Invitation to Nonlinear Algebra

Mateusz Michałek

Bernd Sturmfels

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

E-mail address: `michalek@mis.mpg.de`

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

E-mail address: `bernd@mis.mpg.de`

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Preface

This book grew out of the lecture notes for a graduate course we taught during the Summer Semester 2018 at the Max-Planck Institute (MPI) for Mathematics in the Sciences in Leipzig, Germany. This was part of the general lecture series (called *Ringvorlesung* in German) which is offered biannually by the International Max-Planck Research School (IMPRS). The aim of our course was to introduce the theme of *Nonlinear Algebra*, which is also the name of the research group that started at MPI Leipzig in early 2017.

Linear algebra is the foundation of much of mathematics, particularly in applied mathematics. Numerical linear algebra is the basis of scientific computing, and its importance for the sciences and engineering can hardly be overestimated. The ubiquity of linear algebra masks the fairly recent growth of nonlinear models across the mathematical sciences. There has been a proliferation of methods based on systems of multivariate polynomial equations and inequalities. This is fueled by recent theoretical advances, efficient software, and an increased awareness of these tools. At the heart of this lies algebraic geometry, but there are links to many other branches, such as combinatorics, algebraic topology, commutative algebra, convex and discrete geometry, tensors and multilinear algebra, number theory, representation theory, and symbolic and numerical computation. Application areas include optimization, statistics, complexity theory, among many others.

Nonlinear algebra is not simply a rebranding of algebraic geometry. It is a recognition that a focus on computation and applications, and the theoretical needs that this requires, results in a body of inquiry that is complementary to the existing curriculum. The term nonlinear algebra is intended to capture these trends, and to be more friendly to applied scientists. A special research semester with that title, held in the fall of 2018 at the Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University, explored the theoretical and computational challenges that have arisen, and it charted the course for the future. This book supports this effort by offering a warm welcome to nonlinear algebra.

Our presentation is structured into 13 chapters, one for each week in a semester. Many of our chapters are rather ambitious in that they promise a first introduction to an area of mathematics that would normally be covered in a full-year course. But what we offer is really just an invitation. Our readers are hence encouraged to go further in their studies with other sources. We hope that students will find our presentation of use and that nonlinear algebra will encourage them to think critically and deeply, and to question the historic boundaries between "pure" and "applied" mathematics.

Mateusz Michałek and Bernd Sturmfels

Chapter 1

Polynomial Rings

"*Algebra is but written geometry*", Sophie Germain

A natural next step after linear algebra is commutative algebra. In that subject area one studies algebraic structures such as fields, rings and ideals. In this first chapter we introduce the relevant basics, with a focus on polynomials and Gröbner basis. We show how to use these for computing basic invariants of a polynomial ideal, like dimension or degree. The formalism we develop now will be applied to geometric situations in later chapters.

1.1. Ideals

Our most basic algebraic structure is that of a *field*. The elements of the field serve as numbers, also called scalars. We can add, subtract, multiply and divide them. It is customary to denote fields by the letter K, for the German word *Körper*. Our favorite field is the set $K = \mathbb{Q}$ of rational numbers. Another important field is the set $K = \mathbb{R}$ of real numbers. In practise, these two fields are very different. Numbers in $\mathbb{Q}$ can be manipulated by exact *symbolic computation*, whereas numbers in $\mathbb{R}$ are approximated by floating point representations and manipulated by *numerical computation*.

Other widely used fields are the complex numbers $\mathbb{C}$ and the finite field $\mathbb{F}_q$ with q elements. If K is not algebraically closed then we write $\overline{K}$ for its algebraic closure. This is the smallest field in which every non-constant polynomial with coefficients in K has a root. For instance, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}_q}$ are the algebraic closures of the two fields above. Another important example is the field of rational functions $\mathbb{Q}(t)$. Its algebraic closure $\overline{\mathbb{Q}(t)}$ is contained in the field of *Puiseux series*, denoted $\mathbb{C}\{\{t\}\}$, which is also algebraically closed.

In this section we study the ring of polynomials in n variables $x_1, \dots, x_n$ with coefficients in our field K. It is denoted $K[\mathbf{x}] = K[x_1, \dots, x_n]$. If the number n is small then we typically use letters without indices to denote the variables. For instance, we write $K[x], K[x,y]$, or $K[x,y,z]$ for the polynomial ring when $n = 1, 2, 3$.

Many of the constructions we present work not just for the polynomial ring $K[\mathbf{x}]$ but for an arbitrary commutative ring R with unit 1. We allow $1 = 0$, i.e. R as a set may contain just one element 0. For the most part, the reader may assume $R = K[\mathbf{x}]$. But, it would not hurt to peruse a standard text book on *abstract algebra* and look up the axioms of a *ring* and the formal definitions of *commutative* and *unit.* Important examples of commutative rings are the integers $\mathbb{Z}$, the polynomial ring of the integers $\mathbb{Z}[\mathbf{x}]$, or the quotient of a polynomial ring by an ideal. The latter will be discussed soon.

The polynomial ring $K[\mathbf{x}]$ is an infinite-dimensional K-vector space. A distinguished basis of this vector space consists of the monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. There is one monomial for each nonnegative integer vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. Every polynomial $f \in K[\mathbf{x}]$ is written uniquely as a finite K-linear combination of monomials:

$$f \;=\; \sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}}.$$

The *degree* of f is the maximum of the quantities $|\mathbf{a}| = a_1 + \cdots + a_n$, where $c_{\mathbf{a}} \neq 0$. For polynomials of degree $1, 2, 3, 4, 5, 6$ we use the words *linear*, *quadratic*, *cubic*, *quartic*, *quintic*, *sextic*. These can be adjectives or nouns.

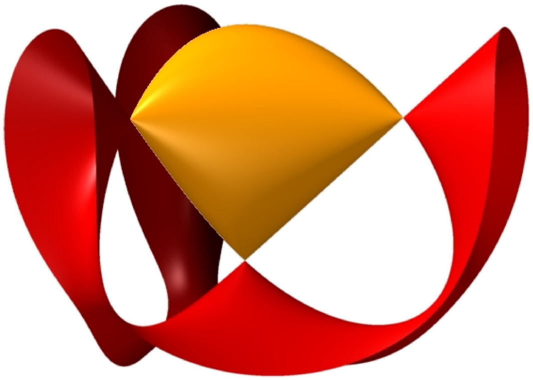


Figure 1. A cubic surface with four singular points.

For example, the following is a cubic polynomial in $n = 3$ variables:

$$f \;=\; \det \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \;=\; 2xyz - x^2 - y^2 - z^2 + 1. \tag{1.1}$$

The zero set of this f is the surface in $\mathbb{R}^3$ that is shown in Figure 1. It consists of all points at which the rank of the 3×3-matrix in (1.1) drops.

Our cubic determinantal surface has four singular points, namely the points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$. These four points are the common zeros in $\mathbb{R}^3$ of the cubic f and its three partial derivatives

$$\frac{\partial f}{\partial x} = 2yz - 2x\,, \quad \frac{\partial f}{\partial y} = 2xz - 2y\,, \quad \frac{\partial f}{\partial z} = 2xy - 2z.$$

These are the points at which the rank of the 3×3-matrix in (1.1) equals 1.

Definition 1.1. An *ideal* in a ring R is a nonempty subset I of R such that

(a) if $f \in R$ and $g \in I$ then $fg \in I$;

(b) if $g, h \in I$ then $g + h \in I$.

If $R = K[\mathbf{x}]$ then an ideal I is a nonempty subset of $K[\mathbf{x}]$ that is closed under taking linear combinations with polynomial coefficients. An alternative definition is as follows: A subset I of a ring R is an ideal if and only if there exists a ring homomorphism $\phi : R \to S$ whose kernel $\ker \phi = \phi^{-1}(0)$ is equal to I. For instance, if $R = \mathbb{Z}$ then the set I of even integers is an ideal. It is the kernel of the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ that takes an integer to either 0 or 1, depending on its parity.

Ideals in a ring play the same role as normal subgroups in a group. They are the subobjects used to define quotients. Consider the quotient of abelian groups R/I. Its elements are the congruence classes $f + I$ modulo I. The axioms (a) and (b) in Definition 1.1 ensure that the following identities hold:

$$(1.2) \qquad (f + I) + (g + I) \,=\, (f + g) + I \quad \text{and} \quad (f + I)(g + I) \,=\, fg + I.$$

Corollary 1.2. *If $I \subset R$ is an ideal then the quotient R/I is a ring.*

Given any subset $\mathcal{F}$ of a ring R, we write $\langle \mathcal{F} \rangle$ for the smallest ideal containing $\mathcal{F}$. This is the *ideal generated by* $\mathcal{F}$. If $R = K[\mathbf{x}]$ then the ideal $\langle \mathcal{F} \rangle$ is the set of all polynomial linear combinations of finite subsets of $\mathcal{F}$.

Proposition 1.3. *If I and J are ideals in a ring R then the following subsets of R are ideals as well: the sum $I + J$, the intersection $I \cap J$, the product IJ, and the quotient $(I : J)$. The latter two subsets of R are defined as follows:*

$$IJ \,=\, \langle\, fg \,:\, f \in I, g \in J \,\rangle \quad \text{and} \quad (I : J) \,=\, \bigl\{ f \in R : fJ \subseteq I \bigr\}.$$

Proof. The product IJ is an ideal by definition. For the others one checks that conditions (a) and (b) hold. We shall carry this out for the ideal quotient $(I : J)$. To show (a), suppose that $f \in R$ and $g \in (I : J)$. We have:

$$(fg)J = f(gJ) \,\subset\, fI \,\subset\, I.$$

For (b), suppose f and g are in $(I : J)$. We have:

$$(f+g)J \;\subset\; fJ + gJ \;\subset\; I + I \;=\; I.$$

This implies $f+g \in (I : J)$. We have shown that $(I : J)$ is an ideal. □

The *Euclidean algorithm* works in the polynomial ring $K[x]$ in one variable x over a field K. This implies that $K[x]$ is a *principal ideal domain* (PID), i.e. every ideal I in $K[x]$ is generated by one element. That generator can be uniquely factored into irreducible polynomials.

Unique factorization of polynomials also holds when the number of variables satisfies $n \geq 2$. We say that the polynomial ring $K[\mathbf{x}]$ is a *unique factorization domain* (UFD). However, $K[\mathbf{x}]$ is not a PID when $n \geq 2$.

Example 1.4 ($n = 1$)**.** Consider the following two ideals in $\mathbb{Q}[x]$:

$$I \;=\; \langle\, x^3 + 6x^2 + 12x + 8 \,\rangle \quad \text{and} \quad J \;=\; \langle\, x^2 + x - 2 \,\rangle.$$

We compute the four ideals in Proposition 1.3. For this, it helps to factor:

$$I \;=\; \langle\, (x+2)^3 \,\rangle \qquad \text{and} \qquad J \;=\; \langle\, (x-1)(x+2) \,\rangle.$$

The four new ideals are

$$\begin{array}{lclclcl} I \cap J &=& \langle\, (x-1)(x+2)^3 \,\rangle & \qquad & IJ &=& \langle\, (x-1)(x+2)^4 \,\rangle \\ I + J &=& \langle\, x+2 \,\rangle & & I : J &=& \langle\, (x+2)^2 \,\rangle. \end{array}$$

We see that arithmetic in $\mathbb{Q}[x]$ is just like arithmetic in the ring of integers $\mathbb{Z}$.

A non-zero element f in a ring R is called

- *a nilpotent* if $f^m = 0$ for some positive integer m,
- *a zero divisor* if there exists $0 \neq g \in R$ such that $gf = 0$.

A ring R is called *an integral domain* if it has no zero divisors and $1 \neq 0$ in R, i.e. 0 is a ring but not an integral domain.

