Wavelets transformation

Václav Hlaváč

Czech Technical University in Prague
Faculty of Electrical Engineering, Department of Cybernetics
Center for Machine Perception
http://cmp.felk.cvut.cz/~hlavac, hlavac@fel.cvut.cz

Deficiencies of Fourier transform

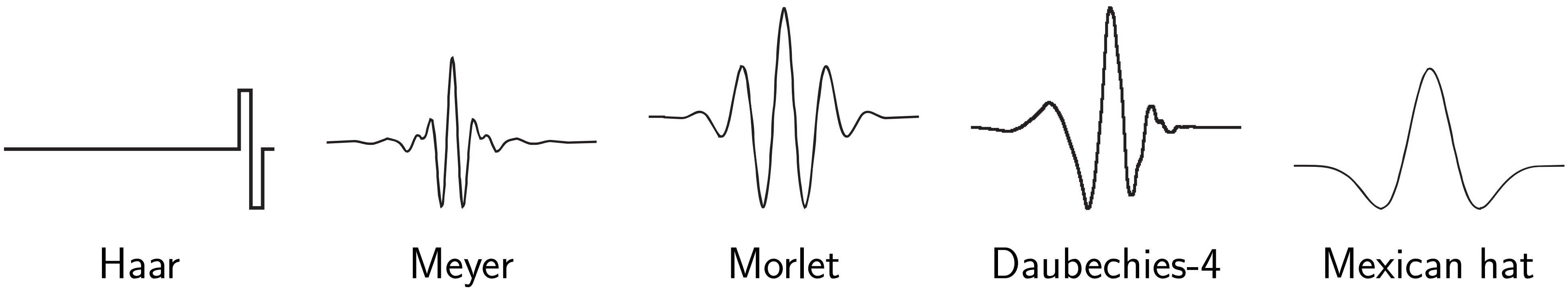
- Fourier transform and similar ones have a principal disadvantage: only information about the frequency spectrum is provided, and no information is available on the *time* in 1D (or *location in the image* in 2D) at which events occur.
- One solution to the problem of localizing changes in the signal (image) is to use the short time Fourier transform, where the signal is divided into small windows and treated locally as it were periodic.
- The uncertainty principle provides guidance on how to select the windows to minimize negative effects, i.e., windows have to join neighboring windows smoothly.
- The window dilemma remains—a narrow window yields poor frequency resolution, while a wide window provides poor localization.

A more complex basis functions – wavelets

- The wavelet transform goes further than the short time Fourier transform.
- It also analyzes the signal (image) by multiplying it by a window function and performing an orthogonal expansion, analogously to other linear integral transformations.
- Formally, a wavelet series represents a square-integrable function with respect a complete, orthonormal set of basis functions called wavelets, meaning a small wave.
- There are two directions in which the analysis is extended with respect to Fourier transformation.
  1. The used basis functions (wavelets) are more complicated than sines and cosines applied in Fourier transform.
  2. The analysis is performed at multiple scales.

Wavelets

- Wavelets provide localization in space to a certain degree.
- The entire space-frequency localization is still not possible due to the Heisenberg's uncertainty principle.
- In 1D, the shape of five commonly used basis functions in a single scale of many scales (mother wavelets) is illustrated pictorially in a qualitative manner;



Multiple scales

- Modeling a spike in a function (a noise dot in an image, for example) with a sum of a huge number of functions will be hard because of the spike's strict locality.
- Functions that are already local will be naturally suited to the task.
- Such functions lend themselves to more compact representation via wavelets—sharp spikes and discontinuities normally take fewer wavelet bases to represent as compared to the sine-cosine basis functions.
- Localization in the spatial domain together with the wavelet's localization in frequency yields a sparse representation of many practical signals (images).

- Sparseness opens the door to successful applications in data/image compression, noise filtering and detecting features in images.

