Lecture 1 – Basic methods of dynamical modelling (Newton-Euler equations)

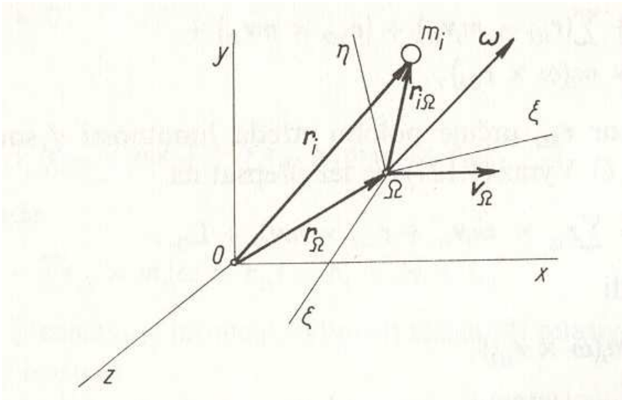
The following several pages summarize the sequence of the most important formulas necessary for the preparation of Newton-Euler equations for dynamic modelling of the system of rigid bodies.

The presented text isn‘t the substitution of the lecture (!!!), it is only the basic material for the lecture.

<u>Prerequisities:</u>

- Basic laws of Newton mechanics
- Kinematics of planar and spatial mechanisms
- Free body diagram in statics of mechanisms

Rigid body - limit state of system of many small mass particles



System of mass particles

Moment of momentum

$$\mathbf{L}_O = \mathbf{r}_\Omega \times \mathbf{H} + (\mathbf{r}_{S\Omega} \times m\mathbf{v}_\Omega) + \sum \mathbf{r}_{i\Omega} \times m_i(\omega \times \mathbf{r}_{i\Omega})\,.$$

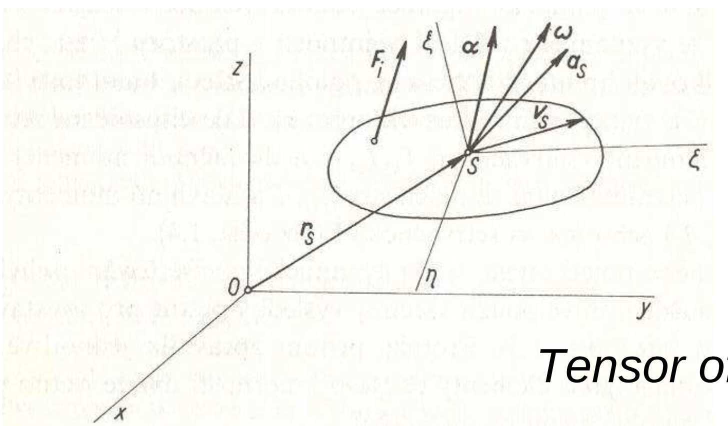
$$\mathbf{L}_O = \mathbf{r}_S \times \mathbf{H} + \sum \mathbf{r}_{iS} \times m_i(\omega \times \mathbf{r}_{iS}) = \mathbf{r}_S \times \mathbf{H} + \mathbf{L}_{Sr}$$

Rigid body

Moment of momentum

$$\mathbf{H} = \int_m \mathbf{v}\,\mathrm{d}m = m\mathbf{v}_S \qquad \mathbf{L}_O = \int_m (\mathbf{r} \times \mathbf{v})\,\mathrm{d}m$$

$$\mathbf{L}_O = \mathbf{r}_S \times \mathbf{H} + \mathbf{L}_S$$



$$\mathbf{L}_S = \int_m [\varrho \times (\omega \times \varrho)]\,\mathrm{d}m = \int_m [\omega\varrho^2 - \varrho(\omega\varrho)]\,\mathrm{d}m =$$

$$= \omega \int_m (\xi^2 + \eta^2 + \zeta^2)\,\mathrm{d}m - \int_m \varrho(\omega_\xi \xi + \omega_\eta \eta + \omega_\zeta \zeta)\,\mathrm{d}m$$

