The formal system for modal logics

and some facts we can rely on

Axioms of propositional modal logics

The formal system K_n for of language with n agents:

1. **Propositional tautologies**
2. **Distribution Axiom** (denoted as **K**) $(K_i A \wedge K_i(A \rightarrow B)) \rightarrow K_i B$

Derivation rules:

R1. From the formulas α and $\alpha \rightarrow \beta$ derive β (**Modus Ponens**)

R2. From the formula α derive $K_i\, \alpha$ (**Knowledge Generalization**)

can be complemented by additional axioms corresponding to properties of the used admissibility relations - **r**eflexivity, **t**ransitivity or **s**ymmetry

3. **Knowledge Axiom** (denoted as **T**) $K_i A \rightarrow A$ — r
4. **Positive Introspection Axiom** (den.as **4**) $K_i A \rightarrow K_i K_i A$ — t
5. **Negative Introspection Axiom** (den.as **5**) $\neg K_i A \rightarrow K_i \neg K_i A$ — s+t
6. **Consistency Axiom** (den.as **D**) $\neg K_i\, false$

Propositional logics: its formal system
and some of its useful tautologies

Prop-Ax1: $\varphi \rightarrow (\psi \rightarrow \varphi)$

Prop-Ax2: $(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma))$

Prop-Ax3: $(\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi)$

Modus ponens:

$$\frac{\varphi,\ (\varphi \rightarrow \psi)}{\psi}$$

Prop-T1: $(\varphi \rightarrow (\psi \rightarrow \tau)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \tau))$

Prop-T2: $(\alpha 1 \rightarrow \beta) \rightarrow ((\alpha 2 \rightarrow \beta) \rightarrow ((\alpha 1 \vee \alpha 2) \rightarrow \beta))$

Prop-T3: $(\alpha \rightarrow \beta) \rightarrow (\neg \beta \rightarrow \neg \alpha)$

Prop-T4a: $\neg$ false

Prop-T4b: $\neg \alpha \vee \alpha$

Prop-T4c: $\alpha \rightarrow (\neg \alpha \rightarrow \tau)$

Prop-T5: $((\alpha \rightarrow \beta) \rightarrow (\neg\alpha \vee \beta)) \ \& \ ((\neg\alpha \vee \beta) \rightarrow (\alpha \rightarrow \beta))$

Prop-T6: $((\neg\alpha \vee \neg\beta) \rightarrow \neg(\alpha \ \& \ \beta)) \ \& \ (\neg(\alpha \ \& \ \beta) \rightarrow (\neg\alpha \vee \neg\beta))$

Prop-T7a: $\alpha \rightarrow (\alpha \vee \beta)$

Prop-T7b: $(\alpha \rightarrow \beta) \rightarrow ((\alpha \vee \varphi) \rightarrow (\beta \vee \varphi))$

We can prove the following formulas:

M-T1a: $\mathbf{K}_n$, $(\varphi \rightarrow \psi)$ |- $K_i\, \varphi \rightarrow K_i\, \psi$ (see the lecture)

M-T1b: Let φ ,ψ be two equiv. fomulas (ie. $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ is a tautology and this is denoted as $\varphi \equiv \psi$) then $\mathbf{K}_n$ |- $K_i\, \varphi \equiv K_i\, \psi$

M-T2a: $\mathbf{K}_n$ |- $K_i\, (\alpha \,\&\, \beta) \rightarrow K_i\, \alpha$ (see the lecture)

M-T2b: $\mathbf{K}_n$ |- $K_i\, (\alpha \,\&\, \beta) \equiv K_i\, \alpha \,\&\, K_i\, \beta$ (see the lecture)

M-T3: $(\mathbf{K}_n + \mathbf{A3})$ |- $K_i\, (\neg \text{false})$ **A6**

M-T4a: $(\mathbf{K2} + \mathbf{A3})$ |- $\neg K_i\, \alpha \vee \neg K_i\, \neg K_i\, \alpha$

M-T4b: $(\mathbf{K2} + \mathbf{A3})$ |- $\neg K_i\, (\alpha \,\&\, \neg K_i\, \alpha)$

M-T5: $(\mathbf{K}_n + \mathbf{A6})$ |- $\neg (K_i\, \alpha \,\&\, K_i \neg \alpha)$

M-T6: $\mathbf{K}_n$ |- $(K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i\, p \wedge K_i\, (\neg K_i\, p))$

M-T7: In $(\mathbf{K}_n + \mathbf{A3})$, it is impossible to prove the formula $K_i \neg (p \rightarrow K_i p)$

We know that $\mathbf{K}_n, (\alpha \rightarrow \beta) \vdash K_i \alpha \rightarrow K_i \beta$

$\mathbf{K}_n \vdash K_i(\alpha \,\&\, \beta) \equiv K_i \alpha \,\&\, K_i \beta$

Can one prove $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \equiv K_i \alpha \vee K_i \beta$?