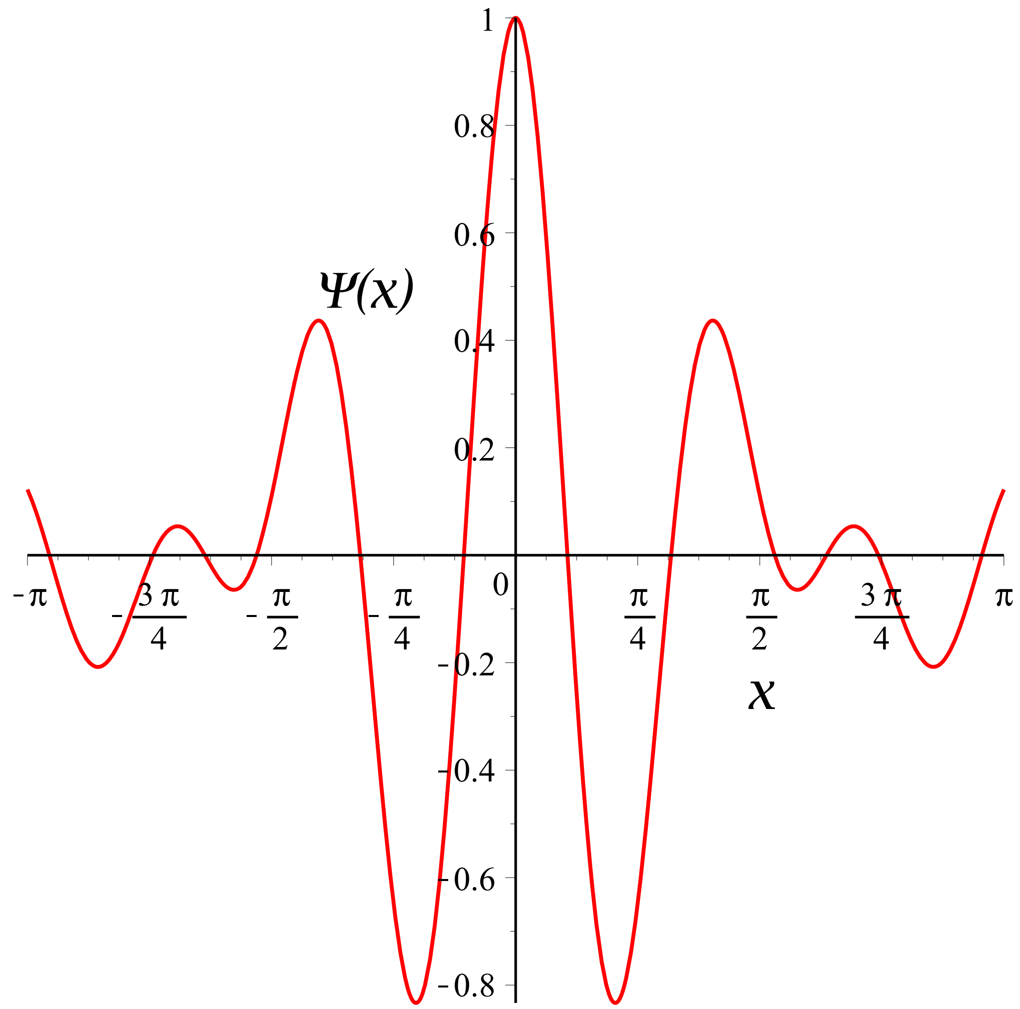
Continuous wavelet transforms (CWT)

- Continuous shift and scale parameters are considered.
- A given signal of finite energy is projected on a continuous family of frequency bands (subspaces of the function space L^p in functional analysis).
- For instance the signal may be represented on every frequency band of the form $[f, 2f]$ for all positive frequencies $f > 0$.
- The original signal can be reconstructed by a suitable integration over all the resulting frequency components.
- The frequency bands are scaled versions of a subspace at scale 1.
- This subspace is generated by the shifts of one generating function Ψ, called the mother wavelet.

The mother wavelet illustration, an example

For example, let us demonstrate the Shannon mother wavelet in one frequency band $[1, 2]$,

$$\Psi(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}\,.$$



1D continuous wavelet transform

- A function $f(t)$ is decomposed into a set of basis functions Ψ—wavelets

$$c(s,\tau) = \int_R f(t)\,\Psi^*_{s,\tau}(t)\,\mathrm{d}t\,, \quad s \in R^+ - \{0\}\,, \quad \tau \in R\,.$$

$c(s,\tau)$ are wavelet coefficients. A complex conjugation is denoted by *.

- The subscripts s, τ denote scale and shift (translation), respectively.
- Wavelets are generated from the single mother wavelet $\Psi(t)$ by scaling s and shifting τ; $s > 1$ dilates, $s < 1$ contracts the signal,

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\Psi\left(\frac{t-\tau}{s}\right)\,.$$

- The coefficient $1/\sqrt{s}$ is used because the energy has to be normalized across different scales.