We examine these properties for the quotient ring R/I where I is an ideal in R. Properties of the ideal I correspond to properties of the ring R/I. This summarized in the following table:

property	**definition**	**the quotient ring** R/I
I is *maximal*	no other proper ideal contains I	is a *field*
I is *prime*	$fg \in I \;\Rightarrow\; f \in I$ or $g \in I$	is an *integral domain*
I is *radical*	$(\exists s\,:\, f^s \in I) \;\Rightarrow\; f \in I$	has no *nilpotent* elements
I is *primary*	$fg \in I$ and $g \notin I \;\Rightarrow\; (\exists s\,:\, f^s \in I)$	all *zero divisor* are nilpotent

Maximal, prime and primary ideals are proper. In other words, the ring R itself is an ideal in R, but it is neither maximal, nor prime, nor primary.

Example 1.5. The ideal $I = \langle x^2 + 10x + 34, 3y - 2x - 13\rangle$ is maximal in the polynomial ring $\mathbb{R}[x,y]$. The field $\mathbb{R}[x,y]/I$ is isomorphic to the field of complex numbers $\mathbb{C} = \mathbb{R}[i]/\langle i^2+1\rangle$. One isomorphism is gotten by sending $i = \sqrt{-1}$ to $\frac{1}{13}(x+5y)$. The square of that expression is -1 mod I. The principal ideal $J = \langle x^2+10x+34\rangle$ is prime but not maximal in $\mathbb{R}[x,y]$. The quotient $\mathbb{R}[x,y]/J$ is an integral domain. It is isomorphic to $\mathbb{C}[y]$.

Examples for the other two classes of ideals are given in the next proof.

Proposition 1.6. *We have the following implications for an ideal I in R:*

$$I \text{ maximal} \Rightarrow I \text{ prime} \begin{array}{l} \Rightarrow I \text{ radical}, \\ \Rightarrow I \text{ primary}. \end{array}$$

None of these implications is reversible. However, every ideal that is both radical and primary is prime. Every intersection of prime ideals is radical.

Proof. The first implication holds because there are no zero divisors in a field. To see that prime implies radical, we take $g = f^{s-1}$ and we use induction on s. Prime implies primary is clear. To prove that every radical primary ideal is prime assume $fg \in I$ and $f \notin I$. Then, as I is primary, we have $g^s \in I$ for some $s \in \mathbb{N}$. As I is radical, we now conclude that $g \in I$.

To see that no implication is reversible, we consider the following three ideals in the polynomial ring $\mathbb{R}[x,y]$ with $n=2$ variables:

- $I = \langle\, x^2 \,\rangle$ is primary but not radical,
- $I = \langle\, x(x-1) \,\rangle$ is radical but not primary,
- $I = \langle\, x \,\rangle$ is prime but not maximal.

The last statement holds since intersections of radical ideals are radical. □

We now revisit the surface in Figure 1 from the perspective of ideals.

Example 1.7 ($n = 3$)**.** We consider the ideal generated by the partial derivatives of the cubic $f = 2xyz - x^2 - y^2 - z^2 + 1$. This is the ideal

$$I \;=\; \Big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \Big\rangle \;=\; \langle\, xy - z\,,\; xz - y\,,\; yz - x \,\rangle \quad\subset\; \mathbb{R}[x,y,z].$$

The cubic f is not in this ideal because every polynomial in I has zero constant term. The ideal I is radical because we can write it as the intersection of five maximal ideals. Namely, using a computer algebra system, we find that I equals the intersection

$$\begin{aligned} I \;=\;& \langle x,y,z\rangle \cap \langle x-1,y-1,z-1\rangle \cap \langle x-1,y+1,z+1\rangle \\ & \cap\; \langle x+1,y-1,z+1\rangle \cap \langle x+1,y+1,z-1\rangle. \end{aligned} \tag{1.3}$$

The cubic f lies in the last four maximal ideals. Their intersection is equal to $I + \langle f\rangle$. The zero set of the radical ideal $I+\langle f\rangle$ consists of the four singular points on the surface seen in in Figure 1. The *Chinese Remainder Theorem* implies that the quotient ring is a product of fields. Namely, we have an isomorphism $\mathbb{R}[x,y]/I \simeq \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. It takes each polynomial modulo I to its residue classes modulo in the intersectands in (1.3).

1.2. Gröbner Bases

Every ideal has many different generating sets. There is no canonical notion of basis for an ideal. For instance, the set $\mathcal{F} = \{x^6-1, x^{10}-1, x^{15}-1\}$ minimally generates the ideal $\langle x-1\rangle$ in the polynomial ring $\mathbb{Q}[x]$ in one variable. Of course, the singleton $\{x-1\}$ is a preferable generating set for that ideal. Recall that every ideal in $\mathbb{Q}[x]$ is *principal* for $n=1$. The *Euclidean algorithm* transforms the set $\mathcal{F}$ into the set $\{x-1\}$.

Here is a certificate for the fact that $x-1$ is in the ideal generated by $\mathcal{F}$:

$$x^5\cdot(x^6-1) \;-\; (x^5+x)\cdot(x^{10}-1) \;+\; 1\cdot(x^{15}-1) \;=\; x-1.$$

Such identities can be found with the *Extended Euclidean Algorithm.* Please google this. Finding certificates for ideal membership when $n\geq 2$ is a harder problem. This topic comes up when we discuss Hilbert's Nullstellensatz in Chapter 6. In this section we introduce the basics for computing with ideals.

Gaussian elimination is familar from linear algebra. It gives a process for manipulating ideals that are generated by linear polynomials. For example, the following two ideals are identical in the polynomial ring $\mathbb{Q}[x,y,z]$:

$$\begin{aligned} &\langle\, 2x+3y+5z+7,\; 11x+13y+17z+19,\; 23x+29y+31z+37\,\rangle \\ = \;&\langle\, 7x-16,\; 7y+12,\; 7z+9\,\rangle. \end{aligned}$$

Undergraduate linear algebra taught us how to transform the three generators on the left into the simpler ones on the right. This is the process of solving a system of linear equations. In our example there is a unique solution, namely the point $\left(\frac{16}{7}, -\frac{12}{7}, -\frac{9}{7}\right)$ in $\mathbb{R}^3$.

We next introduce Gröbner bases. The framework of Gröbner bases offers practical methods for computing with ideals in a polynomial ring $K[\mathbf{x}]$ in n variables. Here K is a field whose arithmetic we can compute. Implementations of Gröbner bases are available in many computer algebra systems. We strongly encourage our readers to experiment with these tools.

Informally, we can think of Gröbner bases as a version of the Euclidean algorithm for polynomials in $n\geq 2$ variables, or as a version of Gaussian elimination for polynomials of degree ≥ 2. Gröbner bases for ideals in $K[\mathbf{x}]$ are fundamental in nonlinear algebra, just like Gaussian elimination for matrices is fundamental when one studies linear algebra. The premise of this book is that **nonlinear algebra is the next step after linear algebra**.

We identify the set $\mathbb{N}^n$ of non-negative integer vectors with the monomial basis of the polynomial ring $K[\mathbf{x}]$. The coordinatewise partial order on $\mathbb{N}^n$

corresponds to divisibility of monomials. To be precise, we have $\mathbf{a} \leq \mathbf{b}$ in $\mathbb{N}^n$ if and only if the monomial $\mathbf{x}^\mathbf{a}$ divides the monomial $\mathbf{x}^\mathbf{b}$.

Theorem 1.8 (Dickson's Lemma)**.** *Any infinite subset of $\mathbb{N}^n$ contains a pair $\{\mathbf{a}, \mathbf{b}\}$ that satisfies $\mathbf{a} \leq \mathbf{b}$.*

Proof. We proceed by induction on n. The statement is trivial for $n = 1$. Any subset of cardinality at least two in $\mathbb{N}$ contains a comparable pair. Suppose now that Dickson's Lemma has been proved for $n-1$, and consider an infinite subset $\mathcal{M}$ of $\mathbb{N}^n$. For each $i \in \mathbb{N}$ let $\mathcal{M}_i$ denote the set of all vectors $\mathbf{a} \in \mathbb{N}^{n-1}$ such that $(\mathbf{a}, i) \in \mathcal{M}$. If some $\mathcal{M}_i$ is infinite then we are done by the induction hypothesis. Hence each $\mathcal{M}_i$ is a finite subset of $\mathbb{N}^{n-1}$, and we have $\mathcal{M}_i \neq \emptyset$ for infinitely many i.

The infinite subset $\cup_{i=0}^{\infty}\mathcal{M}_i$ of $\mathbb{N}^{n-1}$ satisfies the assertion. This means that its subset of minimal elements with respect to the coordinatewise order is finite. Hence there exists an index j such that all minimal elements are contained in the finite set $\cup_{i=0}^{j}\mathcal{M}_i$. Pick any element $(\mathbf{b}, k) \in \mathcal{M}_k$ for $k > j$. Since $\mathbf{b}$ is not minimal in $\cup_{i=0}^{\infty}\mathcal{M}_i$, there exists an index i with $i \leq j < k$ and an element $\mathbf{a} \in \mathcal{M}_i$ with $\mathbf{a} \leq \mathbf{b}$. Then we have $(\mathbf{a}, i) \leq (\mathbf{b}, k)$ in $\mathcal{M}$. □

Corollary 1.9. *For any nonempty set $\mathcal{M} \subset \mathbb{N}^n$, its subset of coordinatewise minimal elements is finite and nonempty.*

Proof. The fact that it is nonempty follows by induction on n. The set is finite by Dickson's Lemma. □

Definition 1.10. Consider a total ordering $\prec$ of the set $\mathbb{N}^n$. We write $\mathbf{a} \preceq \mathbf{b}$ if $\mathbf{a} \prec \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$. The ordering $\prec$ is a *monomial order* if, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^n$,

- $(0, 0, \ldots, 0) \preceq \mathbf{a}$;
- $\mathbf{a} \preceq \mathbf{b}$ implies $\mathbf{a} + \mathbf{c} \preceq \mathbf{b} + \mathbf{c}$.

This gives a total order on monomials in $K[\mathbf{x}]$. Three standard examples are:

- *the lexicographic ordering*: we set $\mathbf{a} \prec_{\text{lex}} \mathbf{b}$ if the leftmost non-zero entry of $\mathbf{b} - \mathbf{a}$ is positive.
- *the degree lexicographic order*: we set $\mathbf{a} \prec_{\text{deglex}} \mathbf{b}$ if either $|\mathbf{a}| < |\mathbf{b}|$, or $|\mathbf{a}| = |\mathbf{b}|$ and the leftmost non-zero entry of $\mathbf{b} - \mathbf{a}$ is positive.
- *the degree reverse lexicographic order*: we set $\mathbf{a} \prec_{\text{revlex}} \mathbf{b}$ if either $|\mathbf{a}| < |\mathbf{b}|$, or $|\mathbf{a}| = |\mathbf{b}|$ and the rightmost non-zero entry of $\mathbf{b} - \mathbf{a}$ is negative.

All three orders satisfy $x_1 \succ x_2 \succ \cdots \succ x_n$, but they differ on monomials of higher degree. We recommend that the reader list the 10 quadratic monomials for $n = 4$ in each of the three orderings above.

Throughout this book we specify a monomial order by giving the name of the order and how the variables are sorted. For instance, we might say: "let $\prec$ denote the degree lexicographic order on $K[x, y, z]$ given by $y \prec z \prec x$". Further choices of monomial orderings can be obtained by assigning positive weights to the variables. See [**10**, Exercise 11 in §2.4]. We also note that any monomial order is a refinement of the coordinatewise partial order on $\mathbb{N}^n$:

$$\text{if } \mathbf{x}^{\mathbf{a}} \text{ divides } \mathbf{x}^{\mathbf{b}} \text{ then } \mathbf{a} \preceq \mathbf{b}.$$

Remark 1.11. Fix a monomial order $\prec$ and let $\mathcal{M}$ be any nonempty subset of $\mathbb{N}^n$. Then $\mathcal{M}$ has a unique minimal element with respect to $\prec$. To show this, we apply Dickson's Lemma as in Corollary 1.9. Our set $\mathcal{M}$ has a finite, nonempty subset of minimal elements in the componentwise order on $\mathbb{N}^n$. This finite subset is linearly ordered by $\prec$. We select its minimal element.

