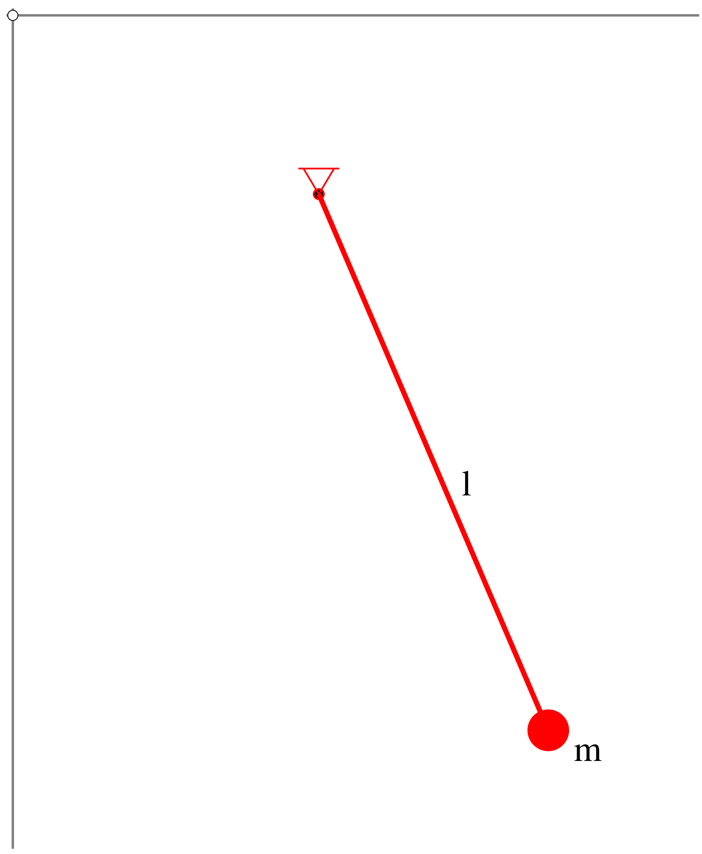
Pendullum Dynamics - Given



Prepare dynamic simulation model of the pendulum:

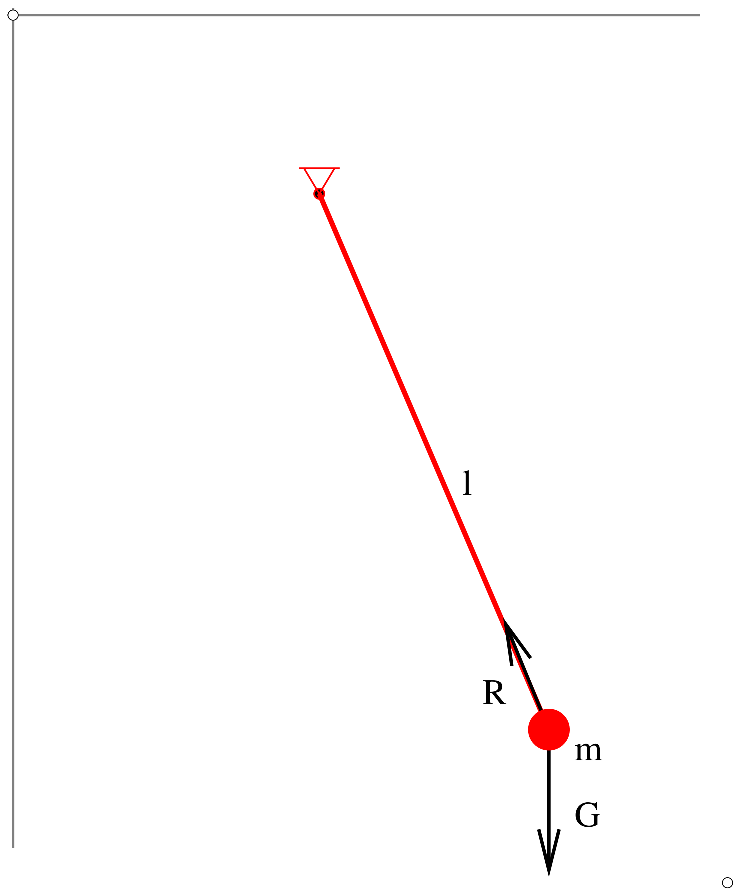
1. with the fixed hinge,
2. with the controlled hinge position.