Meaning of wavelet coefficients $c(s,\tau)$

- The integral $\int_R f(t)\,\Psi^*_{s,\tau}(t)\,\mathrm{d}t$ from the previous slide can be interpreted as the scalar (inner) product of the signal $f(t)$ and the particular wavelet (basis function) $\Psi^*_{s,\tau}(t)$.
- This scalar product tells to what degree is the shape of the signal similar (correlated) to the local probe given by the particular wavelet.
- The space of scales s and shifts τ is discretized in real use. We will deal with the discretization later.

Inverse continuous wavelet transform

- The inverse continuous wavelet transform serves to synthesize the 1D signal $f(t)$ of finite energy from wavelet coefficients $c(s, \tau)$

$$f(t) = \int_{R^+} \int_{R} c(s, \tau)\, \Psi_{s,\tau}(t)\, \mathrm{d}s\, \mathrm{d}\tau\ .$$

- Note:
  The wavelet transform was defined generally without the need to specify a particular mother wavelet: the user can select or design the basis of the expansion according to application needs.

Q: Can any function be a wavelet? A: No.

- The wavelet should be oscillatory

$$\int_{-\infty}^{\infty} \Psi(t) \mathrm{d}\, t = 0\,.$$

- Wavelet has to have a finite energy

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 \mathrm{d}\, t \leq \infty\,.$$

Dyadic (octave) grid for scale and shift (1)

- The continuous change of the scale and shift parameters would lead to a very redundant signal representation.
- Scale and shift parameters will be changed in discrete steps.
- This is step towards discrete wavelet transformation (DWT).

Dyadic (octave) grid for scale and shift (2)

- It is advantageous to use special values for shift τ and scale s while defining the wavelet basis, i.e. introducing the scale step j and the shift step k: $s = 2^{-j}$ and $\tau = k\,2^{-j}$; $j = 1, \ldots$; $k = 1, \ldots$;

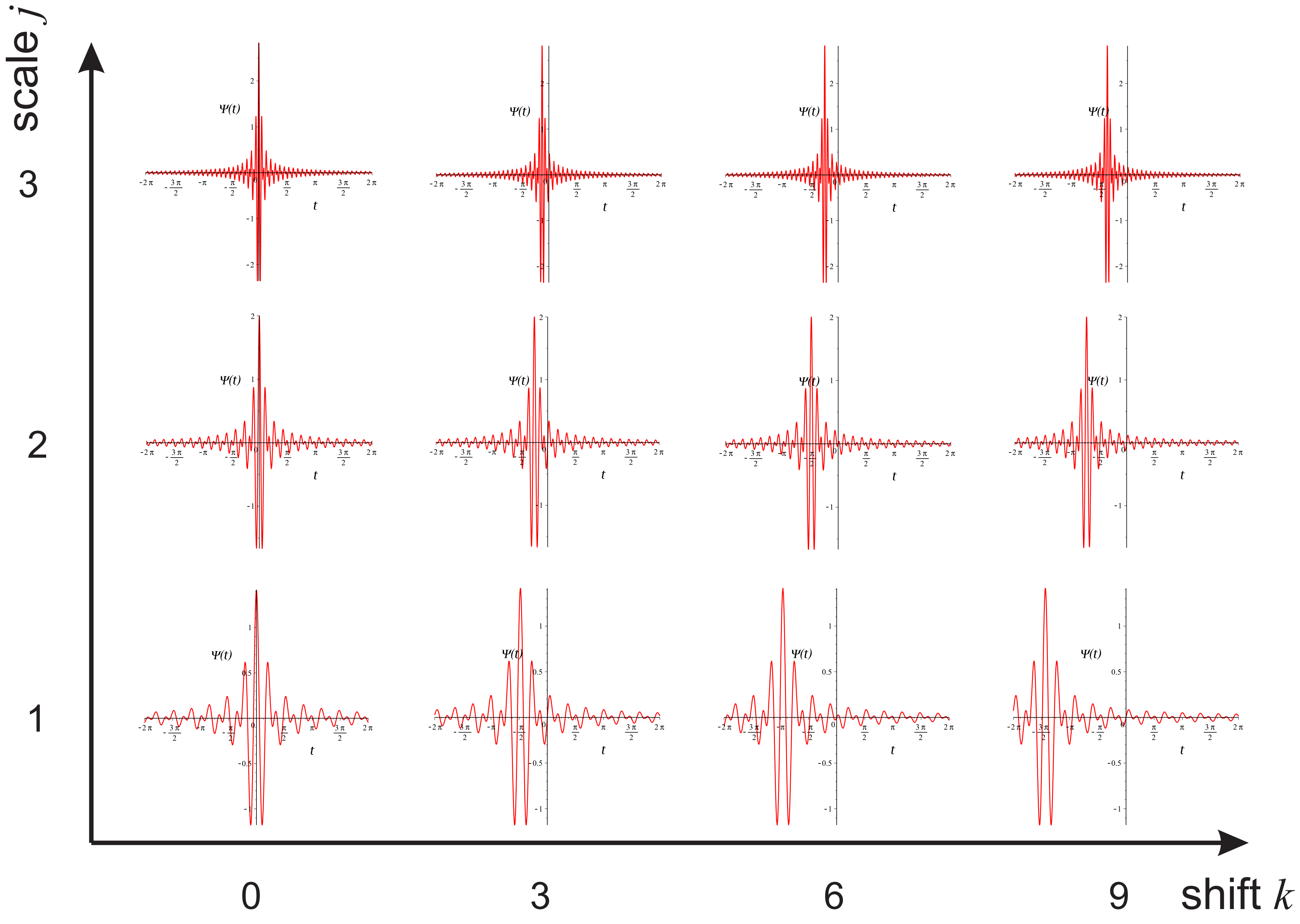
$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\Psi\left(\frac{t-\tau}{s}\right) = \frac{1}{\sqrt{2^{-j}}}\,\Psi\left(\frac{t-k\,2^{-j}}{2^{-j}}\right) = 2^{\frac{j}{2}}\,\Psi\left(2^j\,t-k\right) .$$

