



The formal system for modal logics and some facts we can rely on

The formal system K_n for of language with n agents:

1. Propositional tautologies
2. Distribution Axiom (denoted as **K**) $(K_i A \wedge K_i (A \rightarrow B)) \rightarrow K_i B$

Derivation rules:

R1. From the formulas α and $\alpha \rightarrow \beta$ derive β (**Modus Ponens**)

R2. From the formula α derive $K_i \alpha$ (**Knowledge Generalization**)

can be complemented by additional axioms corresponding to properties of the used admissibility relations - reflexivity, transitivity or symmetry

3. **Knowledge Axiom** (denoted as **T**) $K_i A \rightarrow A$ r
4. **Positive Introspection Axiom** (den.as **4**) $K_i A \rightarrow K_i K_i A$ t
5. **Negative Introspection Axiom** (den.as **5**) $\neg K_i A \rightarrow K_i \neg K_i A$ s+t
6. **Consistency Axiom** (den.as **D**) $\neg K_i \text{false}$

Propositional logics: its formal system and some of its useful tautologies

Prop-Ax1: $\varphi \rightarrow (\psi \rightarrow \varphi)$

Prop-Ax2: $(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$

Prop-Ax3: $(\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi)$

Modus ponens:

$$\frac{\varphi, (\varphi \rightarrow \psi)}{\psi}$$

Prop-T1: $(\varphi \rightarrow (\psi \rightarrow \tau)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \tau))$

Prop-T2: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \vee \alpha) \rightarrow \beta))$

Prop-T3: $(\alpha \rightarrow \beta) \rightarrow (\neg \beta \rightarrow \neg \alpha)$

Prop-T4a: $\neg \text{false}$

Prop-T4b: $\neg \alpha \vee \alpha$

Prop-T4c: $\alpha \rightarrow (\neg \alpha \rightarrow \tau)$

Prop-T5: $((\alpha \rightarrow \beta) \rightarrow (\neg \alpha \vee \beta)) \& ((\neg \alpha \vee \beta) \rightarrow (\alpha \rightarrow \beta))$

Prop-T6: $((\neg \alpha \vee \neg \beta) \rightarrow \neg(\alpha \& \beta)) \& (\neg(\alpha \& \beta) \rightarrow (\neg \alpha \vee \neg \beta))$

Prop-T7a: $\alpha \rightarrow (\alpha \vee \beta)$

Prop-T7b: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \vee \varphi) \rightarrow (\beta \vee \varphi))$

We can prove the following formulas:

M-T1a: \mathbf{K}_n , $(\varphi \rightarrow \psi) \vdash K_i \varphi \rightarrow K_i \psi$ (see the lecture)

M-T1b: Let φ, ψ be two equiv. fomulas (ie. $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ is a tautology and this is denoted as $\varphi \equiv \psi$) then $\mathbf{K}_n \vdash K_i \varphi \equiv K_i \psi$

M-T2a: $\mathbf{K}_n \vdash K_i (\alpha \ \& \ \beta) \rightarrow K_i \alpha$ (see the lecture)

M-T2b: $\mathbf{K}_n \vdash K_i (\alpha \ \& \ \beta) \equiv K_i \alpha \ \& \ K_i \beta$ (see the lecture)

M-T3: $(\mathbf{K}_n + \mathbf{A3}) \vdash K_i (\neg \text{false})$ **A6**

M-T4a: $(\mathbf{K2} + \mathbf{A3}) \vdash \neg K_i \alpha \vee \neg K_i \neg K_i \alpha$

M-T4b: $(\mathbf{K2} + \mathbf{A3}) \vdash \neg K_i (\alpha \ \& \ \neg K_i \alpha)$

M-T5: $(\mathbf{K}_n + \mathbf{A6}) \vdash \neg (K_i \alpha \ \& \ K_i \neg \alpha)$

M-T6: $\mathbf{K}_n \vdash (K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \ \wedge \ K_i (\neg K_i p))$

M-T7: In $(\mathbf{K}_n + \mathbf{A3})$, it is impossible to prove the formula $K_i \neg (p \rightarrow K_i p)$

We know that $\mathbf{K}_n, (\alpha \rightarrow \beta) \vdash K_i \alpha \rightarrow K_i \beta$

$$\mathbf{K}_n \vdash K_i(\alpha \ \& \ \beta) \equiv K_i \alpha \ \& \ K_i \beta$$

Can one prove $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \equiv K_i \alpha \vee K_i \beta$?

