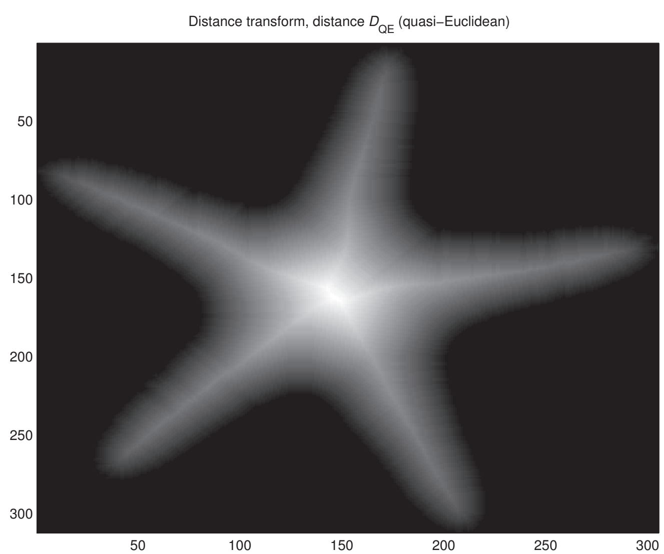
DT, starfish example, results



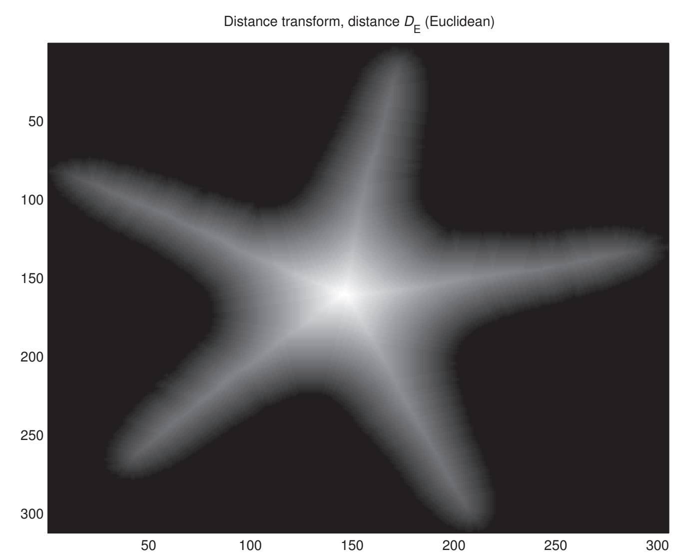
D4



D8



quazieuclidean



euclidean