Tensor of inertia

$$L_{S\xi} = I_\xi \omega_\xi - D_{\xi\eta}\omega_\eta - D_{\xi\zeta}\omega_\zeta$$
$$L_{S\eta} = I_\eta \omega_\eta - D_{\eta\zeta}\omega_\zeta - D_{\eta\xi}\omega_\xi$$
$$L_{S\zeta} = I_\zeta \omega_\zeta - D_{\zeta\xi}\omega_\xi - D_{\zeta\eta}\omega_\eta$$

$$\mathbf{L}_S = \mathbf{I}_S \boldsymbol{\omega}$$

$$L_{S\xi} = I_\xi \omega_\xi$$
$$L_{S\eta} = I_\eta \omega_\eta$$
$$L_{S\zeta} = I_\zeta \omega_\zeta$$

S … body centroid

Moment of momentum with respect to general point

$$L_{\Omega\xi} = I_\xi\omega_\xi - D_{\xi\eta}\omega_\eta - D_{\xi\zeta}\omega_\zeta$$
$$L_{\Omega\eta} = I_\eta\omega_\eta - D_{\eta\zeta}\omega_\zeta - D_{\eta\xi}\omega_\xi$$
$$L_{\Omega\zeta} = I_\zeta\omega_\zeta - D_{\zeta\xi}\omega_\xi - D_{\zeta\eta}\omega_\eta$$

$$\mathbf{L}_O = (\mathbf{r}_\Omega \times m\mathbf{v}_S) + (\mathbf{r}_{S\Omega} \times m\mathbf{v}_\Omega) + \mathbf{L}_{\Omega\mathrm{r}}$$

Centroid motion equation

$$m\mathbf{a}_S = \sum \mathbf{F}_i$$

$$\mathbf{H} - \mathbf{H}_0 = \sum \int_0^t \mathbf{F}_i \, \mathrm{d}t$$

Moment of momentum equation

$$\frac{\mathrm{d}\mathbf{L}_O}{\mathrm{d}t} = \sum \mathbf{M}_{iO}$$

$$\frac{\mathrm{d}\mathbf{L}_O}{\mathrm{d}t} = \mathbf{r}_S \times m\mathbf{a}_S + \frac{\mathrm{d}\mathbf{L}_S}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}\mathbf{L}_S}{\mathrm{d}t} = \frac{\mathrm{d}'\mathbf{L}_S}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{L}_S,$$

For principal axes of inertia

$$\left(\frac{\mathrm{d}\mathbf{L}_S}{\mathrm{d}t}\right)_\xi = \frac{\mathrm{d}L_{S\xi}}{\mathrm{d}t} + \omega_\eta L_{S\zeta} - \omega_\zeta L_{S\eta} =$$
$$= I_\xi\dot{\omega}_\xi - D_{\zeta\eta}\dot{\omega}_\eta - D_{\xi\zeta}\dot{\omega}_\zeta + \omega_\eta(I_\zeta\omega_\zeta - D_{\zeta\xi}\omega_\xi - D_{\zeta\eta}\omega_\eta) -$$
$$- \omega_\zeta(I_\eta\omega_\eta - D_{\eta\zeta}\omega_\zeta - D_{\eta\xi}\omega_\xi),$$

$$\left(\frac{\mathrm{d}\mathbf{L}_S}{\mathrm{d}t}\right)_\xi = I_\xi\dot{\omega}_\xi + \omega_\eta\omega_\zeta(I_\zeta - I_\eta)$$

Euler equations

Simplified version for principal axes of inertia and reference point in centroid

$$I_\xi \dot{\omega}_\xi + \omega_\eta \omega_\zeta (I_\zeta - I_\eta) = \sum M_{i\xi}$$
$$I_\eta \dot{\omega}_\eta + \omega_\zeta \omega_\xi (I_\xi - I_\zeta) = \sum M_{i\eta}$$
$$I_\zeta \dot{\omega}_\zeta + \omega_\xi \omega_\eta (I_\eta - I_\xi) = \sum M_{i\zeta}$$