$$\Psi_{j,k}(t) = 2^{\frac{j}{2}}\,\Psi\left(2^j\,t-k\right) .$$

- Example (Shannon wavelet) expanded from the slide on the page 7:

$$\Psi_{j,k}(t) = 2^{\frac{j}{2}}\,\frac{(\sin(2\pi(2^j\,t+k)) - \sin(\pi 2^j\,t+k))}{\pi\,(2^j\,t+k)} .$$

Example, Shannon wavelet, multiple scales, shifts



Computation of the Continuous Wavelet Transformation

1. Begins at a finest scale and a zero shift.

2. The wavelet is placed at beginning of the signal, the inner product of the signal and the wavelet is calculated, and integrated for all times.

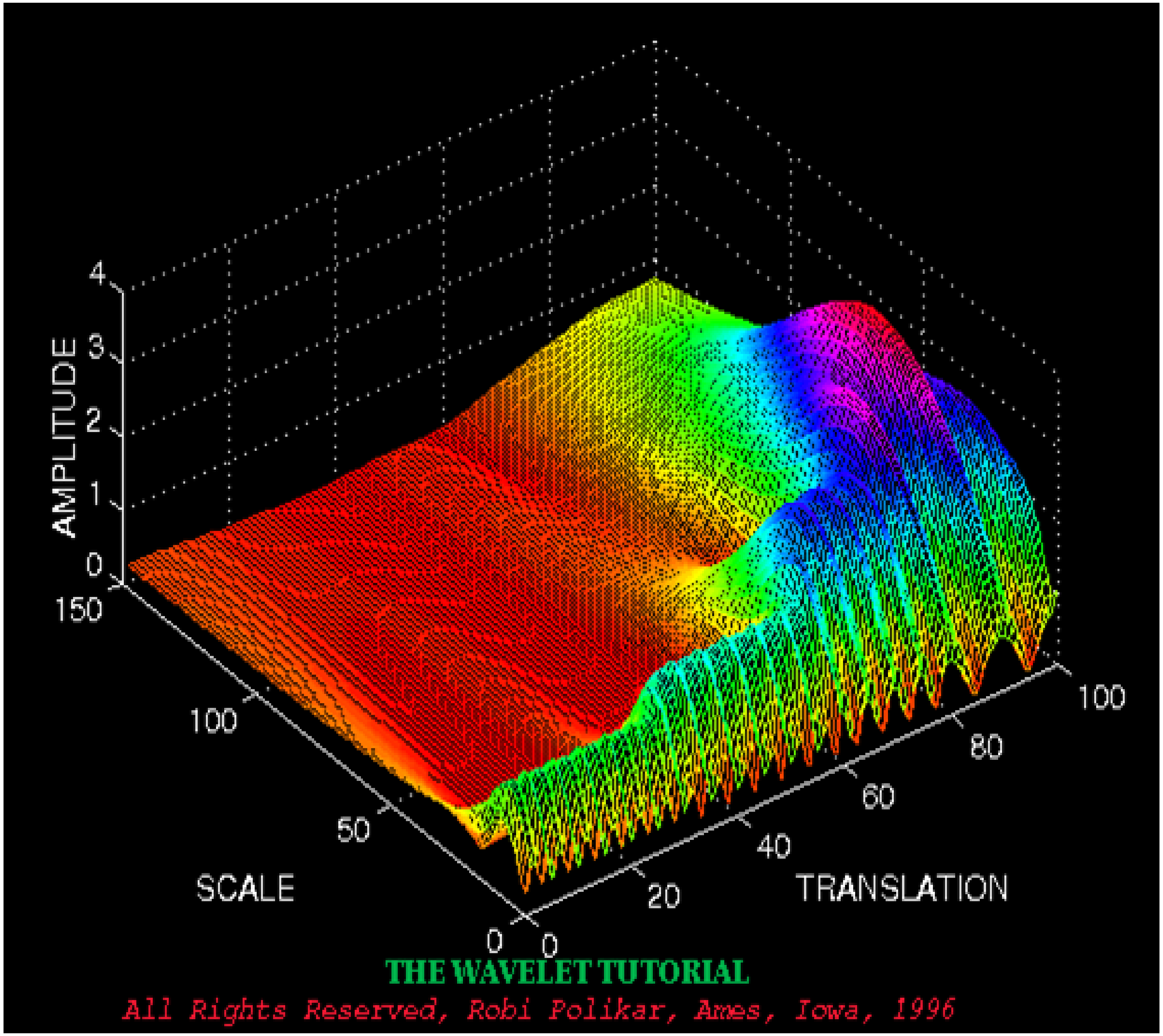
   The result is one value of $c(j, k)$ providing the 'local similarity' of a part of a signal and the wavelet.

3. The wavelet is shifted and the step 2 is repeated until end of the signal.

4. Steps 2-3 repeated for different scales.

The output is a matrix of c values for all scales and shifts, so called spectrogram.

Wavelet spectrogram example



Wavelets properties from a user point of view (1)

Simultaneous localization in time and space.

- The location of the wavelet allows to explicitly represent the location of events in time (with a theoretical limit given by Heisenberg's uncertainty principle).
- The shape of the wavelet allows to represent different detail or resolution.

Sparsity – for signals appearing in practice: Many of the coefficients in a wavelet representation are either zero or very small.

Linear-time complexity – many 1D wavelet transformations can be accomplished in $\mathcal{O}(\mathcal{N})$ time.

Wavelets properties from a user point of view (2)

Adaptability – wavelets can be adapted to represent a wide variety of signals (e.g., functions with discontinuities, functions defined on bounded domains, etc.).

- Suited, e.g., for tasks involving images, closed or open curves, and very different surfaces.
- Wavelets can represent functions with discontinuities or corners rather efficiently (Recall that some wavelets have sharp corners themselves).

Discrete wavelet transformation (DWT)

- Uses discrete dyadic (octave) grid for scale parameter j and shift parameter k as introduced on slide 13.
- Forward DWT:

$$a_{j,k} = \sum_t f(t)\, \Psi^*_{j,k}(t)\ , \text{ where } \Psi_{j,k}(t) = 2^{\frac{j}{2}}\, \Psi\left(2^j\, t - k\right)\ .$$

as was introduced on the slide number 13.