Certainly not! Why?

Step 1. We can prove that $\mathbf{K}_n \vdash K_i \alpha \rightarrow K_i(\alpha \vee \beta)$ and $\mathbf{K}_n \vdash K_i \beta \rightarrow K_i(\alpha \vee \beta)$. These too formulas can be combined using following prop. tautology

Prop-T2 :$(\alpha 1 \rightarrow \beta) \rightarrow ((\alpha 2 \rightarrow \beta) \rightarrow ((\alpha 1 \vee \alpha 2) \rightarrow \beta))$

and we get $\mathbf{K}_n \vdash (K_i \alpha \vee K_i \beta) \rightarrow K_i(\alpha \vee \beta)$

Step 2. To show that the inverse implication $\mathbf{K}_n \vdash K_i(\alpha \vee \beta) \rightarrow (K_i \alpha \vee K_i \beta)$ **is not valid** it is enough to construct simple Kripke structure with 3 states, where this formula is not valid.

Some proofs:

M-T3: c1) K2, T(A3) |- $\neg K_i \mathit{false}$

1. $K_i \mathit{false} \rightarrow \mathit{false}$ [A3]
2. $\neg \mathit{false} \rightarrow (\neg K_i \mathit{false})$ [eq.transf.of 1 based on Prop-T3]
3. $\neg \mathit{false}$ [Prop-T4]
4. $\neg K_i \mathit{false}$ [Modus Ponens: 2,3]

M-T4a: K2, T |- $\neg K_i \alpha \vee \neg K_i \neg K_i \alpha$

[eq.transf. based on Prop-T5 "replacement of $\rightarrow$ by $\vee$ " in the axiom T $K_i (\neg K_i \alpha) \rightarrow \neg K_i \alpha$]

M-T4b: K2, T |- $\neg K_i (\alpha \,\&\, \neg K_i \alpha)$

1. $K_i \neg K_i \alpha \rightarrow \neg K_i \alpha$ [A3, the axiom T]
2. $\neg K_i \neg K_i \alpha \vee \neg K_i \alpha$ [eq.transf.of 1 based on Prop-T5]
3. $\neg (K_i \neg K_i \alpha \,\&\, K_i \alpha)$ [eq.transf.of 2 based on Prop-T6]
4. $\neg K_i (\neg K_i \alpha \,\&\, \alpha)$ [eq.transf.of 3 based on Mol-T2b], *viz* a2

Proof of M-T5: (K_n + A6) |- $\neg(K_i \alpha \wedge K_i \neg \alpha)$.

1. $(K_i \alpha \wedge K_i \neg \alpha) \equiv K_i(\alpha \wedge \neg \alpha)$ **M-T2b:**
2. $\neg K_i(\mathit{false})$ [A6]
3. $\mathit{false} \equiv (\alpha \wedge \neg \alpha)$ [tautology: A1]
4. $K_i\ \mathit{false} \equiv K_i(\alpha \wedge \neg \alpha)$ **M-T1b**
5. $\neg K_i(\alpha \wedge \neg \alpha)$ [MP for 2 and 4]
6. $\neg(K_i \alpha \wedge K_i \neg \alpha) \equiv \neg K_i(\alpha \wedge \neg \alpha)$ [taut.transcript of l. 1]
7. $\neg(K_i \alpha \wedge K_i \neg \alpha)$ [MP for 6 and 5]

M-T6: $\mathbf{K}_n \vdash (K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \,\&\, K_i (\neg K_i p))$

Crucial steps of the proof

a) $\neg (p \rightarrow K_i p) \equiv (p \wedge \neg K_i p)$ **[Taut]**

b) $K_i \neg (p \rightarrow K_i p) \equiv K_i (p \,\&\, \neg K_i p)$ **[M-T1b]**

c) $K_i (p \,\&\, \neg K_i p) \rightarrow (K_i p \,\&\, K_i (\neg K_i p))$ **[M-T2b]**

d) $(K_i \neg (p \rightarrow K_i p)) \rightarrow (K_i p \,\&\, K_i (\neg K_i p))$ [lines b,c and transitivity of "$\rightarrow$"]

Mod_T7: The formula $K_i \neg (p \rightarrow K_i p)$ is not provable in (K_n + Ax3).