We now fix a monomial order $\prec$. Given any nonzero polynomial $f \in K[\mathbf{x}]$, its *initial monomial* $\mathrm{in}_\prec(f)$ is the $\prec$-largest monomial $\mathbf{x}^{\mathbf{a}}$ among those that appear in f with non-zero coefficient. To illustrate this for the orders above, let $n = 3$ with variable order $x \succ y \succ z$: Fix the polynomial $f = x^2+xz^2+y^3$. Then $\mathrm{in}_{\prec_{\mathrm{lex}}}(f) = x^2$, $\mathrm{in}_{\prec_{\mathrm{deglex}}}(f) = xz^2$ and $\mathrm{in}_{\prec_{\mathrm{revlex}}}(f) = y^3$.

For any ideal $I \subset K[\mathbf{x}]$, we define the *initial ideal* of I with respect to a given monomial order $\prec$ as follows:

$$\mathrm{in}_\prec(I) \;=\; \langle\, \mathrm{in}_\prec(f) \,:\, f \in I \,\rangle.$$

This is a *monomial ideal*, i.e. it is generated by a set of monomials. A priori, this generating set is infinite. However, it turns out that we can always choose a finite subset that suffices to generate this monomial ideal.

Proposition 1.12. *Fix a monomial order $\prec$. Every ideal I in the polynomial ring $K[\mathbf{x}]$ has a finite subset $\mathcal{G}$ such that*

$$\mathrm{in}_\prec(I) \;=\; \langle\, \mathrm{in}_\prec(f) \,:\, f \in \mathcal{G} \,\rangle.$$

Such a finite subset $\mathcal{G}$ of I is called a Gröbner basis *for I with respect to $\prec$.*

Proof. Suppose no such finite set $\mathcal{G}$ exists. Then we can create a list of infinitely many polynomials $f_1, f_2, f_3, \ldots$ in I such that none of the initial monomials $\mathrm{in}_\prec(f_i)$ divides any other initial monomial $\mathrm{in}_\prec(f_j)$. This would be a contradiction to Dickson's Lemma. □

We next show that every Gröbner basis actually generates its ideal.

Theorem 1.13. *If $\mathcal{G}$ is a Gröbner basis for an ideal I in $K[\mathbf{x}]$ then $I = \langle \mathcal{G} \rangle$.*

Proof. Suppose that $\mathcal{G}$ does not generate I. Among all elements in the set-theoretic difference $I \backslash \langle \mathcal{G} \rangle$, there exists a polynomial f whose initial monomial $\mathbf{x}^{\mathbf{b}} = \mathrm{in}_\prec(f)$ is minimal with respect to $\prec$. This follows from Remark

1.11. Since $\mathbf{x^b} \in \mathrm{in}_\prec(I)$, there exists an element $g \in \mathcal{G}$ whose initial monomial divides $\mathbf{x^b}$, say $\mathbf{x^b} = \mathbf{x^c} \cdot \mathrm{in}_\prec(g)$. Now, $f - \mathbf{x^c}g$ is a polynomial with strictly smaller initial monomial. It lies in I but it does not lie in the ideal $\langle \mathcal{G} \rangle$. This is a contradiction to the choice of f. □

Corollary 1.14 (Hilbert's Basis Theorem)**.** *Every ideal I in $K[\mathbf{x}]$ is finitely generated.*

Proof. Fix any monomial order $\prec$. By Proposition 1.12, the ideal I has a finite Gröbner basis $\mathcal{G}$. By Theorem 1.13, the finite set $\mathcal{G}$ generates I. □

Gröbner bases are not unique. If $\mathcal{G}$ is a Gröbner basis of an ideal I with respect to a monomial order $\prec$ then so is every other finite subset of I that contains $\mathcal{G}$. In that sense, Gröbner bases differ from the bases we know from linear algebra. The issue of minimality and uniqueness is addressed next.

Definition 1.15. Fix I and $\prec$. A Gröbner basis $\mathcal{G}$ is *reduced* if the following two conditions hold:

(a) The leading coefficient of each polynomial $g \in \mathcal{G}$ is 1.

(b) For distinct $g, h \in \mathcal{G}$, no monomial in g is a multiple of $\mathrm{in}_\prec(h)$.

In what follows we fix an ideal $I \subset K[\mathbf{x}]$ and a monomial ordering $\prec$.

Theorem 1.16. *The ideal I has a unique reduced Gröbner basis for $\prec$.*

Proof idea. We refer to [**10**, §2.7, Theorem 5]. The idea is as follows. We start with any Gröbner basis $\mathcal{G}$ and we turn it into a reduced Gröbner basis by applying the following steps. First we divide each $g \in \mathcal{G}$ by its leading coefficient to make it monic, so that (a) holds. We then remove all elements g from $\mathcal{G}$ whose initial monomial is not a minimal generator of $\mathrm{in}_\prec(I)$. For any pair of polynomials with the same initial monomial we delete one of them. Next we apply the division algorithm [**10**, §2.3] to any trailing monomial until no more trailing monomial is divisible by any leading monomial. The resulting set is the reduced Gröbner basis. □

Let $\mathcal{S}_\prec(\mathcal{I})$ be the set of all monomials $\mathbf{x^b}$ that are not in the initial ideal $\mathrm{in}_\prec(I)$. We call these $\mathbf{x^b}$ the *standard monomials* of I with respect to $\prec$.

Theorem 1.17. *The set $\mathcal{S}_\prec(I)$ of standard monomials is a basis for the K-vector space $K[\mathbf{x}]/I$.*

Proof. The image of $\mathcal{S}_\prec(I)$ in $K[\mathbf{x}]/I$ is linearly independent because every non-zero polynomial f has at least one monomial, namely $\mathrm{in}_\prec(f)$, that is not in $\mathcal{S}$. We next prove that $\mathcal{S}_\prec(I)$ spans $K[\mathbf{x}]/I$. Suppose not. Then there exists a monomial $\mathbf{x^c}$ which is not in the K-span of $\mathcal{S}_\prec(I)$ modulo I. We may assume that $\mathbf{x^c}$ is minimal with respect to the monomial order $\prec$.

Since $\mathbf{x}^{\mathbf{c}}$ is not in $\mathcal{S}_{\prec}(I)$, it lies in the initial ideal $\mathrm{in}_{\prec}(I)$. Hence there exists $h \in I$ with $\mathrm{in}_{\prec}(h) = \mathbf{x}^{\mathbf{c}}$. Each monomial in h other than $\mathbf{x}^{\mathbf{c}}$ is smaller with respect to $\prec$, so it lies in the K-span of $\mathcal{S}_{\prec}(I)$ modulo I. Hence $\mathbf{x}^{\mathbf{c}}$ has the same property. This is a contradiction. $\square$

Software for Gröbner bases rests on *Buchberger's Algorithm* [**10**, §2.7]. This is implemented in all major computer algebra systems. It takes as its input a monomial order $\prec$ and a finite set $\mathcal{F}$ of polynomials in $K[\mathbf{x}]$. The output of Buchberger's Algorithm is the unique reduced Gröbner basis $\mathcal{G}$ for the ideal $I = \langle \mathcal{F} \rangle$ with respect to $\prec$. Experimenting with such an implementation is an essential step for any student in nonlinear algebra.

In what follows we present some examples of input-output pairs $(\mathcal{F}, \mathcal{G})$ for $n = 3$. Here we take the lexicographic monomial order with $x \succ y \succ z$.

Example 1.18. A computer algebra system, like `Maple`, `Mathematica`, `Magma`, `Macaulay2`, or `Singular`, transforms the input $\mathcal{F} \subset \mathbb{Q}[x, y, z]$ into its reduced Gröbner basis $\mathcal{G}$. The initial monomials are always underlined:

- For $n = 1$, computing the reduced Gröbner basis means computing the greatest common divisor of the input polynomials: $\mathcal{F} = \{x^3 - 6x^2 - 5x - 14, 3x^3 + 8x^2 + 11x + 10, 4x^4 + 4x^3 + 7x^2 - x - 2\}$, $\mathcal{G} = \{\underline{x^2} + x + 2\}$.
- For linear polynomials, running Buchberger's algorithm amounts to Gaussian elimination: For $\mathcal{F} = \{2x + 3y + 5z + 7,\ 11x + 13y + 17z + 19,\ 23x + 29y + 31z + 37\}$, the reduced Gröbner basis is found to be $\mathcal{G} = \{\underline{x} - \frac{16}{7},\ \underline{y} + \frac{12}{7},\ \underline{z} + \frac{9}{7}\}$.
- Here is another ideal we saw earlier: $\mathcal{F} = \{xy - z, xz - y, yz - x\}$ yields $\mathcal{G} = \{\underline{x} - yz, \underline{y^2} - z^2, \underline{yz^2} - y, \underline{z^3} - z\}$. There are precisely five standard monomials: $\mathcal{S}_{\prec}(I) = \{1, y, z, yz, z^2\}$. This is consistent with Example 1.7, where we saw that $\mathcal{F}$ has five zeros in $\mathbb{C}^3$.
- This input is a curve in the (y, z)-plane parametrized by two cubics in one variable x. We write this as $\mathcal{F} = \{y - x^3 + 4x, z - x^3 - x + 1\}$. The Gröbner basis has the implicit equation of this curve as its second element: $\mathcal{G} = \{\,\underline{x} + \frac{1}{5}y + \frac{1}{5}z - \frac{1}{5},\ \underline{y^3} - 3y^2z - 3y^2 + 3yz^2 + 6yz + 28y - z^3 - 3z^2 + 97z + 99\,\}$.
- Let z be the sum of $x = \sqrt[3]{7}$ and $y = \sqrt[4]{5}$. We encode this in the set $\mathcal{F} = \{x^3 - 7, y^4 - 5, z - x - y\}$. The real number $z = \sqrt[3]{7} + \sqrt[4]{5}$ is algebraic of degree 12 over $\mathbb{Q}$. Its minimal polynomial is the first element in the Gröbner basis $\mathcal{G} = \{\underline{z^{12}} - 28z^9 - 15z^8 + 294z^6 - 1680z^5 + 75z^4 - 1372z^3 - 7350z^2 - 2100z + 2276, \ldots\}$.
- The elementary symmetric polynomials $\mathcal{F} = \{x + y + z, xy + xz + yz, xyz\}$ have the reduced Gröbner basis $\mathcal{G} = \{\,\underline{x} + y + z,\ \underline{y^2} +$

$yz + z^2,\ \underline{z^3}\}$. There are six standard monomials. The quotient $\mathbb{Q}[x, y, z]/I$ is the regular representation of the symmetric group S_3.

For each of the six ideals above, what is the reduced Gröbner basis for the degree lexicographic order? What are the possible initial monomial ideals?

In general, the choice of monomial order can make a huge difference in the complexity of the reduced Gröbner basis, even for two input polynomials.

Example 1.19 (Intersecting two quartic surfaces in projective 3-space $\mathbb{P}^3$)**.** A random homogeneous polynomial of degree four in $n = 4$ variables has 35 monomials. Consider the ideal I generated by two such random polynomials. If $\prec$ is the degree reverse lexicographic order then the reduced Gröbner basis $\mathcal{G}$ contains 5 elements of degree up to 7. If $\prec$ is the lexicographic order then $\mathcal{G}$ contains 150 elements of degree up to 73.

Naturally, one uses a computer to find the 150 polynomials above. Many computer algebra systems offer an implementation of Buchberger's algorithm for Gröbner bases. We reiterate: our readers are strongly encouraged to experiment with a computer algebra system while studying this book.