General version for arbitrary axes and reference point out of centroid

$$m(a_{\Omega\zeta}\eta_S - a_{\Omega\eta}\zeta_S) + I_\xi \dot{\omega}_\xi + (I_\zeta - I_\eta)\,\omega_\eta\omega_\zeta +$$
$$+ D_{\xi\eta}(\omega_\xi\omega_\zeta - \dot{\omega}_\eta) - D_{\xi\zeta}(\omega_\xi\omega_\eta + \dot{\omega}_\zeta) + D_{\eta\zeta}(\omega_\zeta^2 - \omega_\eta^2) = \sum M_{i\xi}$$
$$m(a_{\Omega\xi}\zeta_S - a_{\Omega\zeta}\xi_S) + I_\eta \dot{\omega}_\eta + (I_\xi - I_\zeta)\,\omega_\zeta\omega_\xi +$$
$$+ D_{\eta\zeta}(\omega_\eta\omega_\xi - \dot{\omega}_\zeta) - D_{\eta\xi}(\omega_\eta\omega_\zeta + \dot{\omega}_\xi) + D_{\zeta\xi}(\omega_\xi^2 - \omega_\zeta^2) = \sum M_{i\eta}$$
$$m(a_{\Omega\eta}\xi_S - a_{\Omega\xi}\eta_S) + I_\zeta \dot{\omega}_\zeta + (I_\eta - I_\xi)\,\omega_\xi\omega_\eta +$$
$$+ D_{\zeta\xi}(\omega_\zeta\omega_\eta - \dot{\omega}_\xi) - D_{\zeta\eta}(\omega_\zeta\omega_\xi + \dot{\omega}_\eta) + D_{\xi\eta}(\omega_\eta^2 - \omega_\xi^2) = \sum M_{i\zeta}$$

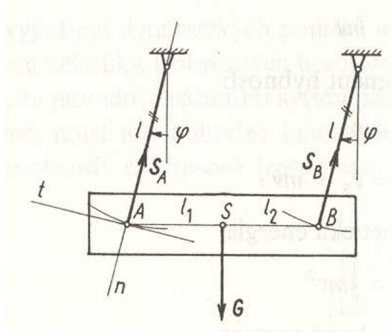
Newton equations

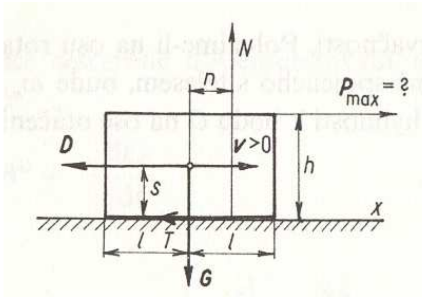
$$m a_{S\xi} = \sum F_{i\xi}$$
$$m a_{S\eta} = \sum F_{i\eta}$$
$$m a_{S\zeta} = \sum F_{i\zeta}$$

Newton-Euler equations for simplified motions

Translational motion

$$
\begin{aligned}
ma_x &= \sum F_{ix} ,\\
ma_y &= \sum F_{iy} ,\\
ma_z &= \sum F_{iz} ,\\
m(y_S a_z - z_S a_y) &= \sum M_{ix} ,\\
m(z_S a_x - x_S a_z) &= \sum M_{iy} ,\\
m(x_S a_y - y_S a_x) &= \sum M_{iz} ,
\end{aligned}
$$

