3D Computer Vision

Radim Šára Martin Matoušek

Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague

https://cw.fel.cvut.cz/wiki/courses/tdv/start

http://cmp.felk.cvut.cz
mailto:sara@cmp.felk.cvut.cz
phone ext. 7203

rev. November 21, 2023

Open Informatics Master's Course

Module VI

3D Structure and Camera Motion

covered by

[1] [H&Z] Secs: 9.5.3, 10.1, 10.2, 10.3, 12.1, 12.2, 12.4, 12.5, 18.1

[2] Triggs, B. et al. Bundle Adjustment—A Modern Synthesis. In *Proc ICCV Workshop on Vision Algorithms*. Springer-Verlag. pp. 298–372, 1999.

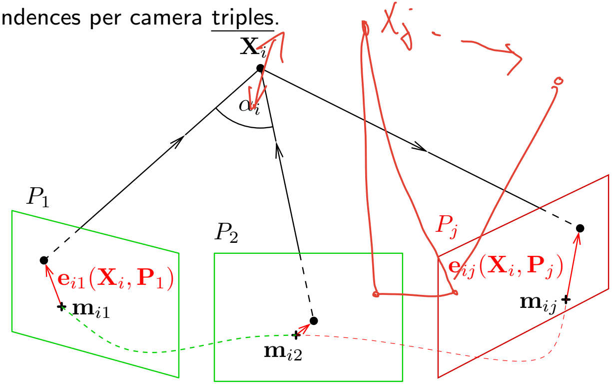
additional references

D. Martinec and T. Pajdla. Robust Rotation and Translation Estimation in Multiview Reconstruction. In *Proc CVPR*, 2007

M. I. A. Lourakis and A. A. Argyros. SBA: A Software Package for Generic Sparse Bundle Adjustment. ACM Trans Math Software 36(1):1–30, 2009.

▶Reconstructing Camera System by Gluing Camera Triples

Given: Calibration matrices $\mathbf{K}_j$ and tentative correspondences per camera triples.



Initialization

1. initialize camera cluster $\mathcal{C}$ with a pair P_1, P_2
2. find essential matrix $\mathbf{E}_{12}$ and matches M_{12} by the 5-point algorithm ← select scale →89
3. construct camera pair

$$\mathbf{P}_1 = \mathbf{K}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{P}_2 = \mathbf{K}_2 \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

4. triangulate $\{X_i\}$ per match from M_{12} →108
5. initialize point cloud $\mathcal{X}$ with $\{X_i\}$ satisfying chirality constraint $z_i > 0$ and apical angle constraint $|\alpha_i| > \alpha_T$

Attaching camera $P_j \notin \mathcal{C}$

1. select points $\mathcal{X}_j$ from $\mathcal{X}$ that have matches to P_j
2. estimate $\mathbf{P}_j$ using $\mathcal{X}_j$, RANSAC with the 3-pt alg. (P3P), projection errors $\mathbf{e}_{ij}$ in $\mathcal{X}_j$ →66
3. reconstruct 3D points from all tentative matches from P_j to all P_l, $l \neq k$ that are not in $\mathcal{X}$
4. filter them by the chirality and apical angle constraints and add them to $\mathcal{X}$
5. add P_j to $\mathcal{C}$
6. perform bundle adjustment on $\mathcal{X}$ and $\mathcal{C}$ coming next →142

►The Projective Reconstruction Theorem

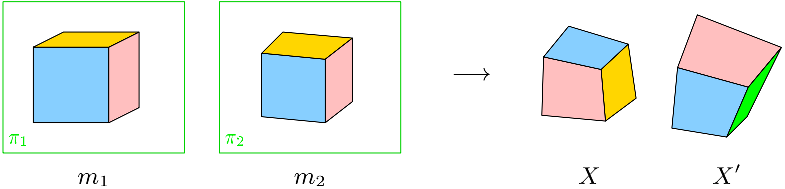
- We can run an analogical procedure when the cameras remain uncalibrated. But:

Observation: Unless $\mathbf{P}_j$ are constrained, then for any number of cameras $j = 1, \ldots, k$

$$\underline{\mathbf{m}}_{ij} \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \underbrace{\mathbf{P}_j \mathbf{H}^{-1}}_{\mathbf{P}'_j} \underbrace{\mathbf{H} \underline{\mathbf{X}}_i}_{\underline{\mathbf{X}}'_i} = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

- when $\mathbf{P}_i$ and $\underline{\mathbf{X}}$ are both determined from correspondences (including calibrations $\mathbf{K}_i$), they are given up to a common 3D homography $\mathbf{H}$

(translation, rotation, scale, shear, pure perspectivity)