For an introduction to Buchberger's algorithm and many further details regarding Gröbner bases, the reader is referred to the textbooks by Cox-Little-O'Shea [**10**], Greuel-Pfister [**25**] and Kreuzer-Robbiano [**31**]. In later chapters we shall freely use concepts from this area, like S-polynomials and Buchberger's Criterion. After all, our book is nothing but an "Invitation".

1.3. Dimension and Degree

The two most important invariants of an ideal I in a polynomial ring $K[\mathbf{x}]$ are its dimension and its degree. We shall define these invariants, starting with the case of monomial ideals. In this section we focus on combinatorial aspects. The geometric interpretation of will be presented in Chapter 2.

Definition 1.20 (Hilbert function)**.** Let $I \subset K[\mathbf{x}]$ be a monomial ideal. The *Hilbert function* h_I takes nonnegative integers to nonnegative integers. The value $h_I(q)$ is the number of monomials of degree q *not* belonging to I.

A convenient way to represent a function $\mathbb{N} \to \mathbb{N}$ is by its generating function. This is a formal power series with nonnegative integer coefficients. The generating function for the Hilbert function is known as *Hilbert series.*

Definition 1.21 (Hilbert series)**.** Let $I \subset K[\mathbf{x}]$ be a monomial ideal. We fix a formal variable z. The Hilbert series of I is the generating function

$$\mathrm{HS}_I(z) \quad = \quad \sum_{q=0}^{\infty} h_I(q) z^q .$$

We begin with zero ideal $I = \{0\}$. Here we count all monomials in $K[\mathbf{x}]$.

Example 1.22. The Hilbert series of the zero ideal is the rational function

$$\mathrm{HS}_{\{0\}}(q) \;=\; \frac{1}{(1-z)^n} \;=\; \sum_{q=0}^{\infty} \binom{n+q-1}{n-1} z^q.$$

The number of monomials of degree q in n variables equals $h_I(q) = \binom{n+q-1}{n-1}$. Note that the Hilbert function $h_{\{0\}}(q)$ is a polynomial of degree $n-1$ in q.

We next consider the case of a principal ideal.

Example 1.23. Let $I = \langle\, x_1^{a_1} \cdots x_n^{a_n} \rangle$, where $\sum_{i=1}^n a_i = e$. We must count monomials of degree q that are not divisible by the generator of I. To do this, we can count all monomials and then subtract those that are in I. This yields

$$\mathrm{HS}_I(z) \;=\; \frac{1-z^e}{(1-z)^n} \;=\; \sum_{q=0}^{\infty} \left[\binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1} \right] z^q.$$

The second binomial coefficient is zero when $q < e$. For all $q \geq e$, the Hilbert function $h_I(q) = \binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1}$ is a polynomial in q of degree $n-2$.

Our third example concerns ideals generated by two monomials:

Example 1.24. Fix an ideal $I = \langle m_1, m_2 \rangle$ in $K[\mathbf{x}]$, where m_i is a monomial of degree e_i for $i = 1, 2$. We count the monomials in I of degree q by

(1) computing the number of monomials divisible by m_1,

(2) adding the number of monomials divisible by m_2,

(3) subtracting the number of monomials divisible both by m_1 and m_2.

Case (3) concerns monomials that are divisible by the least common multiple $m_{12} = \mathrm{lcm}(m_1, m_2)$. Let e_{12} denote the degree of m_{12}. Then the Hilbert series equals

$$\mathrm{HS}_I(z) \;=\; \frac{1 - z^{e_1} - z^{e_2} + z^{e_{12}}}{(1-z)^n}.$$

Therefore, the Hilbert function is an alternating sum of binomial coefficients:

$$h_I(q) \;=\; \binom{n+q-1}{n-1} - \binom{n+q-e_1-1}{n-1} - \binom{n+q-e_1-1}{n-1} + \binom{n+q-e_{12}-1}{n-1}.$$

This expression agrees with a polynomial in q, provided $q \geq e_{12}$.

Theorem 1.25. *The Hilbert series of a monomial ideal $I \subset K[\mathbf{x}]$ equals*

$$\mathrm{HS}_I(z) \;=\; \frac{\kappa_I(z)}{(1-z)^n}, \tag{1.4}$$

where $\kappa_I(z)$ is polynomial with integer coefficients and $\kappa_I(0) = 1$. There exists a polynomial HP *in one unknown q of degree $\leq n-1$, known as the*

Hilbert polynomial *of the ideal* I*, such that* $\mathrm{HP}(q) = h_I(q)$ *for all values of the integer* q *that are sufficiently large.*

Proof. We prove this result by counting monomials using inclusion-exclusion, as hinted at in the three examples above. Let $m_1, m_2, \ldots, m_r$ be the monomials that minimally generate I. For any subset τ of the index set $\{1, 2, \ldots, r\}$, we write m_τ for the least common multiple of the set $\{m_i : i \in \tau\}$, and we set $e_\tau = \mathrm{degree}(m_\tau)$. This includes the empty set $\tau = \emptyset$, for which $m_\emptyset = 1$ and $e_\emptyset = 0$. The desired numerator polynomial (1.4) can be written as an alternating sum of 2^r powers of z:

$$\kappa_I(z) \quad = \sum_{\tau \subseteq \{1,2,\ldots,r\}} (-1)^{|\tau|} \cdot z^{e_\tau}.$$

The cases $r = 0, 1, 2$ were seen above. The general case is inclusion-exclusion. Note that $\kappa_I \in \mathbb{Z}[z]$ with $\kappa_I(0) = 1$. By regrouping the terms of (1.4),

$$h_I(q) \quad = \sum_{\tau \subseteq \{1,2,\ldots,r\}} (-1)^{|\tau|} \binom{n + q - e_\tau - 1}{n - 1}. \tag{1.5}$$

This expression is a polynomial for $q \gg 0$. More precisely, the Hilbert function $h_I(q)$ coincides with the Hilbert polynomial $\mathrm{HP}_I(q)$ for all q that exceed $e_{\{1,2,\ldots,r\}}$. This number is the degree of the least common multiple of all generators of I. □

Remark 1.26. The inclusion-exclusion principle carried out in the proof of Theorem 1.25 is a powerful idea, but it also hints at possible simplifications. We wrote the numerator polynomial $\kappa_I(z)$ and the Hilbert polynomial $\mathrm{HP}_I(q)$ as an alternating sum of 2^r terms. However, in most applications r is much larger than n, and the vast majorities of terms will cancel each other. Doing the correct bookkeeping leads us the the topic of *minimal free resolutions* of monomial ideals. This is a main theme in a subject area known as *combinatorial commutative algebra*. Yes, please google this.

Example 1.27. Let $n = 2$ and consider the monomial ideal

$$I \;=\; \langle x \rangle \cap \langle y \rangle \cap \langle x, y \rangle^{r+1} \quad = \quad \langle x^r y, x^{r-1}y^2, x^{r-2}y^3, \ldots, x^2 y^{r-1}, xy^r \rangle.$$

Our formula for κ_I involves 2^r terms. After cancellations, only $2r$ remain:

$$\kappa_I(z) \;=\; 1 \,-\, rz^{r+1} \,+\, (r-1)z^{r+2}.$$

The Hilbert polynomial is the constant $h_I(q) \equiv 2$. This is also the value of the Hilbert function $\mathrm{HF}_I(q)$ for $q > r$. Note that $\mathrm{HF}_I(q) = q + 1$ for $q \leq r$.

Definition 1.28 (Dimension, Degree)**.** Let I be a monomial ideal and write

$$\mathrm{HP}_I(q) \quad = \quad \frac{g}{(d-1)!} q^{d-1} \,+\, \text{lower order terms in } q.$$

The *dimension* of I is d and the *degree* of I is g. Here g is a positive integer.

Remark 1.29. The fact that g is a positive integer is a non-trivial piece of combinatorics. This proof is omitted here. From the inclusion-exclusion formulas above, one can show that the numerator of the Hilbert series factors as $\kappa_I(z) = \lambda_I(z) \cdot (1-z)^{n-d}$, where $\lambda_I(z)$ is also a polynomial with integer coefficients. The degree of I equals $g = \lambda_I(1)$.

Example 1.30. Let I be a principal ideal as in Example 1.23, generated by a monomial of degree $e > 0$. Then the dimension of I is $n-1$ and the degree of I is e. This follows from the formula we gave for the Hilbert series.

Example 1.31. Let $n = 2m$ be even and consider the monomial ideal

$$I \;=\; \langle\, x_1x_2,\, x_3x_4,\, x_5x_6,\, \ldots\,,\, x_{2m-3}x_{2m-2},\, x_{2m-1}x_{2m}\,\rangle.$$

The dimension of I equals m and the degree of I equals 2^m. It is instructive to work out the Hilbert series and the Hilbert polynomial of I for $m = 3, 4$.

We now consider an arbitrary ideal I in $K[\mathbf{x}]$. We now longer assume that I is generated by monomials. Let $\prec$ be any *degree-compatible* monomial order. This means that $|\mathbf{a}| < |\mathbf{b}|$ implies $\mathbf{a} \prec \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in I$.

Lemma 1.32. *The number of standard monomials of I in a given degree q is independent of the choice of monomial order $\prec$, provided $\prec$ is degree-compatible.*

Proof. Let $K[\mathbf{x}]_{\leq q}$ denote the vector space of polynomials of degree $\leq q$. We write $I_{\leq q} := I \cap K[\mathbf{x}]_{\leq q}$ for the subspace of polynomials that lie in the ideal I. Also, consider the set of standard monomials of degree at most q:

$$\mathcal{S}_\prec(I)_{\leq q} \;=\; \mathcal{S}_\prec(I) \,\cap\, K[\mathbf{x}]_{\leq q}$$

We claim that $\mathcal{S}_\prec(I)_{\leq q}$ is a K-vector space basis for the quotient space $K[\mathbf{x}]_{\leq q}/I_{\leq q}$. It is linearly independent since no K-linear combination of $\mathcal{S}_\prec(I)$ lies in I. But, given that $\prec$ is degree compatible, it also spans because taking the normal form of a polynomial modulo the Gröbner basis can only decrease the total degree. □

Remark 1.33. The function that associates to q the dimension of the quotient space $\dim K[\mathbf{x}]_{\leq q}/I_{\leq q}$ is known as *the affine Hilbert function*. We will see its relations to the Hilbert function, as in Definition 1.20, in Chapter 2, after introducing projective varieties and homogenization.

Definition 1.34. Given any ideal I in a polynomial ring $K[\mathbf{x}]$, we define its *Hilbert function* h_I to be that of its initial ideal $\mathrm{in}_\prec(I)$, where $\prec$ is any degree-compatible term order. For all $q \in \mathbb{N}$ we have

$$\begin{aligned} h_I(q) \;=\; h_{\mathrm{in}_\prec(I)}(q) \quad &= \quad |\mathcal{S}_\prec(I)_{\leq q}| \;-\; |\mathcal{S}_\prec(I)_{\leq q-1}| \\ &= \quad \dim(K[\mathbf{x}]_{\leq q}/I_{\leq q}) - \dim(K[\mathbf{x}]_{\leq q-1}/I_{\leq q-1}). \end{aligned}$$

This is the number of standard monomials whose degree is exactly q. This number is independent of $\prec$, thanks to Lemma 1.32. We also define the *Hilbert series* and the *Hilbert polynomial* to be that of any degree-compatible initial monomial ideal:

$$\mathrm{HS}_I(z) \;=\; \mathrm{HS}_{\mathrm{in}_\prec(I)}(z) \quad \text{and} \quad \mathrm{HP}_I(q) \;=\; \mathrm{HP}_{\mathrm{in}_\prec(I)}(q).$$

Finally, we define the dimension of I as the definition of $\mathrm{in}_\prec(I)$, and similarly for the *degree* of I. Here $\prec$ is any degree-compatible monomial ordering. All of these concepts are now well-defined, thanks to Lemma 1.32.