Certainly not! Why?

Step 1. We can prove that $\mathbf{K}_n \vdash K_i \alpha \rightarrow K_i(\alpha \vee \beta)$ and $\mathbf{K}_n \vdash K_i \beta \rightarrow K_i(\alpha \vee \beta)$.

These two formulas can be combined using following prop. tautology

$$\text{Prop-T2} : (\alpha_1 \rightarrow \beta) \rightarrow ((\alpha_2 \rightarrow \beta) \rightarrow ((\alpha_1 \vee \alpha_2) \rightarrow \beta))$$

and we get $\mathbf{K}_n \vdash (K_i \alpha \vee K_i \beta) \rightarrow K_i(\alpha \vee \beta)$

Step 2. To show that the inverse implication $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \rightarrow (K_i \alpha \vee K_i \beta)$ is **not valid** it is enough to construct simple Kripke structure with 3 states, where this formula is not valid.

M-T3: c1) K2, T(A3) |- $\neg K_i \text{false}$

1. $K_i \text{false} \rightarrow \text{false}$ [A3]
2. $\neg \text{false} \rightarrow (\neg K_i \text{false})$ [eq.transf.of 1 based on Prop-T3]
3. $\neg \text{false}$ [Prop-T4]
4. $\neg K_i \text{false}$ [Modus Ponens: 2,3]

M-T4a: K2, T |- $\neg K_i \alpha \vee \neg K_i \neg K_i \alpha$

[eq.transf. based on Prop-T5 “replacement of \rightarrow by \vee “ in the axiom T
 $K_i(\neg K_i \alpha) \rightarrow \neg K_i \alpha$]

M-T4b: K2, T |- $\neg K_i (\alpha \& \neg K_i \alpha)$

1. $K_i \neg K_i \alpha \rightarrow \neg K_i \alpha$ [A3, the axiom T]
2. $\neg K_i \neg K_i \alpha \vee \neg K_i \alpha$ [eq.transf.of 1 based on Prop-T5]
3. $\neg (K_i \neg K_i \alpha \& K_i \alpha)$ [eq.transf.of 2 based on Prop-T6]
4. $\neg K_i (\neg K_i \alpha \& \alpha)$ [eq.transf.of 3 based on Mol-T2b], viz a2

Proof of M-T5: $(K_n + A6) \vdash \neg (K_i \alpha \wedge K_i \neg \alpha)$.

1. $(K_i \alpha \wedge K_i \neg \alpha) \equiv K_i (\alpha \wedge \neg \alpha)$ **M-T2b:**
2. $\neg K_i (false)$ **[A6]**
3. $false \equiv (\alpha \wedge \neg \alpha)$ **[tautology: A1]**
4. $K_i false \equiv K_i (\alpha \wedge \neg \alpha)$ **M-T1b**
5. $\neg K_i (\alpha \wedge \neg \alpha)$ **[MP for 2 and 4]**
6. $\neg (K_i \alpha \wedge K_i \neg \alpha) \equiv \neg K_i (\alpha \wedge \neg \alpha)$ **[taut.transcript of l. 1]**
7. $\neg (K_i \alpha \wedge K_i \neg \alpha)$ **[MP for 6 and 5]**

M-T6: $\mathbf{K}_n \vdash (K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \ \& \ K_i (\neg K_i p))$

Crucial steps of the proof

- a) $\neg (p \rightarrow K_i p) \equiv (p \wedge \neg K_i p)$ [Taut]
- b) $K_i \neg (p \rightarrow K_i p) \equiv K_i (p \ \& \ \neg K_i p)$ [M-T1b]
- c) $K_i (p \ \& \ \neg K_i p) \rightarrow (K_i p \ \& \ K_i (\neg K_i p))$ [M-T2b]
- d) $(K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \ \& \ K_i (\neg K_i p))$ [lines b,c and transitivity of “ \rightarrow ”]

Mod_T7: The formula $K_i \neg (p \rightarrow K_i p)$ is not provable in $(K_n + Ax3)$.