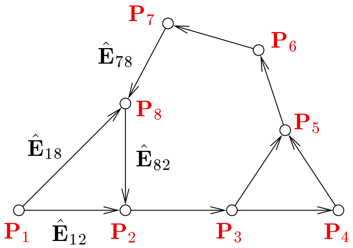
- when cameras are internally calibrated ($\mathbf{K}_j$ known) then $\mathbf{H}$ is restricted to a similarity since it must preserve the calibrations $\mathbf{K}_j$ [H&Z, Secs. 10.2, 10.3], [Longuet-Higgins 1981]
(translation, rotation, scale)

→137 for an indirect proof

▸Reconstructing Camera System from Pairs (Correspondence-Free)

Problem: Given a set of p decomposed pairwise essential matrices $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_\times \hat{\mathbf{R}}_{ij}$ and calibration matrices $\mathbf{K}_i$ reconstruct the camera system $\mathbf{P}_i$, $i = 1, \ldots, k$

→82 and →154 on representing $\mathbf{E}$



We construct calibrated camera pairs $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$ see (19)

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1}\hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1}\hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons i, j correspond to graph nodes — k nodes
- pairs ij correspond to graph edges — p edges

$\hat{\mathbf{P}}_{ij}$ are in different coordinate systems but these are related by similarities $\hat{\mathbf{P}}_{ij}\mathbf{H}_{ij} = \mathbf{P}_{ij}$ — $\mathbf{H}_{ij} \in \mathrm{SIM}(3)$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\in \mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\top & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \stackrel{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\in \mathbb{R}^{6,4}} \tag{31}$$

$\in$ SIM(3)

- (31) is a system of $24p$ eqs. in $7p + 6k$ unknowns — $24 = 6 \cdot 4,\ 7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij}),\ 6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each $\hat{\mathbf{P}}_i = (\mathbf{R}_i, \mathbf{t}_i)$ appears on the RHS as many times as is the degree of node $\mathbf{P}_i$ — eg. P_5 3×

►cont'd

Eq. (31) implies $\begin{bmatrix} \mathbf{R}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{R}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i \\ \mathbf{R}_j \end{bmatrix}$ and $\begin{bmatrix} \mathbf{t}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{t}_{ij} + s_{ij}\hat{\mathbf{t}}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{t}_j \end{bmatrix}$

- $\mathbf{R}_{ij}$ and $\mathbf{t}_{ij}$ can be eliminated:

$$\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j, \qquad \hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} = \mathbf{t}_j, \qquad s_{ij} > 0 \tag{32}$$

- note transformations that do not change these equations — assuming no error in $\hat{\mathbf{R}}_{ij}$

  1. $\mathbf{R}_i \mapsto \mathbf{R}_i\mathbf{R}$, 2. $\mathbf{t}_i \mapsto \sigma\,\mathbf{t}_i$ and $s_{ij} \mapsto \sigma s_{ij}$, 3. $\mathbf{t}_i \mapsto \mathbf{t}_i + \mathbf{R}_i\mathbf{t}$

- the global frame is fixed, e.g. by selecting

$$\mathbf{R}_1 = \mathbf{I}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \frac{1}{p}\sum_{i,j} s_{ij} = 1 \tag{33}$$

- rotation equations are decoupled from translation equations
- in principle, s_{ij} could correct the sign of $\hat{\mathbf{t}}_{ij}$ from essential matrix decomposition →82
  but $\mathbf{R}_i$ cannot correct the α sign in $\hat{\mathbf{R}}_{ij}$ ⇒ therefore make sure all points are in front of cameras and constrain $s_{ij} > 0$; →84

+ pairwise correspondences are sufficient
− suitable for well-distributed cameras only (dome-like configurations) otherwise intractable or numerically unstable

1. **Poor Man's Algorithm:**
   a) create a Minimum Spanning Tree of $\mathcal{G}$ from →136
   b) propagate rotations from $\mathbf{R}_1 = \mathbf{I}$ via $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$ from (32)

2. **Rich Man's Algorithm:**
   Consider $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$, $(i,j) \in E(\mathcal{G})$, where $\mathbf{R}$ are a 3×3 rotation matrices
   Errors per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\mathbf{e}_{ij}^c = \hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c, \qquad \text{for all } i,\ j,\ c$$

   Solve

$$\arg\min \sum_{(i,j)\in E(\mathcal{G})} \sum_{c=1}^{3} (\mathbf{e}_{ij}^c)^\top \mathbf{e}_{ij}^c \quad \text{s.t.} \quad (\mathbf{r}_i^k)^\top(\mathbf{r}_j^l) = \begin{cases} 1 & i = j \wedge k = l \\ 0 & i \neq j \wedge k = l \\ 0 & i = j \wedge k \neq l \end{cases}$$

   this is a quadratic programming problem

3. **SVD-Lover's Algorithm:**
   Ignore the constraints and project the solution onto rotation matrices see next

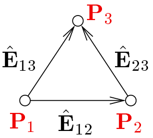
SVD Algorithm (cont'd)

Per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \qquad \text{for all } i,\ j\ ,c \tag{34}$$

