Material for lecture 2 – Basic methods of dynamical modeling - principle of virtual work, Lagrange equations

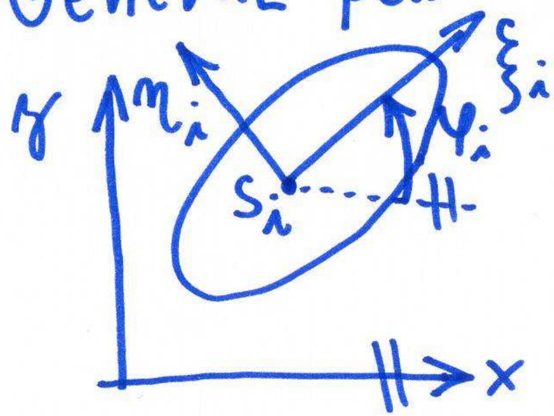
Prerequisites: - kinematics of mechanisms
- kinetic energy of moving bodies
- linear algebra methods (SVD, cond)
- basic methods of numerical solution of ordinary DE

Literature: V. Stejskal, M. Valášek: Kinematics and Dynamics of Machinery, Marcel Dekker 1996, New York.

$$E_K = \sum_i E_{Ki} \qquad i\text{-th body}$$

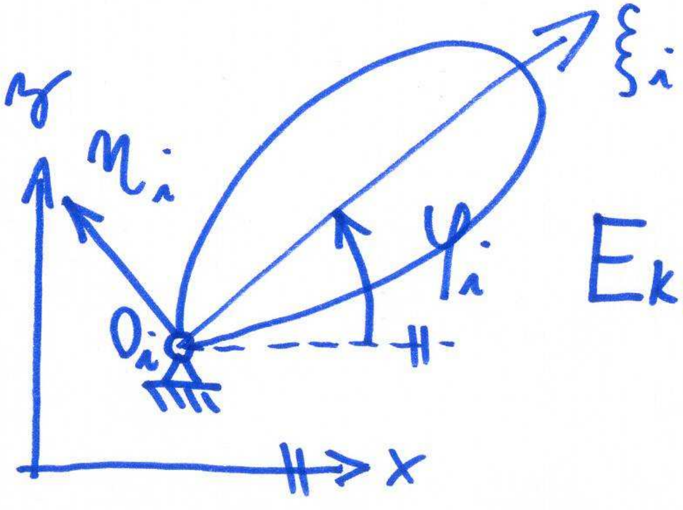
$$E_{Ki} = \frac{1}{2} m_i |\vec{v}_{Si}|^2 + \frac{1}{2} \vec{\omega}_i^T \underline{I}_{Si} \vec{\omega}_i$$

General planar motion



$$E_{Ki} = \frac{1}{2} m_i \left(\dot{x}_{Si}^2 + \dot{y}_{Si}^2\right) + \frac{1}{2} I_{Si} \dot{\varphi}_i^2$$

Revolute motion



$$E_{Ki} = \frac{1}{2} I_{Oi} \dot{\varphi}_i^2$$

Lagrange's equations of second type

n degrees of freedom, $i = 1, 2, \dots, n$

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{q}_i}\right) - \frac{\partial E_K}{\partial q_i} = Q_i$$

q_i ... independent coordinates

$$\sum_i Q_i \dot{q}_i = \sum_l \vec{F_l} \cdot \underline{v_{F_l}} + \sum_r M_r \omega_r$$

Problem: $E_K(\vec{q_k}, \vec{\dot{q}_k})$ very complex

Lagrange's equation of mixed type

n ... degrees of freedom, $i = 1, 2, \dots, n$

m coordinates, $j = 1, 2, \dots, m$

s_j

$r = m - n$ constraints, $k = 1, 2, \dots, r$

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{s}_j}\right) - \frac{\partial E_K}{\partial s_j} = \sum_{k=1}^{r} \frac{\partial f_k}{\partial s_j} \lambda_k$$

λ_k ... Lagrange's multipliers

Advantage: $E_K(\underrightarrow{s}, \underrightarrow{\dot{s}})$ simplified

Problem: Not in state-space form

Lagrange's equations of mixed type - matrix form

$$\underline{M}(s)\,\underrightarrow{\ddot{s}} = \underline{\Phi}^{T}(s)\,\underrightarrow{\lambda} + \underrightarrow{\mu}(s,\dot{s})$$

$$\underline{f}(\underrightarrow{s}) = \underrightarrow{0} \quad \text{... constraints}$$

$$\underline{\Phi}(s)\,\underrightarrow{\dot{s}} = \underrightarrow{0}$$

$$\underline{\Phi}(s)\,\underrightarrow{\ddot{s}} + \underline{\dot{\Phi}}(s)\,\underrightarrow{\dot{s}} = \underrightarrow{0}$$

$$\begin{bmatrix} \underline{M}(s) & \underline{\Phi}^{T}(s) \\ \underline{\Phi}(s) & \underline{0} \end{bmatrix} \begin{bmatrix} \underrightarrow{\ddot{s}} \\ -\underrightarrow{\lambda} \end{bmatrix} = \begin{bmatrix} \underrightarrow{\mu}(\underline{s},\underline{\dot{s}}) \\ -\underline{\dot{\Phi}}(s)\,\underrightarrow{\dot{s}} \end{bmatrix}$$