Example 1.35. Let I be a principal ideal generated by a polynomial f of degree e in n variables. The dimension of I is $n-1$ and the degree of I is e. This follows from Example 1.30 because the singleton $\{f\}$ is a Gröbner basis, and its initial monomial $\mathrm{in}_\prec(f)$ has degree e in any degree-compatible monomial order $\prec$.

What we have accomplished in this section is to give a purely combinatorial definition of dimension and degree of an ideal I. In Chapter 2 we shall see that that notion of dimension agrees with the intuitive one for the associated algebraic variety $V(I)$. Namely, a variety has dimension 0 if and only it consists of finitely many points. The number of these points is counted by the degree of the corresponding radical ideal. Likewise, the ideal of a curve has dimension 1, the ideal of a surface has dimension 2 etc. The degree is a measure for how curvy these shapes are. One can show that a prime ideal has degree 1 if and only if it is generated by linear polynomials.

Example 1.36. Fix the polynomial ring $K[x,y,z]$ and let $f = xyz - x^2 - y^2 - z^2 + 1$ as in (1.1). The ideal $\langle f \rangle$ has dimension 2 and degree 3. Let I be the ideal generated by its partial derivatives, as in Example 1.7. Then I has dimension 0 and degree 5. The ideal $I + \langle f \rangle$, whose zeros are the four singular points of the surface in Figure 1, has dimension 0 and degree 4.

Exercises

(1) The polynomial $f = 5x^3 - 25x^2y + 25y^3 + 15xy - 50y^2 - 5x + 25y - 1$ is a product of three linear factors in $\mathbb{R}[x,y]$. Prove this and draw the plane curve $\{f = 0\}$.

(2) For $n = 2$, define a monomial order $\prec$ such that $(2,3) \prec (4,2) \prec (1,4)$.

(3) Let $n = 2$ and fix the monomial ideals $I = \langle x, y^2 \rangle$ and $J = \langle x^2, y \rangle$. Compute the ideals $I + J$, $I \cap J$, IJ and $I^3J^4 = IIIJJJJ$. How many minimal generators does the ideal $I^{123}J^{234}$ have?

(4) The *radical* $\sqrt{I}$ of an ideal I in a ring R is the smallest radical ideal containing I. Prove that the radical of a primary ideal is prime. For ideals in a polynomial ring $K[\mathbf{x}]$, prove that
- the radical of a principal ideal is principal;
- the radical of a monomial ideal is a monomial ideal.

(5) Show that the following inclusions always hold and are strict in general:

$$\sqrt{I}\sqrt{J} \subseteq \sqrt{IJ} \quad \text{and} \quad \mathrm{in}_\prec\big(\sqrt{I}\big) \subseteq \sqrt{\mathrm{in}_\prec(I)}.$$

(6) Using Gröbner bases, find the minimal polynomials of $\sqrt[5]{6} + \sqrt[7]{8}$ and $\sqrt[5]{6} - \sqrt[7]{8}$. This is analogous to the fifth item in Example 1.18.

(7) Find the implicit equation of the curve $\{(x^5 - 6, x^7 - 8) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.

(8) Study the ideal $I = \langle x^3 - yz, y^3 - xz, z^3 - xy\rangle$. Is it radical? If not, find $\sqrt{I}$. Regarding I as a triple of equations, what are its solutions in $\mathbb{R}^3$?

(9) For the ideals I and $\sqrt{I}$ in the previous exercise, determine the Hilbert function, the Hilbert series, Hilbert polynomial, dimension, and degree.

(10) Find an ideal in $\mathbb{Q}[x, y]$ whose reduced Gröbner basis (in lexicographic order) has cardinality 5 and there are precisely 19 standard monomials.

(11) Prove: An ideal in a polynomial ring $K[\mathbf{x}]$ is principal if and only if its reduced Gröbner basis is a singleton.

(12) Let I be the ideal generated by the n elementary symmetric polynomials in $x_1, \ldots, x_n$. Pick a monomial order and find the initial ideal $\mathrm{in}_\prec(I)$.

(13) Let X be 2×2-matrix whose entries are variables. Let I_s be the ideal generated by the entries of the matrix power X^s for $s = 2, 3, 4, \ldots$. Investigate these ideals. What are the dimension and the degree of I_s?

(14) A symmetric 3×3-matrix with unknown entries has seven principal minors: three of size 1×1, three of size 2×2, and one of size 3×3. Does there exist an algebraic relation among these minors? Hint: lexicographic Gröbner basis.

(15) Prove that if $\mathrm{in}_\prec(I)$ is radical then I is radical. Does the converse hold?

(16) Determine all straight lines that lie on the cubic surface in Figure 1.

(17) Identify maximal, prime, radical and primary ideals in the ring $R = \mathbb{Z}$.

(18) Let I be the ideal generated by all 2×2 minors of a $2 \times n$ matrix filled with $2n$ variables. What is the degree and dimension of I for $n = 2, 3, 4$?

(19) Find a prime ideal of degree three and dimension one in n variables for
- $n = 2$,
- $n = 3$,
- $n = 3$ with further assumption that $h_I(1) = 4$.

(20) Compute the dimension and degree of the ideal generated by two random degree four polynomials in $n = 4$ variables, as in Example 1.19.

Chapter 2

Varieties

"*Geometry is but drawn algebra*", Sophie Germain

A *variety* is the set of solutions to.a system of polynomial equations in several unknowns. These are the main objects of study in algebraic geometry. Varieties are the geometric counterparts to ideals in a polynomial ring. The latter live on the algebraic side. We distinguish between *affine varieties* and *projective varieties*. The former arise from arbitrary polynomials, while the latter are the zero sets of systems of *homogeneous* polynomials. Geometers prefer projective varieties because of their nice properties, explained in some of the results we present, like Theorem 2.22. But, for starters, our readers are invited to peruse the pictures shown in this chapter.

2.1. Affine varieties

Algebraic varieties represent solutions of a system of polynomial equations. Fix a field K and consider polynomials $f_1, \ldots, f_k$ in $K[\mathbf{x}] = K[x_1, \ldots, x_n]$. The *variety* defined by these polynomials is the set of their common zeros:

$$\mathcal{V}(f_1, \ldots, f_k) \ := \ \big\{ \mathbf{p} = (p_1, \ldots, p_n) \in K^n : f_1(\mathbf{p}) = \cdots = f_k(\mathbf{p}) = 0 \big\}.$$

Different sets of polynomials can define the same variety. For instance,

$$\mathcal{V}(f_1, f_2) \ = \ \mathcal{V}(f_1^2, f_2^5) \ = \ \mathcal{V}(f_1, f_1 + f_2). \tag{2.1}$$

Instead of thinking about the polynomials themselves, we consider the ideal they generate, $I = \langle f_1, \ldots, f_k \rangle$, and we define $\mathcal{V}(I) := \mathcal{V}(f_1, \ldots, f_k)$. A subset of K^n is a *variety* if it has the form $\mathcal{V}(I)$ for some ideal $I \subset K[\mathbf{x}]$. Given any ideal $I \subset K[\mathbf{x}]$, by Hilbert's Basis Theorem 1.14, we can always find a *finite* set of generators. By Exercise 1, the definition of $\mathcal{V}(I)$ does not depend on the choice of generators of I.

Remark 2.1. Two distinct ideals may define the same variety. For instance, for two non-constant polynomials f_1 and f_2, the ideal $\langle f_1^2, f_2^5\rangle$ is strictly contained in $\langle f_1, f_2\rangle = \langle f_1, f_1 + f_2\rangle$, but they define the same variety in (2.1). Chapter 6 on the Nullstellensätze deals with this issue for fields K that are either algebraically closed, like the complex numbers $K = \mathbb{C}$, or real closed, like the reals $K = \mathbb{R}$.

Algebraic geometry is the study of the geometry of varieties. As in many branches of mathematics, one considers the basic, irreducible building blocks for the objects of study. A variety $\mathcal{V}(I)$ is called *irreducible* if it cannot be written as a union of proper subvarieties in K^n. In symbols, $\mathcal{V}(I)$ is irreducible if and only if, for any ideals J and J' in $K[\mathbf{x}]$ we have

$$\mathcal{V}(I) = \mathcal{V}(J) \cup \mathcal{V}(J') \implies \mathcal{V}(I) = \mathcal{V}(J) \text{ or } \mathcal{V}(I) = \mathcal{V}(J').$$

Any variety can be decomposed into irreducible varieties. The relevant algebraic tool is primary decomposition. This is our topic in the next chapter.

Example 2.2. Consider the ideal $I = \langle xy\rangle \subset \mathbb{R}[x, y]$. Its variety $\mathcal{V}(I) = \mathcal{V}(x) \cup \mathcal{V}(y)$ is a union of two lines in the plane $\mathbb{R}^2$. Hence, this is a reducible variety. Algebraically, I is the intersection of two larger ideals $\langle x\rangle$ and $\langle y\rangle$. Their respective varieties $\mathcal{V}(x)$ and $\mathcal{V}(y)$ are irreducible. This follows from Proposition 2.3 because $\langle x\rangle$ and $\langle y\rangle$ are prime ideals.

For any field K, we can turn K^n into a topological space, using the *Zariski topology*. In this topology, the closed set are the varieties in K^n. In this setting, the definition of an irreducible variety coincides with the definition of an irreducible topological space. If $K = \mathbb{R}$ or $K = \mathbb{C}$ then we also have the classical Euclidean topology on K^n. The Euclidean topology is much finer than the Zariski topology because it has many more open sets.

Our aim is to relate geometric properties of the variety $\mathcal{V}(I)$ to algebraic properties of the ideal I. Consider a maximal ideal of the form $m := \langle x_1 - p_1, \ldots, x_n - p_n\rangle$ in $K[\mathbf{x}]$. The point $(p_1, \ldots, p_n)$ lies in $\mathcal{V}(I)$ if and only if $I \subseteq m$. Given any subset $W \subset K^n$, we consider the set of all polynomials that vanish on W. This set is a radical ideal, denoted

$$\mathcal{I}(W) := \{ f \in K[\mathbf{x}] : f(\mathbf{p}) = 0 \text{ for all } \mathbf{p} \in W \}.$$

The set W is a variety if and only if $W = \mathcal{V}(\mathcal{I}(W))$. Furthermore, given any two varieties V and W in K^n, we have $V \subseteq W$ if and only if $\mathcal{I}(W) \subseteq \mathcal{I}(V)$.

Proposition 2.3. *A variety $W \subset K^n$ is irreducible if and only if its ideal $\mathcal{I}(W)$ is prime.*

Proof. Suppose $\mathcal{I}(W)$ is prime and $W = \mathcal{V}(J) \cup \mathcal{V}(J')$. If $W \neq \mathcal{V}(J)$ then there exists $f \in J$ and $v \in W$ such that $f(v) \neq 0$. Therefore, $f \notin \mathcal{I}(W)$. For any $g \in J'$ we know that fg vanishes on $\mathcal{V}(J)$ and $\mathcal{V}(J')$, hence on W. Thus $fg \in \mathcal{I}(W)$. As $\mathcal{I}(W)$ is prime, we have $g \in \mathcal{I}(W)$. we conclude that $J' \subset \mathcal{I}(W)$. By Exercise 2, this implies $W = \mathcal{V}(\mathcal{I}(W)) \subset \mathcal{V}(J')$.