- fix c and denote $\mathbf{r}^c = \left[\mathbf{r}_1^c, \mathbf{r}_2^c, \dots, \mathbf{r}_k^c\right]^\top$ c-th columns of all rotation matrices stacked; $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (34) becomes $\mathbf{D}\,\mathbf{r}^c = \mathbf{0}$ $\qquad \mathbf{D} \in \mathbb{R}^{3p,3k}$
- $3p$ equations for $3k$ unknowns $\to p \geq k$ $\qquad$ in a 1-connected graph we have to fix $\mathbf{r}_1^c = [1, 0, 0]$

Ex: $(k = p = 3)$



$\to$

$$\begin{aligned}
\hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c &= \mathbf{0} \\
\hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c &= \mathbf{0} \\
\hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c &= \mathbf{0}
\end{aligned}$$

$\to$

$$\mathbf{D}\,\mathbf{r}^c = \begin{bmatrix} \hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I} \\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^c \\ \mathbf{r}_2^c \\ \mathbf{r}_3^c \end{bmatrix} = \mathbf{0}$$

- must hold for any c

$\|D\,r^c\|^2 \to \min$

Idea: [Martinec & Pajdla CVPR 2007]

1. find the space of all $\mathbf{r}^c \in \mathbb{R}^{3k}$ that solve (34) $\qquad$ $\mathbf{D}$ is sparse, use `[V,E] = eigs(D'*D,3,0);` (Matlab)
2. choose 3 unit orthogonal vectors in this space $\qquad$ 3 smallest eigenvectors
3. find closest rotation matrices per cam. using SVD $\qquad$ because $\|\mathbf{r}^c\| = 1$ is necessary but insufficient
   $\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$, where $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$

- global world rotation is arbitrary

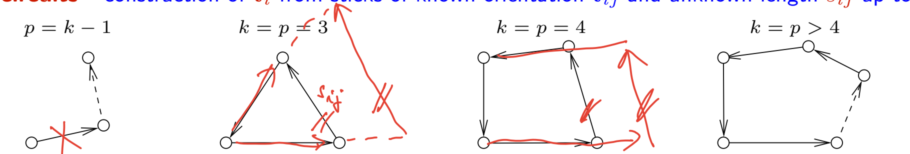
Finding The Translation Component in Eq. (32)

From (32) and (33): $0 < d \le 3$ – rank of camera center set, p – #pairs, k – #cameras

(a): $\hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}$, (b): $\sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}$, (c): $\sum_{i,j} s_{ij} = p$, $s_{ij} > 0$, $\mathbf{t}_i \in \mathbb{R}^d$

- in rank d: $\underbrace{d \cdot p}_{\text{(a)}} + \underbrace{d}_{\text{(b)}} + \underbrace{1}_{\text{(c)}}$ indep. eqns for $\underbrace{d \cdot k}_{\mathbf{t}_i} + \underbrace{p}_{s_{ij}}$ unknowns $\rightarrow p \ge \frac{d(k-1)-1}{d-1} \stackrel{\text{def}}{=} Q(d,k)$

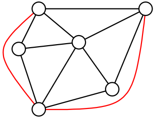
Ex: Chains, circuits construction of $\mathbf{t}_i$ from sticks of known orientation $\hat{\mathbf{t}}_{ij}$ and unknown length s_{ij} up to overall scale?



$k \le 2$ for any d $3 \ge d \ge 2$: non-collinear ok $3 \ge d \ge 3$: non-planar ok $3 \ge d \ge k-1$: impossible

- equations insufficient for chains, trees, or when $d = 1$ collinear cameras
- 3-connectivity implies sufficient equations for $d = 3$ cams. in general pos. in 3D
  - s-connected graph has $p \ge \lceil \frac{sk}{2} \rceil$ edges for $s \ge 2$, hence $p \ge \lceil \frac{3k}{2} \rceil \ge Q(3,k) = \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any k when $d = 2$ coplanar cams
  - since $p \ge \lceil 2k \rceil \ge Q(2,k) = 2k - 3$
  - maximal planar tringulated graphs have $p = 3k - 6$ and give a solution for $k \ge 3$

maximal planar triangulated graph example:



Linear equations in (32) and (33) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \qquad \mathbf{t} = \left[\mathbf{t}_1^\top, \mathbf{t}_2^\top, \ldots, \mathbf{t}_k^\top, s_{12}, \ldots, s_{ij}, \ldots\right]^\top$$

assuming measurement errors $\mathbf{D}\mathbf{t} = \boldsymbol{\epsilon}$ and $d = 3$, we have

$$\mathbf{t} \in \mathbb{R}^{3k+p}, \quad \mathbf{D} \in \mathbb{R}^{3p,3k+p} \quad \text{sparse}$$

and

$$\mathbf{t}^* = \arg\min_{\mathbf{t},\, s_{ij}>0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D}\, \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

Thank You