Numerical transformation to independent coordinates

$$\vec{f}(\vec{s}) = \vec{0}$$

$$\underline{\Phi}(\vec{s})\,\dot{\vec{s}} = \vec{0}$$

$$\vec{q} = \underline{B}\,\vec{s}\,, \quad \underline{B} = \text{const.}$$

$\rightarrow$ subjective choice

$\rightarrow$ objective choice (singular decomposit. of $\underline{\Phi}(\vec{s}_0)$)

$\underline{B}$ ... $n \times m$

$\underline{\Phi}(\vec{s}_0)$ ... $r \times m$

$$\begin{bmatrix} \underline{\Phi}(\vec{s}_0) \\ \underline{B} \end{bmatrix} \dot{\vec{s}} = \begin{bmatrix} \vec{0} \\ \dot{\vec{q}} \end{bmatrix}, \quad \dot{\vec{s}} = \begin{bmatrix} \underline{\Phi}(\vec{s}_0) \\ \underline{B} \end{bmatrix}^{-1} \begin{bmatrix} \vec{0} \\ \dot{\vec{q}} \end{bmatrix} = \underline{R}^{*}\vec{0} + \underline{R}\,\dot{\vec{q}}$$

$$\underline{\Phi}(\underrightarrow{s})\,\underrightarrow{\dot{s}} = \underrightarrow{0} \qquad \underbrace{\underline{\Phi}(\underrightarrow{s})\,\underline{R}}_{\underline{0}}\,\underrightarrow{\dot{q}} = \underrightarrow{0}$$

independent velocities

$$\underline{M}\,\underrightarrow{\ddot{s}} = \underline{\Phi}^T\underrightarrow{\lambda} + \underrightarrow{f} \qquad \underrightarrow{\dot{s}} = \underline{R}\,\underrightarrow{\dot{q}}\,,\quad \underrightarrow{\ddot{s}} = \underline{R}\,\underrightarrow{\ddot{q}} + \underline{\dot{R}}\,\underrightarrow{\dot{q}}$$

$$\underline{M}\,\underline{R}\,\underrightarrow{\ddot{q}} = \underline{\Phi}^T\underrightarrow{\lambda} - \underline{M}\,\underline{\dot{R}}\,\underrightarrow{\dot{q}} + \underrightarrow{f}$$

$$\underbrace{\underline{R}^T\underline{M}\,\underline{R}}_{\underline{M_q}}\,\underrightarrow{\ddot{q}} = \underbrace{\underline{R}^T\underline{\Phi}^T}_{\underline{0}^T}\underrightarrow{\lambda} - \underline{R}^T\underline{M}\,\underline{\dot{R}}\,\underrightarrow{\dot{q}} + \underline{R}^T\underrightarrow{f}$$

$$\underline{\dot{R}}\,\underrightarrow{\dot{q}} = \underrightarrow{\ddot{s}}\left(\underrightarrow{\ddot{q}} = \underrightarrow{0},\ \underrightarrow{\dot{q}},\ \underrightarrow{q}\right)$$

Objective choice of independent coordinates

$$\underline{\Phi}(\vec{s}_0) = \underline{U}\,\underline{S}\,\underline{V}^T \quad (\underline{U}^T\underline{U} = \underline{I},\ \underline{V}^T\underline{V} = \underline{I})$$

$$\underline{\Phi}\,\dot{\vec{s}} = \vec{0} \quad \cdots \quad \underline{U}\,\underline{S}\,\underline{V}^T\dot{\vec{s}} = \vec{0}$$

$$\underline{U}^T /$$

$$\underline{S}\,\underbrace{\underline{V}^T\dot{\vec{s}}}_{\dot{\vec{c}}} = \vec{0} \qquad \begin{array}{c} 1 \\ \vdots \\ r \end{array}\begin{bmatrix} p_1 & & & \vdots & \\ & \ddots & & \vdots & \underline{0} \\ & & p_r & \vdots & \end{bmatrix}\begin{bmatrix} \dot{\vec{c}} \end{bmatrix}$$

$c_{r+1}, \dots, c_m$ ··· independent coordinates