- Make a free body diagram
- Describe the pendulum position in the defined coordinate system
- Write down Newton equations
- Arrange equations to a linear form with respect to angular accelerations and reaction in the rope
- Prepare program code in Matlab (i.e. solve the set of linear equations and integrate results)

The order of reasoning in this type of problems:

1. Physics of the system: Newton's Second Motion Law, gravity force.
2. Further constraints: kinematics, force along the rope.
3. Coordinate system installed.
4. How many degrees of freedom has the system, suitable parametrization: ϕ, ψ.
5. Express kinematics, force along the rope, gravity as function of parameters, substitute to the physics laws. What are unknowns (ϕ, ψ), parameters (m, l), inputs ($\vec{x}_p$).
6. Nature of the equations: set of nonlinear differential equation of the second order. Check the number of equations, number of unknowns.
7. Reformatting into solvable set: set of linear differential equations of the first degree.
8. Implementation in programming language: Matlab.

Pendulum Dynamics - Forces

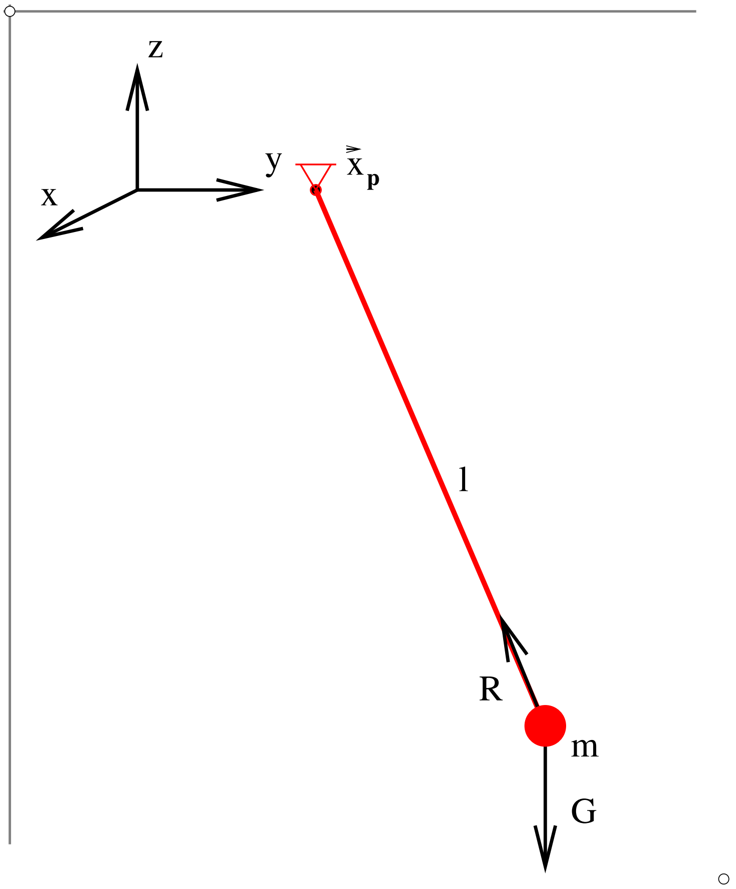


Newton's second law of motion:

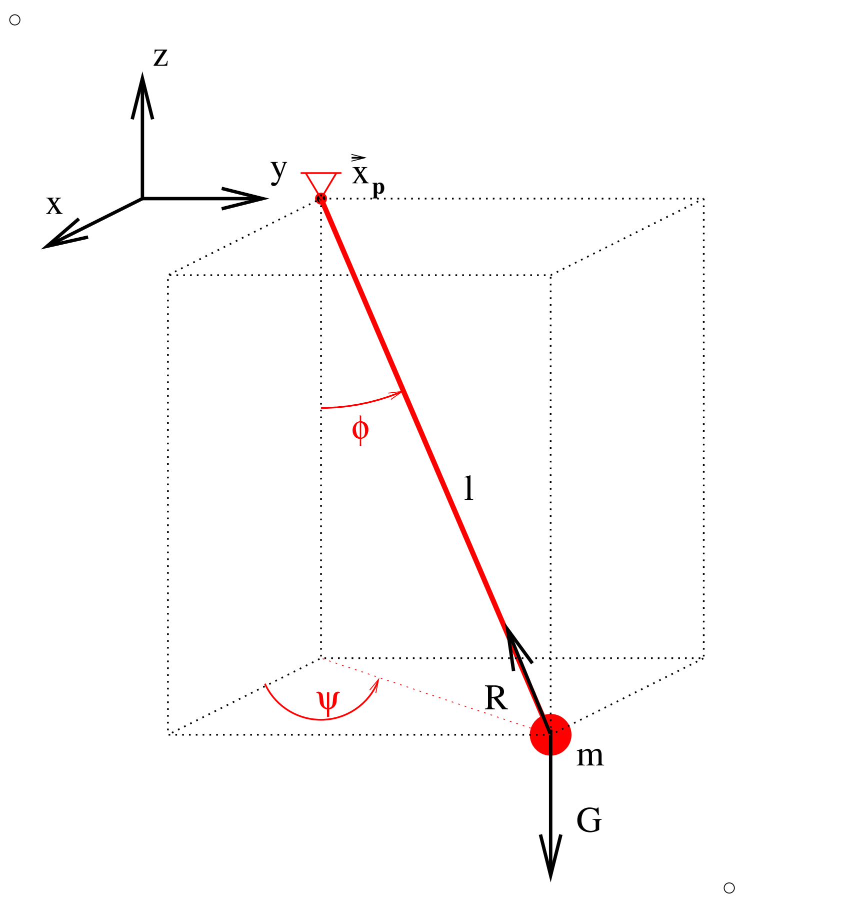
$$\vec{F} = m\vec{a} = m\ddot{\vec{x}} \quad (1)$$