Proof. Suppose $K_i \neg (p \rightarrow K_i p)$ is provable. We could apply modus ponens to **Mod_T6** and $K_i \neg (p \rightarrow K_i p)$ to prove the formula $K_i p \ \& \ K_i (\neg K_i p)$. Let us assume that $K_i p \ \& \ K_i (\neg K_i p)$ holds:

1. $K_i p \ \& \ K_i (\neg K_i p)$ [assumption]
2. $K_i p$ [1 and property of conjunction $(\alpha \ \& \ \beta) \rightarrow \alpha$]
3. $K_i (\neg K_i p)$ [1 and property of conjunction $(\alpha \ \& \ \beta) \rightarrow \beta$]
4. $K_i (\neg K_i p) \rightarrow \neg K_i p$ [**Ax3**]
5. $\neg K_i p$ [Modus ponens: 3,4]
6. *false* [definition of *false* and the lines 2 and 5]

But the axiom system $(K_n + Ax3)$ has a model – that is why it cannot be contradictory and the formula *false* cannot be provable there. What is wrong with our proof? **It must be based a wrong assumption!** There were assumed $K_i \neg (p \rightarrow K_i p)$ and its direct consequence (obtained through **MP** and **Mod_T6**) the formula $K_i p \ \& \ K_i (\neg K_i p)$. The original assumption „the formula $K_i \neg (p \rightarrow K_i p)$ is provable” must be wrong!

Mod_T8a: K_n , **Ax3**, **Ax 5**, $K_i(p \rightarrow K_i p) \vdash K_i p \vee K_i \neg p$

1. $K_i(p \rightarrow K_i p)$ assumption
2. $K_i p \vee \neg K_i p$ Prop_T4b $\alpha \vee \neg \alpha$
3. $K_i p \rightarrow K_i p \vee K_i \neg p$ Prop-T7a $\alpha \rightarrow (\alpha \vee \beta)$
4. $K_i(p \rightarrow K_i p) \rightarrow (p \rightarrow K_i p)$ **Ax 3**
5. $p \rightarrow K_i p$ Modus Ponens for 4 and 1
6. $(p \rightarrow K_i p) \rightarrow (\neg K_i p \rightarrow \neg p)$ Prop_T3
7. $\neg K_i p \rightarrow \neg p$ Modus Ponens for 6 and 5
8. $\neg K_i p \rightarrow K_i \neg K_i p$ **Ax 5**
9. $K_i \neg K_i p \rightarrow K_i \neg p$ M-T1 for the line 7
10. $\neg K_i p \rightarrow K_i \neg p$ transitivity of “ \rightarrow ” for lines 8 and 9
11. $(\alpha \rightarrow \beta) \rightarrow (\gamma \vee \alpha \rightarrow \gamma \vee \beta)$ Prop-T7b
12. $(K_i p \vee \neg K_i p) \rightarrow (K_i p \vee K_i \neg p)$ Modus Ponens for lines 10,11 and $\gamma = K_i p$
13. $(K_i p \vee K_i \neg p)$ Modus Ponens for lines 12 and 2

Mod_T8b: K_n , Ax 4, $(K_i p \vee K_i \neg p) \vdash K_i (p \rightarrow K_i p)$

1. $K_i p \rightarrow (p \rightarrow K_i p)$ propositional axiom
2. $K_i K_i p \rightarrow K_i (p \rightarrow K_i p)$ Mod_T1 for the line 1
3. $K_i p \rightarrow K_i K_i p$ **Ax4**
4. $K_i p \rightarrow K_i (p \rightarrow K_i p)$ transitivity of “ \rightarrow ” for the lines 3 a 2
5. $\neg p \rightarrow (p \rightarrow K_i p)$ Prop_4c
6. $K_i \neg p \rightarrow K_i (p \rightarrow K_i p)$ Mod_T1 for the line 5
7. $(\alpha_1 \rightarrow \beta) \rightarrow ((\alpha_2 \rightarrow \beta) \rightarrow ((\alpha_1 \vee \alpha_2) \rightarrow \beta))$ Prop-T2 for
 $\alpha_1 = K_i p, \alpha_2 = K_i \neg p, \beta = K_i (p \rightarrow K_i p)$
8. $(K_i p \vee K_i \neg p) \rightarrow K_i (p \rightarrow K_i p)$ Modus ponens applied first to 7 and 4 and later to this formula with 6
9. $(K_i p \vee K_i \neg p)$ **assumption**
10. $K_i (p \rightarrow K_i p)$ Modus ponens applied to 8 and 10