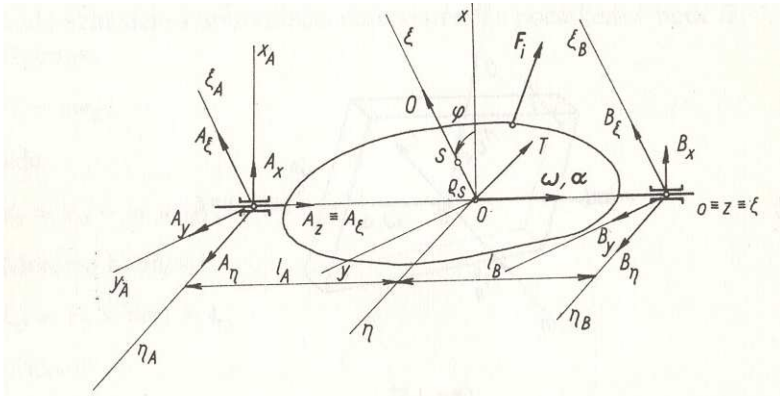




Revolute motion

$$
\begin{aligned}
m\varrho_S \ddot{\varphi} &= \sum F_{it} ,\\
m\varrho_S \dot{\varphi}^2 &= \sum F_{in} ,\\
0 &= \sum F_{ib}
\end{aligned}
$$

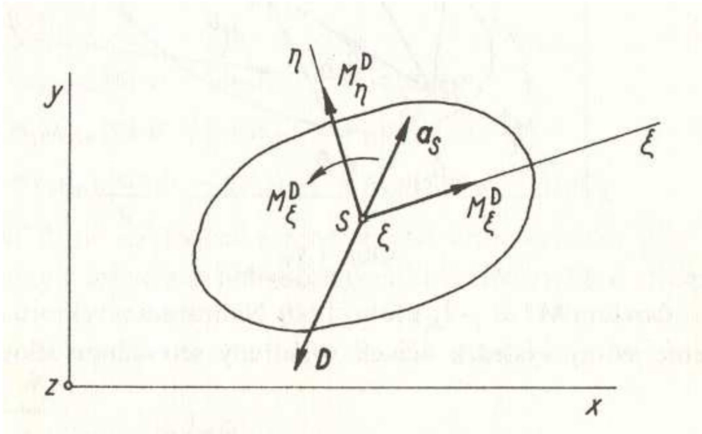
$$
\begin{aligned}
-D_{\xi\zeta}\alpha + D_{\eta\zeta}\omega^2 &= \sum M_{i\xi} ,\\
-D_{\eta\zeta}\alpha - D_{\xi\zeta}\omega^2 &= \sum M_{i\eta} ,\\
I_{\zeta}\alpha &= \sum M_{i\zeta} .
\end{aligned}
$$



General planar motion

$$
\begin{aligned}
ma_{S\xi} &= \sum F_{i\xi}\,, \\
ma_{S\eta} &= \sum F_{i\eta}\,, \\
0 &= \sum F_{i\zeta}\,.
\end{aligned}
$$

$$
\begin{aligned}
-D_{\xi\zeta}\alpha + D_{\eta\zeta}\omega^2 &= \sum M_{i\xi}\,, \\
-D_{\eta\zeta}\alpha - D_{\xi\zeta}\omega^2 &= \sum M_{i\eta}\,, \\
I_\zeta\alpha &= \sum M_{i\zeta}
\end{aligned}
$$