For the converse, suppose that W is irreducible and $fg \in \mathcal{I}(W)$. Hence

$$W \;=\; W \cap \mathcal{V}(fg) \;=\; W \cap (\mathcal{V}(f) \cup \mathcal{V}(g)) \;=\; (W \cap \mathcal{V}(f)) \,\cup\, (W \cap \mathcal{V}(g)).$$

Without loss of generality, $W = W \cap \mathcal{V}(f)$. This means that $W \subseteq \mathcal{V}(f)$ and hence $f \in \mathcal{I}(W)$. This argument proves that $\mathcal{I}(W)$ is a prime ideal. □

Remark 2.4. Proposition 2.3 relates geometry and number theory. Prime ideals in a polynomial ring $K[\mathbf{x}]$ correspond to irreducible varieties, while prime ideals in the ring of integers $\mathbb{Z}$ correspond to prime numbers (or zero). Hence irreducible varieties are to varieties what primes are to all integers.

Prime ideals appear in applications as the constraints satisfied by a *generative model*. Such models are common in statistics. One considers a vector θ of real parameters and one expresses probabilities (or moments of densities) as functions in θ. These functions are often polynomials or rational functions in θ, and one is interested in all valid polynomial constraints among the probabilities in question. Geometrically, this corresponds to computing the closure (in the Zariski topology) of the image of a polynomial map. This closure is an irreducible variety, so its ideal is prime by Proposition 2.3. That prime ideal represents the image and hence the generative model. It is computed as the kernel of the ring map dual to the polynomial map.

Example 2.5. We give an illustration for the most basic generative model, namely the *independence model* for two random variables X and Y, each with state space $\{1, \ldots, m\}$. Probability distribution of X (resp. Y) are vectors $(p_1, \ldots, p_m)$ (resp. $(q_1, \ldots, q_m)$) in $\mathbb{R}^m$ with nonnegative entries that sum to 1. The probability that X (resp. Y) is in state i equals p_i (resp. q_i). The joint random variable (X, Y) has m^2 states. The set of all probability distributions of (X, Y) that are independent is viewed as a variety in $\mathbb{R}^{m^2}$.

Consider the map that takes a distribution of X and a distribution of Y to the joint distribution of (X, Y). This map extends to a polynomial map

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{m^2} \\ (p_1, \ldots, p_m, q_1, \ldots, q_m) & \mapsto & \big(p_1q_1, p_1q_2, \ldots, p_1q_m, p_2q_1, \ldots, p_mq_m\big). \end{array} \tag{2.2}$$

In statistics one incorporates the requirement $\sum p_i = \sum q_i = 1$. We do so by restricting the domain. We write the resulting map explicitly for $m = 3$:

$$\begin{array}{c} (p_1, p_2, q_1, q_2) \;\;\mapsto\;\; \big(p_1q_1,\, p_1q_2,\, p_1(1{-}q_1{-}q_2),\, p_2q_1,\, p_2q_2, \\ p_2(1{-}q_1{-}q_2), (1{-}p_1{-}p_2)q_1, (1{-}p_1{-}p_2)q_2, (1{-}p_1{-}p_2)(1{-}q_1{-}q_2)\big). \end{array} \tag{2.3}$$

In Exercise 9 we ask for the variety and ideal given by the image of this map.

The algebraic study of the independence model was a point of departure for the development of *Algebraic Statistics*. In that subject it is now common to use prime ideals to represent statistical models. This allows for the use of algebraic invariants (like dimension and degree) and algebraic methods (like Gröbner bases) for data analysis and inference. Readers wishing to learn more about Algebraic Statistics should consult the textbooks [**18, 42, 50**].

We have argued that prime ideals are basic building blocks in algebraic geometry and its applications. This motivates the following definition on the algebra side. We now take R to be any commutative ring with unity.

Definition 2.6. The *spectrum* of the ring R is the set of proper prime ideals:

$$\operatorname{Spec}(R) \;:=\; \bigl\{\, p \subsetneq R \,:\, p \text{ is a prime ideal} \,\bigr\}.$$

The set $\operatorname{Spec}(R)$ is a topological space with the Zariski topology. Its closed sets are the *varieties* $\mathcal{V}(I) = \bigl\{\, p \in \operatorname{Spec} R : I \subseteq p \,\bigr\}$, where I is any ideal in R.

The spectrum of the ring remembers a lot of information: all prime ideals and how they are related geometrically. Our most basic example of a ring R is the polynomial ring $K[\mathbf{x}]$ in n variables. Its spectrum is a topological space with many points. Among them are the usual points $(p_1, \ldots, p_n) \in K^n$. These correspond to maximal ideals of the form $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$. However, the spectrum $\operatorname{Spec}(K[\mathbf{x}])$ has points corresponding to *all* irreducible subvarieties of K^n, not just those of dimension 1. In this manner, K^n is a subset of $\operatorname{Spec}(K[\mathbf{x}])$. Exercise 4 asks you to prove that the Zariski topology on K^n is the one induced one the Zariski topology on $\operatorname{Spec} K[\mathbf{x}]$.

Our next example is the coordinate ring $R = K[W]$ of a subvariety $W \subset K^n$. By definition, this is the quotient ring $R = K[\mathbf{x}]/J$ where $J = \mathcal{I}(W)$ is the radical ideal in the polynomial ring $K[\mathbf{x}]$ that encodes the variety. The prime ideals in $K[W]$ are in natural bijection with the prime ideals in $K[\mathbf{x}]$ that contain J. Geometrically, these correspond to irreducible subvarieties of W. Among these are the points $(p_1, \ldots, p_n) \in W$, which correspond to maximal ideals $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ in $K[W]$, just like before. The Zariski topologies on W and $\operatorname{Spec}(K[W])$ are compatible, in the sense of Exercise 4.

Example 2.7. A paraboloid in $\mathbb{R}^3$ is defined by the equation $z = x^2 + y^2$. Its ideal equals $J = \langle z - x^2 - y^2 \rangle$. The ring of (polynomial) functions on the parabola equals $\mathbb{R}[x, y, z]/J$. What are the Gröbner bases of J and what are the standard monomials? How about the dimension and the degree?

The Zariski topology on the paraboloid has various points. First, there are the classical real points on the surface. Second, there are pairs of complex conjugate points satisfying the equation $z = x^2 + y^2$. And, next there are all irreducible curves lying on the surface, one for each non-maximal prime ideal of $\mathbb{R}[x, y, z]$ that strictly contains J. This includes curves that lie on the complex surface but have no real points, like that for $\langle z^2+1, x^2+y^2-z\rangle$.

Remark 2.8. We continue the analogy from Remark 2.4 in order give geometric intuition for *Chinese remainder theorem*. Fix $n_1, \ldots, n_k \in \mathbb{Z}$ pairwise coprime. In the language of varieties, the fact that $(n_i) + (n_j) = \mathbb{Z}$ is equivalent to the fact that the associated varieties do not intersect—recall that the ideal of the intersection is the union of ideals. For each n_i we are given a number $a_i \in \mathbb{Z}/n_i\mathbb{Z}$, i.e. a function on the variety associated to n_i. As these varieties do not intersect, we expect to obtain a unique function on their union that restricts to the given functions on each piece. The union of varieties is given by the intersection of the ideals, which corresponds to the product $N = \prod_{i=1}^{k} n_i$. This is precisely the statement of the Chinese remainder theorem: there exists a unique $x \in \mathbb{Z}/N\mathbb{Z}$ such that $x = a_i \mod n_i$. The reader is encouraged to push the analogy further. If the varieties intersect (i.e. the numbers are not pairwise coprime) we expect the global function to exist if and only if the functions associated to varieties agree on intersections.

Consider two varieties W_1, W_2 and a map $f : W_1 \to W_2$ between them. Given a function on the target variety, say $g : W_2 \to K$, we define its pullback to be the function $f^*(g) := g \circ f$ from W_1 to K. Of course, here were are interested in polynomial functions, so g is an element in the ring $K[W_2]$. Likewise, we want to the pullback $f^*(g)$ to be element in the ring $K[W_1]$.

Hence, given any polynomial map $f : W_1 \to W_2$ between varieties, we would like the map $f^* : K[W_2] \to K[W_1]$ to be a well-defined ring morphism. In Exercise 5 you will show that any ring morphism $K[W_2] \to K[W_1]$ induces a continuous map of topological spaces $\operatorname{Spec} K[W_1] \to \operatorname{Spec} K[W_2]$. Hence, we may think about maps between varieties as homomorphisms between their rings of functions in the opposite direction. In the fancy language of *category theory*: our star $*$ is *a contravariant functor* from varieties to rings. It furnishes an *equivalence of categories* between irreducible affine varieties (over K) and finitely generated K-algebras that are integral domains.

Example 2.9. The following ring homomorphism is an isomorphism:

$$f^* : \mathbb{R}[x, y]/\langle y - x^2\rangle \;\to\; \mathbb{R}[z],\;\; x \mapsto z,\; y \mapsto z^2.$$

It arises from a map of varieties that takes a line to a parabola in the plane:

$$f : \mathbb{R} \;\to\; \mathcal{V}(y - x^2) \;\subset\; \mathbb{R}^2\,,\;\; \lambda \;\mapsto\; (\lambda, \lambda^2).$$

Under this parametrization of the parabola, the coordinate functions x and y on $\mathbb{R}^2$ pull back to the functions z and z^2 on the line $\mathbb{R}$. We use the letter

λ in the parametrization for extra clarity. It can get confusing when you pass to the map of spectra, a continuous map in the Zariski topologies.

Remark 2.10. Textbooks in algebraic geometry usually define affine varieties to be $\operatorname{Spec} R$, with its Zariski topology, for any (commutative, with unity) ring R. Here R need not be a finitely generated algebra over a field K. However, in this book, affine varieties are zero sets of polynomials in $K[\mathbf{x}]$.

The dependence on the field is crucial for geometric properties of maps. Consider the squaring map $K^1 \to K^1, \lambda \mapsto \lambda^2$ from the affine line to itself.

- If $K = \mathbb{C}$ then the squaring map is surjective.
- If $K = \mathbb{R}$ then its image is the set of nonnegative real numbers. In both cases, the Zariski closure of the image is the whole line.
- If $K = \mathbb{F}_p$ and $p \neq 2$, then the image is a proper subset of K^1. It is Zariski closed. What if we replace K by its algebraic closure?
- In each case, is the map $\operatorname{Spec}(K[x]) \to \operatorname{Spec}(K[x])$ surjective?

From the perspective of spectra, it is instructive to study the ideal $I = \langle x^2 + 1 \rangle$ in $K[x]$. Exercise 6 asks the reader to give a description of $\mathcal{V}(I)$.

Example 2.11. Consider the three ideals $I_1 = \langle x^2 - y^2 \rangle$, $I_2 = \langle x^2 - 2y^2 \rangle$ and $I_3 = \langle x^2 + y^2 \rangle$ in $K[x, y]$. The first one is not prime for any K. The second one is not prime for $K = \mathbb{R}$ or $\mathbb{C}$, but it is a prime ideal when $K = \mathbb{Q}$. The ideal I_3 is not prime for $K = \mathbb{C}$, but it is a prime ideal for $K = \mathbb{Q}$ or $\mathbb{R}$.

We prove the last statement. Suppose $fg \in I_3 \subset \mathbb{R}[x, y]$. This means that $fg = (x^2 + y^2) \cdot h$, where $f, g, h \in \mathbb{R}[x, y]$. By the Fundamental Theorem of Algebra, every homogeneous polynomial p in two variables has a unique (up to multiplication by constants) representation as a product of linear forms with complex coefficients $p = \prod l_j$. If p has real coefficients, then the decomposition is stable under complex conjugation: for every j, either l_j has real coefficients or $\overline{l_j}$ must also appear in the decomposition. We have $x^2 + y^2 = (x + iy)(x - iy)$. In the ring $\mathbb{C}[x, y]$, without loss of generality, we may assume $(x + iy)|f$. But then, by the above argument also $(x - iy)|f$. Thus $f = (x+iy)(x-iy)\prod_i \tilde{l}_i$ for $\tilde{l}_i \in \mathbb{C}[x, y]$. However, $\prod_i \tilde{l}_i$ is stable under conjugation, i.e. defines a real polynomial. Thus $x^2 + y^2$ divides f in $\mathbb{R}[x, y]$.