$$\vec{F} = \vec{R} + \vec{G} \quad (2)$$

(3)


z
y
x
$\vec{x}_p$
l
R
m
G

Pendulum Dynamics - Parameterization



Kinematics (DKT):

$$
\begin{aligned}
x &= l \sin\phi \cos\psi + x_p && (4)\\
y &= l \sin\phi \sin\psi + y_p && (5)\\
z &= -l \cos\phi + z_p && (6)\\
& && (7)
\end{aligned}
$$

First and second derivatives of kinematics:

$$
\begin{aligned}
\dot{x} &= l \cos\phi \cos\psi \dot{\phi} - l \sin\phi \sin\psi \dot{\psi} + \dot{x}_p && (8)\\
\dot{y} &= l \cos\phi \sin\psi \dot{\phi} + l \sin\phi \cos\phi \dot{\psi} + \dot{y}_p && (9)\\
\dot{z} &= l \sin\phi \dot{\phi} && (10)\\
\ddot{x} &= l(-\sin\phi \cos\psi \dot{\phi}^2 - \cos\phi \sin\psi \dot{\phi}\dot{\psi} + \cos\phi \cos\psi \ddot{\phi} - \cos\phi \sin\psi \dot{\phi}\dot{\psi} - \sin\phi \cos\psi \dot{\psi}^2 - \sin\phi \sin\psi \ddot{\psi}) + \ddot{x}_p && (11)\\
\ddot{y} &= l(-\sin\phi \sin\psi \dot{\phi}^2 - \cos\phi \cos\psi \dot{\phi}\dot{\psi} + \cos\phi \sin\psi \ddot{\phi} + \cos\phi \cos\psi \dot{\phi}\dot{\psi} - \sin\phi \sin\psi \dot{\psi}^2 + \sin\phi \cos\psi \ddot{\psi}) + \ddot{y}_p && (12)\\
\ddot{z} &= l(\cos\phi \dot{\phi}^2 + \sin\phi \ddot{\phi}) + \ddot{z}_p && (13)\\
& && (14)
\end{aligned}
$$

R as a function of parameters:

$$
\begin{aligned}
R_x &= -R \sin\phi \cos\psi && (15)\\
R_y &= -R \sin\phi \sin\psi && (16)\\
R_z &= R \cos\phi && (17)\\
& && (18)
\end{aligned}
$$

Substitutions into the physics:

$$\begin{aligned} m\ddot{\vec{x}} &= \vec{R} + \vec{G} & (19) \\ m\ddot{x} &= R_x & (20) \\ m\ddot{y} &= R_y & (21) \\ m\ddot{z} &= R_z - mg & (22) \\ && (23) \end{aligned}$$