Proof. Suppose $K_i \neg (p \rightarrow K_i p)$ is provable. We could apply modus ponens to **Mod_T6** and $K_i \neg (p \rightarrow K_i p)$ to prove the formula $K_i p \,\&\, K_i (\neg K_i p)$. Let us assume that $K_i p \,\&\, K_i (\neg K_i p)$ holds:

1. $K_i p \,\&\, K_i (\neg K_i p)$	[assumption]
2. $K_i p$	[1 and property of conjunction $(\alpha \,\&\, \beta) \rightarrow \alpha$]
3. $K_i (\neg K_i p)$	[1 and property of conjunction $(\alpha \,\&\, \beta) \rightarrow \beta$]
4. $K_i (\neg K_i p) \rightarrow \neg K_i p$	**[Ax3]**
5. $\neg K_i p$	[Modus ponens: 3,4]
6. ***false***	[definition of ***false*** and the lines 2 and 5]

But the axiom system (K_n + Ax3) has a model – that is why it cannot be contradictory and the formula ***false*** cannot be provable there. What is wrong with our proof? It must be based a wrong assumption! There were assumed $K_i \neg (p \rightarrow K_i p)$ and its direct consequence (obtained through **MP** and **Mod_T6)** the formula $K_i p \,\&\, K_i (\neg K_i p)$.

The original assumption „the formula $K_i \neg (p \rightarrow K_i p)$ is provable" must be wrong!.

Mod_T8a: K_n , Ax3, Ax 5, $K_i(p \to K_i p) \vdash K_i p \vee K_i \neg p$

1.	$K_i(p \to K_i p)$	assumption
2.	$K_i p \vee \neg K_i p$	Prop_T4b $\alpha \vee \neg \alpha$
3.	$K_i p \to K_i p \vee K_i \neg p$	Prop-T7a $\alpha \to (\alpha \vee \beta)$
4.	$K_i(p \to K_i p) \to (p \to K_i p)$	Ax 3
5.	$p \to K_i p$	Modus Ponens for 4 and 1
6.	$(p \to K_i p) \to (\neg K_i p \to \neg p)$	Prop_T3
7.	$\neg K_i p \to \neg p$	Modus Ponens for 6 and 5
8.	$\neg K_i p \to K_i \neg K_i p$	Ax 5
9.	$K_i \neg K_i p \to K_i \neg p$	M-T1 for the line 7
10.	$\neg K_i p \to K_i \neg p$	transitivity of "$\to$" for lines 8 and 9
11.	$(\alpha \to \beta) \to (\gamma \vee \alpha \to \gamma \vee \beta)$	Prop-T7b
12.	$(K_i p \vee \neg K_i p) \to (K_i p \vee K_i \neg p)$	Modus Ponens for lines 10,11 and $\gamma = K_i p$
13.	$(K_i p \vee K_i \neg p)$	Modus Ponens for lines 12 and 2

Mod_T8b: $\mathbf{K_n}$, **Ax 4** ,$(K_i p \vee K_i \neg p) \vdash K_i(p \rightarrow K_i p)$

1. $K_i p \rightarrow (p \rightarrow K_i p)$ propositional axiom
2. $K_i K_i p \rightarrow K_i(p \rightarrow K_i p)$ Mod_T1 for the line 1
3. $K_i p \rightarrow K_i K_i p$ **Ax4**
4. $K_i p \rightarrow K_i(p \rightarrow K_i p)$ transitivity of "$\rightarrow$" for the lines 3 a 2
5. $\neg p \rightarrow (p \rightarrow K_i p)$ Prop_4c
6. $K_i \neg p \rightarrow K_i(p \rightarrow K_i p)$ Mod_T1 for the line 5
7. $(\alpha 1 \rightarrow \beta) \rightarrow ((\alpha 2 \rightarrow \beta) \rightarrow ((\alpha 1 \vee \alpha 2) \rightarrow \beta))$ Prop-T2 for

   $\alpha 1 = K_i p$, $\alpha 2 = K_i \neg p$, $\beta = K_i(p \rightarrow K_i p)$
8. $(K_i p \vee K_i \neg p) \rightarrow K_i(p \rightarrow K_i p)$ Modus ponens applied first to 7 and 4 and later to this formula with 6
9. $(K_i p \vee K_i \neg p)$ assumption
10. $K_i(p \rightarrow K_i p)$ Modus ponens applied to 8 and 10