Many models arise in applications as the image of a polynomial map f. It is important to note that the image of f need not be closed if $K = \mathbb{C}$. And, it need not be dense in its Zariski closure if $K = \mathbb{R}$. This will be discussed in detail in Chapter 4. The following definition plays an important role.

Definition 2.12. A subset $A \subset K^n$ is *constructible* if it can be described as a finite union of differences of varieties. Over the real numbers, a subset

$B \subset \mathbb{R}^n$ is *semi-algebraic* if it can be described as the solutions of a finite system of polynomial inequalities (both $\geq$ and $>$) or a finite union of such.

Remark 2.13. Every constructible subset of $\mathbb{R}^n$ is semi-algebraic, but the converse is not true. See below. The complement of a constructible set is constructible, and the complement of a semi-algebraic set is semi-algebraic.

Example 2.14. Take $n = 2$ and $K = \mathbb{R}$. The singleton $\mathcal{V}(x, y) = \{(0, 0)\}$ is constructible and hence so is $\mathbb{R}^2\backslash\{(0, 0)\} = \mathcal{V}(0)\backslash\mathcal{V}(x, y)$. The orthant $\mathbb{R}^2_{\geq 0} = \{(u, v) \in \mathbb{R}^2 : u \geq 0 \text{ and } v \geq 0\}$ is semi-algebraic. But it is not constructible, because the Euclidean closure of a constructible set is a variety. Its complement $B = \{(u, v) \in \mathbb{R}^2 : u < 0 \text{ or } v < 0\}$ is also semi-algebraic. Can you write B as set of solutions to a finite list of polynomial inequalities?

The two most important invariants of a variety V in K^n are its *dimension* and its *degree*. We defined these in Section 1.3, via the ideal $\mathcal{I}(V) \subset K[\mathbf{x}]$.

Example 2.15. Let V be a linear subspace of K^n. The dimension of V as a variety equals its dimension as a linear space. The degree of V is one. Indeed, we may assume $\mathcal{I}(V) = \langle x_1, \ldots, x_s\rangle$ where $s = n - \dim(V)$. The result follows from Example 1.22 because $K[\mathbf{x}]/\langle x_1, \ldots, x_s\rangle \simeq K[x_{s+1}, \ldots, x_n]$.

We note an important property of dimension. If $V_1 \subsetneq V_2$ then $\dim(V_1) \leq \dim(V_2)$. The inequality is strict if V_2 is irreducible. The latter is not easy to prove from the definition we gave. It helps to consult a textbook in commutative algebra for alternative (but equivalent) definitions of dimension.

Here is a method for computing the dimension of a variety $\mathcal{V}(I)$. We use that $\mathcal{V}(\mathrm{in}_\prec(I))$ is a union of linear spaces in K^n, for any monomial order $\prec$.

(1) Compute a Gröbner basis of I and hence the monomial ideal $\mathrm{in}_\prec(I)$.

(2) Let $m_1, \ldots, m_k$ monomials that generate $\mathrm{in}_\prec(I)$. Find the smallest (with respect to cardinality) set of variables $S = \{x_{i_1}, \ldots, x_{i_d}\}$ such that every monomial generator m_j is divisible by some variable in S.

(3) The cardinality d of S is dimension of both $\mathcal{V}(\mathrm{in}_\prec(I))$ and $\mathcal{V}(I)$.

The second most important invariant of a variety $V = \mathcal{V}(I) \subset K^n$ is the *degree*. We now provide its geometric interpretation. Suppose K is algebraically closed. A general subspace $L \subset K^n$ with $\dim(L) + \dim(V) = n$ intersects V in finitely many points. This is the degree of V. Indeed, this follows inductively by the fact that a general linear polynomial is not a zero-divisor in $K[V]$. Hence, adding it I changes the Hilbert function in such a way that the dimension drops by one and the degree remains the same. If $\dim(I) = 0$ and I is radical then the degree is the cardinality of $V = \mathcal{V}(I)$.

Some points on a variety are singular, like the four nodes of the cubic surface in Figure 1. Our aim is now to discuss singularities in general. We start with the case of a hypersurface, defined by one polynomial $f \in K[\mathbf{x}]$. A point $p \in \mathcal{V}(f)$ is *singular* if all partial derivatives vanish, i.e. $\frac{\partial f}{\partial x_i}(p) = 0$ for $i = 1, \dots, n$. Thus the *singular locus* of f is the variety of the ideal $\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$. If this ideal has no zeros, we say that the hypersurface $\mathcal{V}(f)$ is *smooth*. Smoothness is a very important condition. It tells us that our variety can be locally approximated by a linear space - the tangent space.

Let $I = \langle f_1, f_2, \dots, f_k \rangle \subset K[\mathbf{x}]$ be a prime ideal, defining an irreducible variety $Y = \mathcal{V}(I)$ in K^n of dimension d. A point $p \in Y$ is singular if and only if the rank of the Jacobian matrix at p is smaller than the codimension:

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix} (p) \quad < \quad n - d.$$

A point that is not singular is called *smooth*. For a smooth point the inequality above turns into equality. The singular locus $\operatorname{Sing}(Y)$ is a variety in K^n. Its ideal is the sum of the ideal I and the ideal generated by $(n-d) \times (n-d)$ minors of the Jacobian matrix. The kernel of this matrix evaluated at the point p is, by definition, the vector space parallel to the tangent space to V at p. Hence, the definition of the smooth point assures that the tangent space and the variety are of the same dimension.

If a variety $X \subset K^n$ is reducible and p lies in more than one irreducible component of X, then p is singular in X. If p belongs to a unique irreducible component Y then p is singular in X if and only if it is singular in Y.

2.2. Projective varieties

The geometric objects we encountered so far are subsets of K^n. We called them *varieties*, but more precisely we should refer to them as *affine varieties*. We now change our perspective by focusing on *projective varieties*.

We start by recalling the construction of the projective space $\mathbb{P}(V)$ over a vector space V of dimension $n+1$. The points of $\mathbb{P}(V)$ are the lines through the origin in V. Hence $[a_0 : \cdots : a_n] \in \mathbb{P}(V)$ represents a line going through the point $(a_0, \dots, a_n) \in V$. Here not all a_i are zero. Formally, $\mathbb{P}(V)$ is the set of equivalence classes $[v]$, for $v \in V \backslash \{0\}$, modulo the relation $v_1 \sim v_2$ if and only if $v_1 = \lambda v_2$ for some $\lambda \in K^* = K \backslash \{0\}$. For the topological construction over $\mathbb{R}$ or $\mathbb{C}$, we note that each line through the origin in V intersects the unit sphere precisely in two points. Thus $\mathbb{P}(V)$ may be regarded as a quotient of the sphere, identifying antipodal points. In particular, $\mathbb{P}(V)$ is compact with respect to the classical topology. On the subset $S_i = \{a_i \neq 0\}$ of $\mathbb{P}(V)$ we rescale to get $a_i = 1$. We thus identify S_i with K^n. The affine spaces $S_i = K^n$ cover $\mathbb{P}^n := \mathbb{P}(V)$, because every point has some nonzero coordinate. We obtain projective n space $\mathbb{P}^n$ by glueing these $n+1$ charts.

As before, we are interested in functions on $\mathbb{P}^n$. The first problem is that, for a polynomial f, it does not make sense to evaluate f on $[a_0 : \cdots : a_n]$, as the result depends on the choice of representative. It may even happen that f vanishes for some representatives, but not for others. Thus, we focus on *homogeneous* polynomials, i.e. linear combinations of monomials of fixed degree. If f is a homogeneous polynomial of degree d in $n+1$ variables, then $f(ta_0, \ldots, ta_n) = t^d f(a_0, \ldots, a_n)$. In particular, f vanishes on some representative of $[a_0 : \cdots : a_n]$ if and only if it vanishes on any representative.

Let $f_1, \ldots, f_k$ be homogeneous polynomials in $K[\mathbf{x}]$. They are allowed to have distinct degrees. We define the associated *projective variety*:

$$\mathcal{V}(f_1, \ldots, f_k) \;=\; \bigl\{[a_0 : \cdots : a_n] \in \mathbb{P}(V) \;:\; f_i(a_0, \ldots, a_n) = 0 \text{ for } i = 1, \ldots, k\bigr\}.$$

An ideal I in $K[\mathbf{x}]$ is *homogeneous* if it is generated by homogeneous polynomials $f_1, \ldots, f_k$. Just like in the affine case, we set $\mathcal{V}(I) := \mathcal{V}(f_1, \ldots, f_k)$.

Remark 2.16. Homogeneous ideals contain (many) nonhomogeneous polynomials. For instance, $\langle x + y^2, y\rangle$ is a homogeneous ideal. See Exercise 11.

For any projective variety $X \subset \mathbb{P}^n$ one defines the *affine cone* $\hat{X}$ over it, i.e. the variety defined by the same ideal, but in $V = K^{n+1}$. The dimension and degree of a projective variety can be defined via its affine cone:

$$\dim(X) \;:=\; \dim(\hat{X}) - 1 \quad \text{and} \quad \operatorname{degree}(X) \;:=\; \operatorname{degree}(\hat{X}). \tag{2.4}$$

It is usually preferable to work with projective varieties. Algebraic geometry is simpler in $\mathbb{P}^n$ than in K^n. For instance, parallel lines in K^2 do not intersect, but any two lines in $\mathbb{P}^2$ intersect. If X is any projective variety of degree ≥ 2 then the affine cone $\hat{X}$ is always singular at the point $0 \in V$. However, if this is the only singular point of $\hat{X}$ then $X \subset \mathbb{P}^n$ is smooth.

If Y is any variety in K^n then there is an associated projective variety $\bar{Y}$ in $\mathbb{P}^n$, called the *projective closure* of Y. This is defined via its ideal. If $I \subset K[x_1, \ldots, x_n]$ is the ideal of Y then the ideal $\bar{I}$ of $\bar{Y}$ lives in $K[x_0, x_1, \ldots, x_n]$. It is generated by the following infinite set of homogeneous polynomials:

$$\bigl\{\, x_0^{\deg(g)} \cdot g\bigl(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\bigr) \;:\; g \in I \,\bigr\}. \tag{2.5}$$

Here is an algorithm for computing the ideal $\bar{I}$ of the projective closure $\bar{Y}$.

Proposition 2.17. *Let I be an ideal in $K[x_1, \ldots, x_n]$ and let $\mathcal{G}$ be its reduced Gröbner basis for a degree-compatible monomial ordering. Then $\bar{I}$ is generated by the homogeneous polynomials in (2.5) where g runs only over $\mathcal{G}$.*

Proof. Let $f = f(x_0, x_1, \ldots, x_n)$ be any homogeneous polynomial in $\bar{I}$. Suppose $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$. The dehomogenization $f(1, x_1, \ldots, x_n)$ lies in I and hence its normal form modulo the Gröbner basis $\mathcal{G}$ is zero. This gives a representation

$$f(1, x_1, \ldots, x_n) \quad = \quad \sum_{i=1}^{s} h_i(x_1, \ldots, x_n) g_i(x_1, \ldots, x_n),$$

where $\deg(h_i g_i) \leq \deg(f)$ for all i. By homogenizing the summands in this identity,

$$f(x_0, x_1, \ldots, x_n) \quad = \quad \sum_{i=1}^{s} x_0^{\deg(f)-\deg(h_i g_i)} \cdot h_i(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \cdot g_i(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

Hence f lies in the ideal generated by the set (2.5) with I replaced by $\mathcal{G}$. □

Corollary 2.18. *The dimension and degree of an affine variety $Y \subset K^n$ are preserved when passing to its projective closure $\bar{Y} \subset \mathbb{P}^n$:*

$$\dim(\bar{Y}) \,=\, \dim(Y) \quad \text{and} \quad \deg(\bar{Y}) \,=\, \deg(Y).$$

Proof. The initial ideal of $\bar{I}$ and the initial ideal of I have the same generators. These monomials in $x_1, \ldots, x_n$ determine dimension and degree. □

Example 2.19. Let I be the ideal generated by $x_i - x_1^i$ for $i = 2, 3, \ldots, n$. Then $Y = \mathcal{V}(I)$ is a curve of degree n in K^n. For the degree-lexicographic monomial order $\prec$, the reduced Gröbner basis has $\binom{n}{2}$ elements, and $\text{in}_\prec(I) = \langle x_1, x_2, \ldots, x_{n-1} \rangle^2$. The ideal $\bar{I}$ is minimally generated by the 2×2-minors of the $2 \times (n-1)$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

The initial monomials of the $\binom{n-1}{2}$ minors are the antidiagonal products. The projective variety $\bar{Y} = \mathcal{V}(\bar{I})$ is the rational normal curve of degree n in $\mathbb{P}^n$.