Final equations:

$$\begin{aligned} ml(\cos\phi\cos\psi\ddot{\phi} - \sin\phi\sin\psi\ddot{\psi}) + R\sin\phi\cos\psi &= ml(\sin\phi\cos\psi(\dot{\phi}^2 + \dot{\psi}^2) + 2\cos\phi\sin\psi\dot{\phi}\dot{\psi}) - m\ddot{x}_p & (24) \\ ml(\cos\phi\sin\psi\ddot{\phi} + \sin\phi\cos\psi\ddot{\psi}) + R\sin\phi\sin\psi &= ml(-\sin\phi\sin\psi(\dot{\phi}^2 + \dot{\psi}^2) - 2\cos\phi\cos\psi\dot{\phi}\dot{\psi}) - m\ddot{y}_p & (25) \\ ml\sin\phi\ddot{\phi} - R\cos\phi &= -ml\cos\phi\dot{\phi}^2 - m\ddot{z}_p - mg & (26) \\ && (27) \end{aligned}$$

Let us consider vector of unknowns:

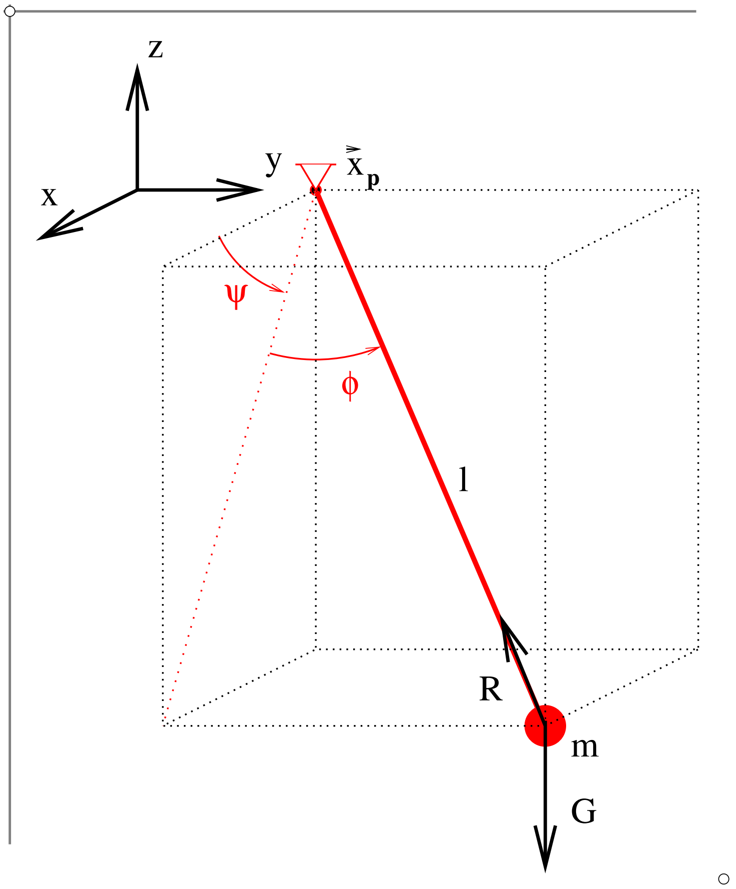
$$\vec{w} = \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \\ R \end{pmatrix}$$

Apparently the set of differential equations is linear in second derivatives so it could be rearranged into $\mathbf{A}\vec{w} = \vec{b}$, which could be further converted into $\vec{w} = \mathbf{A}^{-1}\vec{b}$.

$$\begin{pmatrix} ml\cos\phi\cos\psi & -ml\sin\phi\sin\psi & \sin\phi\cos\psi \\ ml\cos\phi\sin\psi & ml\sin\phi\cos\psi & \sin\phi\sin\psi \\ ml\sin\phi & 0 & -\cos\phi \end{pmatrix} \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \\ R \end{pmatrix} = \begin{pmatrix} ml(\sin\phi\cos\psi(\dot{\phi}^2 + \dot{\psi}^2) + 2\cos\phi\sin\psi\dot{\phi}\dot{\psi}) - m\ddot{x}_p \\ ml(-\sin\phi\sin\psi(\dot{\phi}^2 + \dot{\psi}^2) - 2\cos\phi\cos\psi\dot{\phi}\dot{\psi}) - m\ddot{y}_p \\ -ml\cos\phi\dot{\phi}^2 - m\ddot{z}_p - mg \end{pmatrix} \quad (28)$$

Matlab allows to solve numerically system of first order differential equations so the substitution which converts two second order differential into four first order differential equations.

$$\begin{aligned} z_1 &= \phi & (29) \\ z_2 &= \dot{\phi} & (30) \\ z_3 &= \psi & (31) \\ z_4 &= \dot{\psi} & (32) \\ && (33) \end{aligned}$$



Different parametrization solving problem with singularity in straight down position. The singularity is just moved to another position so the problem is less annoying but still present.