Wavelets transformation

Václav Hlaváč

Czech Technical University in Prague
Faculty of Electrical Engineering, Department of Cybernetics
Center for Machine Perception
`http://cmp.felk.cvut.cz/~hlavac, hlavac@fel.cvut.cz`

Deficiencies of Fourier transform

- Fourier transform and similar ones have a principal disadvantage: only information about the frequency spectrum is provided, and no information is available on the *time (or location in the image)* at which events occur.
- One solution to the problem of localizing changes in the signal (image) is to use the short time Fourier transform, where the signal is divided into small windows and treated locally as it were periodic.
- The uncertainty principle provides guidance on how to select the windows to minimize negative effects, i.e., windows have to join neighboring windows smoothly.
- The window dilemma remains—a narrow window yields poor frequency resolution, while a wide window provides poor localization.

A more complex basis functions – wavelets

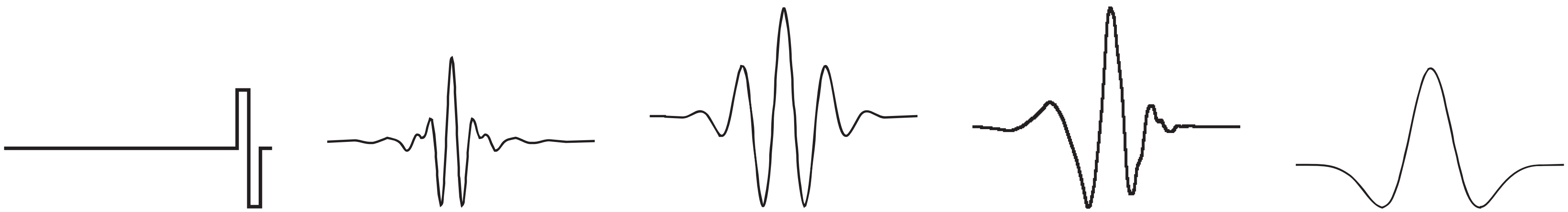
- The wavelet transform goes further than the short time Fourier transform.
- It also analyzes the signal (image) by multiplying it by a window function and performing an orthogonal expansion, analogously to other linear integral transformations.

  There are two directions in which the analysis is extended.

  1. The more complicated basis functions than sines and cosines (called wavelets, meaning a small wave, or mother wavelets) are used.
  2. The analysis is performed at multiple scales.

Wavelets

- Wavelets provide localization in space to a certain degree, not entire localization due to the uncertainty principle.
- The shape of five commonly used mother wavelets is illustrated in a qualitative manner and in a single of many scales;



Haar Meyer Morlet Daubechies-4 Mexican hat

Multiple scales

- Modeling a spike in a function (a noise dot, for example) with a sum of a huge number of functions will be hard because of the spike's strict locality.
- Functions that are already local will be naturally suited to the task.
- such functions lend themselves to more compact representation via wavelets—sharp spikes and discontinuities normally take fewer wavelet bases to represent as compared to the sine-cosine basis functions.
- Localization in the spatial domain together with the wavelet's localization in frequency yields a sparse representation of many practical signals (images).

- Sparseness opens the door to successful applications in data/image compression, noise filtering and detecting features in images.

1D continuous wavelet transform

- A function $f(t)$ is decomposed into a set of basis functions Ψ—wavelets

$$c(s,\tau) = \int_R f(t)\,\Psi^*_{s,\tau}(t)\,\mathrm{d}t\,, \quad s \in R^+ - \{0\}\,, \quad \tau \in R$$

(complex conjugation is denoted by *). The new variables after transformation are s (scale) and τ (translation).

- Wavelets are generated from the single mother wavelet $\Psi(t)$ by scaling s and translation τ

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\Psi\left(\frac{t-\tau}{s}\right)\,.$$

The coefficient $1/\sqrt{s}$ is used because the energy has to be normalized across different scales.

Inverse continuous wavelet transform

- The inverse continuous wavelet transform serves to synthesize the 1D signal $f(t)$ of finite energy from wavelet coefficients $c(s,\tau)$

$$f(t) = \int_{R^+} \int_{R} c(s,\tau)\, \Psi s,\tau(t)\, \mathrm{d}s\, \mathrm{d}\tau\ .$$

- Note:
  The wavelet transform was defined generally without the need to specify a particular mother wavelet: the user can select or design the basis of the expansion according to application needs.

Q: Can any function be a wavelet? A: No.