- Inverse DWT:

$$f(t) = \sum_k \sum_j a_{j,k} \Psi_{j,k}(t)\ .$$

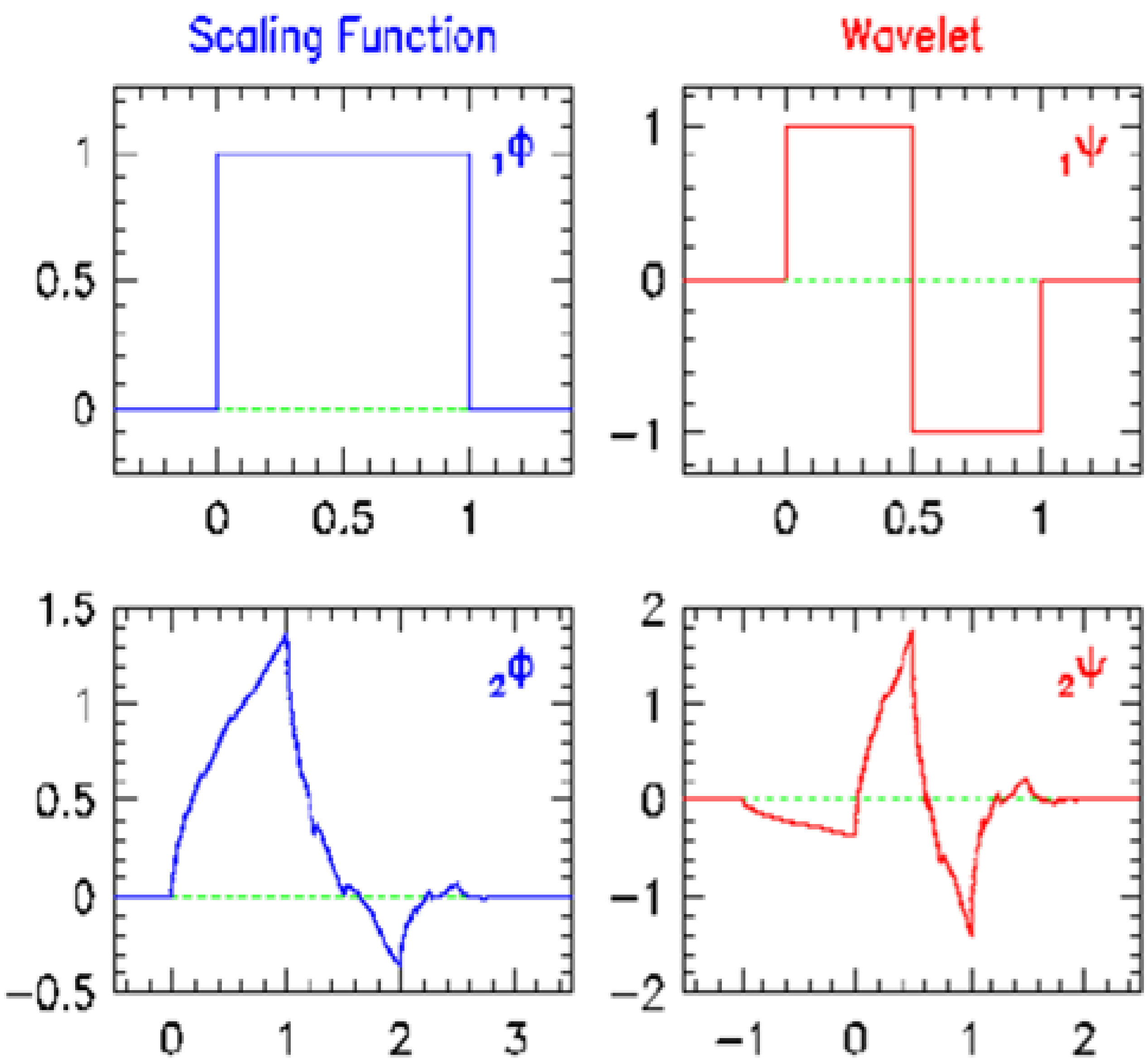
Properties of Daubechies wavelets

Ingrid Daubechies, Communications Pure Applied Math. 41 (1988), 909-996.

- Compact support.
  - Finite number of filter parameters / fast implementations.
  - High compressibility
  - Fine scale amplitudes are very small in regions where the function is smooth / sensitive recognition of structures.
- Identical forward / backward filter parameters.
  - Fast, exact reconstruction.
  - Very asymmetric.