Spherical motion

Euler equations

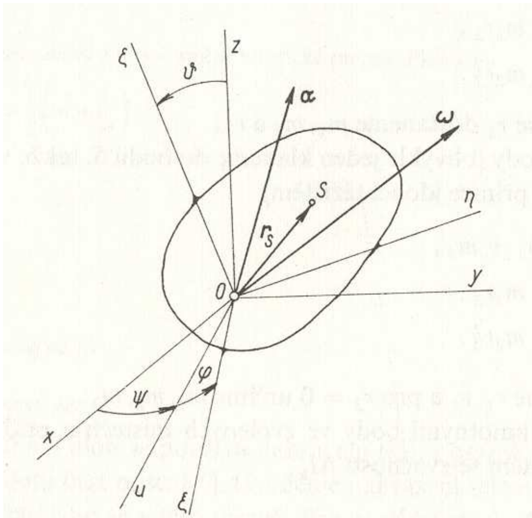
$$
I_\xi \frac{\mathrm{d}\omega_\xi}{\mathrm{d}t} + (I_\zeta - I_\eta)\,\omega_\eta\omega_\zeta + D_{\xi\eta}\left(\omega_\xi\omega_\zeta - \frac{\mathrm{d}\omega_\eta}{\mathrm{d}t}\right) - D_{\xi\zeta}\left(\omega_\xi\omega_\eta + \frac{\mathrm{d}\omega_\zeta}{\mathrm{d}t}\right) + D_{\eta\zeta}(\omega_\zeta^2 - \omega_\eta^2) = \sum M_{i\xi}\,,
$$

$$
I_\eta \frac{\mathrm{d}\omega_\eta}{\mathrm{d}t} + (I_\xi - I_\zeta)\,\omega_\zeta\omega_\xi + D_{\eta\zeta}\left(\omega_\eta\omega_\xi - \frac{\mathrm{d}\omega_\zeta}{\mathrm{d}t}\right) - D_{\eta\xi}\left(\omega_\eta\omega_\zeta + \frac{\mathrm{d}\omega_\xi}{\mathrm{d}t}\right) + D_{\zeta\xi}(\omega_\xi^2 - \omega_\zeta^2) = \sum M_{i\eta}\,,
$$

$$
I_\zeta \frac{\mathrm{d}\omega_\zeta}{\mathrm{d}t} + (I_\eta - I_\xi)\,\omega_\xi\omega_\eta + D_{\zeta\xi}\left(\omega_\zeta\omega_\eta - \frac{\mathrm{d}\omega_\xi}{\mathrm{d}t}\right) - D_{\zeta\eta}\left(\omega_\zeta\omega_\xi + \frac{\mathrm{d}\omega_\eta}{\mathrm{d}t}\right) + D_{\xi\eta}(\omega_\eta^2 - \omega_\xi^2) = \sum M i_\zeta\,.
$$

Newton equations

$$
\begin{aligned}
m[\dot\omega_\eta\zeta_S - \dot\omega_\zeta\eta_S + \omega_\xi(\omega_\eta\eta_S + \omega_\zeta\zeta_S) - (\omega_\eta^2 + \omega_\zeta^2)\,\xi_S] &= \sum F_{i\xi}\,, \\
m[\dot\omega_\zeta\xi_S - \dot\omega_\xi\zeta_S + \omega_\eta(\omega_\zeta\zeta_S + \omega_\xi\xi_S) - (\omega_\zeta^2 + \omega_\xi^2)\,\eta_S] &= \sum F_{i\eta}\,, \\
m[\dot\omega_\xi\eta_S - \dot\omega_\eta\xi_S + \omega_\zeta(\omega_\xi\xi_S + \omega_\eta\eta_S) - (\omega_\xi^2 + \omega_\eta^2)\,\zeta_S] &= \sum F_{i\zeta}\,.
\end{aligned}
$$



Euler equations for principal axes of inertia

$$
\begin{aligned}
I_\xi \frac{\mathrm{d}\omega_\xi}{\mathrm{d}t} + (I_\zeta - I_\eta)\,\omega_\eta\omega_\zeta &= \sum M_{i\xi}\,, \\
I_\eta \frac{\mathrm{d}\omega_\eta}{\mathrm{d}t} + (I_\xi - I_\zeta)\,\omega_\zeta\omega_\xi &= \sum M_{i\eta}\,, \\
I_\zeta \frac{\mathrm{d}\omega_\zeta}{\mathrm{d}t} + (I_\eta - I_\xi)\,\omega_\xi\omega_\eta &= \sum M_{i\zeta}\,,
\end{aligned}
$$