Haar and Daubechies wavelets, pictorial example

- Haar wavelet
- Daubechies wavelet



Mallat's filter scheme, Fast Wavelet Transform

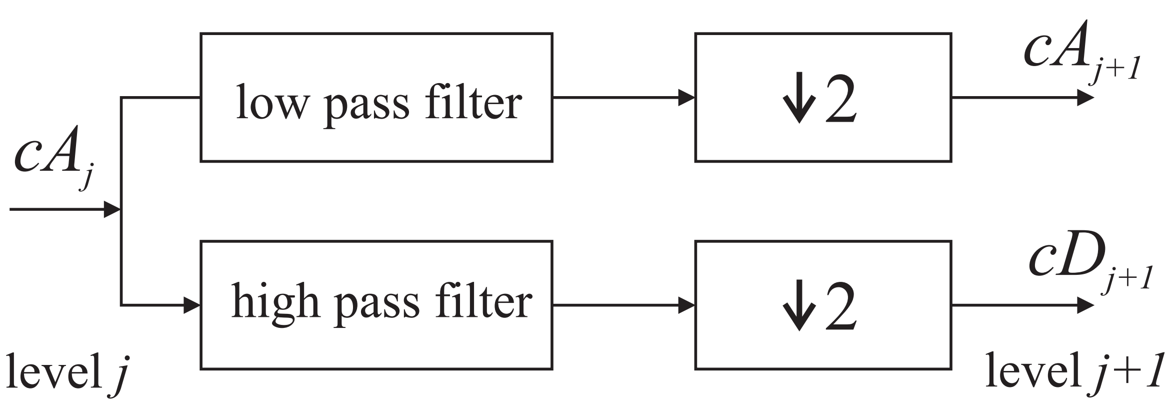
- Stephane G. Mallat: A Theory for Multiresolution Signal Decomposition: The Wavelet Representation, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 11, No. 7, July 1989, pp. 674-693.
- Mallat was the first who implemented the dyadic grid scheme for wavelets using a well known filter design method called 'two channel sub band coder'.
- This yielded a 'Fast Wavelet Transform'.

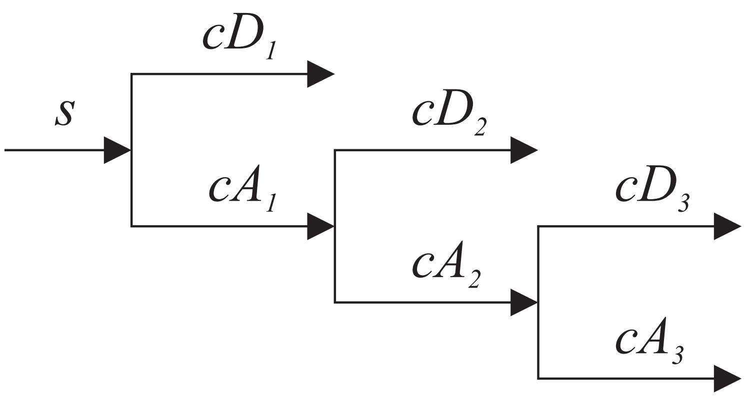
Fast Wavelet Transform (1)

- Consider a discrete 1D signal s of length N which has to be decomposed into wavelet coefficients c.
- The Fast Wavelet Transform consists of $\log_2 N$ steps at most.
- The first decomposition step takes the input and provides two sets of coefficients at level 1: approximation coefficients cA_1 and detail coefficients cD_1.
- The vector s is convolved with a low-pass filter for approximation and with a high-pass filter for detail.
- Dyadic decimation follows which down samples the vector by keeping only its even elements. Such down sampling will be denoted by $\downarrow 2$ in block diagrams.

Fast Wavelet Transform, filter banks (2)

- The coefficients at level $j+1$ are calculated from the coefficients at level j, which is illustrated in the bottom-left figure.
- This procedure is repeated recursively to obtain approximation and detail coefficients at further levels. This yields a tree-like structure of filters called filter banks.
- The structure of coefficients for level $j=3$ is illustrated in the bottom-right figure.





Fast Inverse Wavelet Transformation

- The Fast Inverse Wavelet Transform takes as an input the approximation coefficients cA_j and detail coefficients cD_j and inverts the decomposition step.
- The vectors are extended (up sampled) to double length by inserting zeros at odd-indexed elements and convolving the result with the reconstruction filters. Analogously to down sampling, up sampling is denoted $\uparrow 2$ in the block diagrams.