We now return to our discussion of desirable properties of projective varieties.

Remark 2.20. If $K = \mathbb{C}$ or $K = \mathbb{R}$ then every projective variety is compact in the classical topology. Indeed, the projective space $\mathbb{P}^n$ is compact, and every subvariety X is closed in the classical topology. Hence X is compact. If X is also smooth of dimension d then X is a compact real manifold, of dimension d if $K = \mathbb{R}$ and of dimension $2d$ if $K = \mathbb{C}$. Many interesting manifolds arise in this manner.

The following theorem discussed in greater detail in Chapter 4, Theorem 4.18 shows one aspect of a nice behaviour of projective varieties over $\mathbb{C}$.

Theorem 2.21. *Over an algebraically closed field, the image of a projective variety X under a polynomial map (that is defined on all of X) is Zariski closed.*

Another nice property of projective varieties is their behavior under intersection.

Theorem 2.22. [**46**, 6.2 Theorem 6] *Fix an algebraically closed field K. Let X, Y be two projective varieties in the n-dimensional ambient space $\mathbb{P}^n$, where $d_1 = \dim(X)$ and $d_2 = \dim(Y)$. Then their intersection $X \cap Y$ has dimension at least $d_1 + d_2 - n$. In particular, if $d_1 + d_n \geq n$ then $X \cap Y$ is always non-empty.*

The hypotheses are needed in this theorem. Consider the intersection of two surfaces in $\mathbb{P}^3$, where $n = 3$ and $d_1 = d_2 = 2$. The statement fails in affine space $\mathbb{C}^3$ where we can take two parallel planes. It also fails in $\mathbb{P}^3$ if the field is $K = \mathbb{R}$.

Example 2.23. Consider the two surfaces $X = \mathcal{V}(x_0^2 + x_1^2 - x_2^2 + x_3^2)$ and $Y = \mathcal{V}(x_0^2 + x_1^2 + x_2^2 - x_3^2)$ in $\mathbb{P}^3$. Over $\mathbb{C}$, their intersection is the union of four lines, so $\dim(X \cap Y) = 1 = 2 + 2 - 3$ as expected. However, over $\mathbb{R}$, the intersection consists of two points, so $\dim(X \cap Y) = 0 < 1$, which would violate Theorem 2.22.

Many models in the sciences and engineering are given by homogeneous polynomial equations. Typically, these constraints arise from a construction familiar from linear algebra. Whenever one encounters such a model then it makes much sense to regard it as a projective variety. We close this section with two examples.

Example 2.24 (Nilpotent Matrices)**.** An $n \times n$-matrix A is a point in a projective space $\mathbb{P}^{n^2-1}$. The set of nilpotent matrices A is an irreducible projective variety $X \subset \mathbb{P}^{n^2-1}$. We have $\dim(X) = n^2 - n - 1$ and $\text{degree}(X) = n!$. Indeed, X is a complete intersection. Its prime ideal $\mathcal{I}(X)$ is generated by the coefficients of the characteristic polynomial of A. For instance, if $n = 2$ then $\mathcal{I}(X) = \langle \text{trace}(A), \det(A) \rangle$.

Example 2.25 (Kalman Varieties)**.** In control theory, one is interested in the set of $n \times n$-matrices A that have an eigenvector in a given linear subspace of K^n. This set is a projective variety in $\mathbb{P}^{n^2-1}$. For instance, let $n = 4$ and consider 4×4-matrices that have an eigenvector with the last two coordinates zero. This Kalman variety has dimension 13 and degree 4 in $\mathbb{P}^{15}$. It is defined by the 2×2-minors of

$$\begin{pmatrix} a_{31} & a_{41} & a_{11}a_{31}+a_{21}a_{32}+a_{31}a_{33}+a_{34}a_{41} & a_{11}a_{41}+a_{21}a_{42}+a_{31}a_{43}+a_{41}a_{44} \\ a_{32} & a_{42} & a_{12}a_{31}+a_{22}a_{32}+a_{32}a_{33}+a_{34}a_{42} & a_{12}a_{41}+a_{22}a_{42}+a_{32}a_{43}+a_{42}a_{44} \end{pmatrix}.$$

2.3. Geometry in Low Dimensions

Smooth projective varieties in low dimensions furnish interesting manifolds. Studying the geometry and topology of these manifolds leads to valuable insights that prove to be very useful also for understanding higher-dimensional scenarios.

We work in projective spaces over the real numbers $\mathbb{R}$ and over the complex numbers $\mathbb{C}$. To distinguish these, we use the notations $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n_{\mathbb{C}}$. We regard both of these as compact real manifolds, of dimension n and $2n$ respectively. Students of topology are encouraged to review the homology groups of these manifolds.

Let us start with $n = 1$. The real projective line $\mathbb{P}^1_{\mathbb{R}}$ is a circle. The complex projective line $\mathbb{P}^1_{\mathbb{C}}$ is a sphere, known as the *Riemann sphere*. Every subvariety of $\mathbb{P}^1_{\mathbb{R}}$ or $\mathbb{P}^1_{\mathbb{C}}$ is a finite collection of points, defined by a *binary form* $f(x, y)$, i.e. a homogeneous polynomial in two variables. For instance, let $f = x^{11}y - 11x^6y^6 - xy^{11}$. The variety $X = \mathcal{V}(f)$ has dimension 0 and degree 12 in $\mathbb{P}^1_{\mathbb{C}}$. These 12 points on the Riemann sphere are famous in the history of geometry and arithmetic. They serve as the vertices of the icosahedron in Felix Klein's *Lectures on the Icosahedron.* Out of these 12 complex solutions four are real. The remaining eight come in four conjugate pairs.

We now move on to the $n = 2$ case. The real projective plane $\mathbb{P}^2_{\mathbb{R}}$ is a surface. However, it cannot be embedded homeomorphically in $\mathbb{R}^3$ (only in $\mathbb{R}^4$), thus it is impossible to make a good picture. The simplest curve in the projective plane $\mathbb{P}^2_K$ is a line L, defined by one linear form in three variables. Of course, L is a projective line $L \simeq \mathbb{P}^1_K$, so the discussion in the previous paragraph applies to L. The complement $\mathbb{P}^2_K \backslash L$ is the affine plane K^2. In particular, this complement is connected when $K = \mathbb{R}$. The decomposition into L and K^2 may be used to give a schematic picture of $\mathbb{P}^2_{\mathbb{R}}$. We identify $\mathbb{R}^2$ with the interior of the square. The boundary of the square should represent the line L. However, we need to identify the opposite points of the boundary - this is often represented by putting directed arrows on the boundary as in Figure 1.

For any curve C in $\mathbb{P}^2_{\mathbb{C}}$, the complement $\mathbb{P}^2_{\mathbb{C}} \backslash C$ is connected because C is a surface in the 4-dimensional manifold $\mathbb{P}^2_{\mathbb{C}}$. By contrast, consider a smooth conic C in $\mathbb{P}^2_{\mathbb{R}}$. Then $\mathbb{P}^2_{\mathbb{R}} \backslash C$ has two connected components. One is a disk and the other is a Möbius strip (cf. Figure 2). The former is the *inside* of C and the latter is the *outside* of C. A curve D in $\mathbb{P}^2_{\mathbb{R}}$ is called a *pseudoline* if $\mathbb{P}^2_{\mathbb{R}} \backslash D$ is connected and it is an *oval* otherwise. Every oval behaves like a conic in $\mathbb{P}^2_{\mathbb{R}}$: it has an inside and an outside.

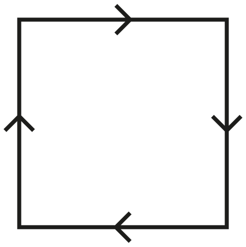


Figure 1. Schematic representation of $\mathbb{P}^2_{\mathbb{R}}$.

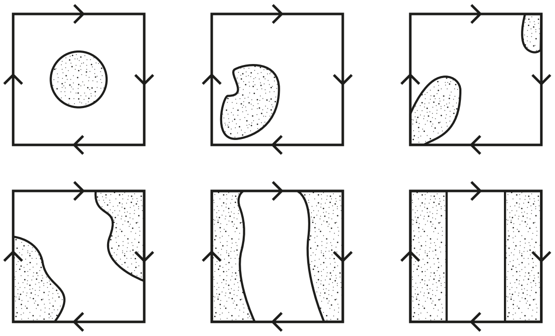


Figure 2. Six topological ovals in $\mathbb{P}^2_{\mathbb{R}}$ with shaded interiors. It is possible to cut out the interior of the lower right oval from the square and glue together the remaining antipodal points on the boundary. This shows that indeed the complement of the interior of the oval is the Möbius strip.

Theorem 2.26. *Let C be a smooth curve of degree d in the projective plane $\mathbb{P}^2_K$. If $K = \mathbb{C}$ then C is an orientable surface of genus $g = \frac{(d-1)(d-2)}{2}$. If $K = \mathbb{R}$ then C is a curve with at most $g+1$ connected components. If d is even then all components are ovals. If d is odd then one component is a pseudoline but all others are ovals.*

Let us illustrate the above theorem for $d = 3$, i.e. $g = 1$. For example consider a cubic curve given in its Weierstrass form:

$$f(x, y, z) = zy^2 - x^3 - xz^2.$$

We decompose the projective space $\mathbb{P}^2_{\mathbb{R}}$ into a line L given by the equation $z = 0$ and its complement: the affine space $A = \mathbb{R}^2$. The cubic has two components: an oval and a pseudoline. We can see both of them by intersecting

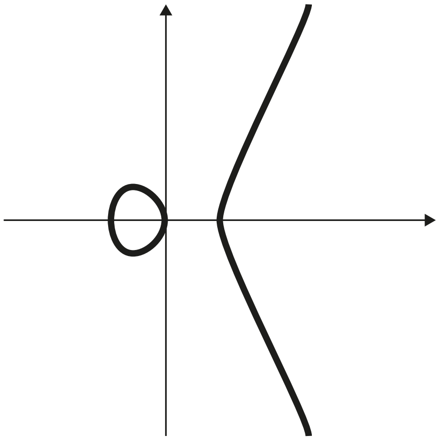


Figure 3. Real elliptic curve $y^2 = x^3 - x$. The component on the left is an oval. The component on the right is a pseudoline. For a more complete picture see Figures 4 and 5.

C with A, as depicted in Figures 3 and 5. There is an additional point P of the curve we do not see on this picture, that belongs to the line L. It is given by $z = x = 0$ and $y = 1$. We may consider the surface in $\mathbb{R}^3$ that is the affine cone over our curve, as in Figure 4.

How can we imagine the complex elliptic curve? This is not simple, as the correct picture just of the affine part would be in $\mathbb{C}^2 \simeq \mathbb{R}^4$. However, there exists a homeomorphism (but not a polynomial map!) of the complex curve C with the real topological torus, i.e. the product of two circles $S^1 \times S^1$. It can be described as follows. We fix a point $p \in C$. For any point $q \in C$ consider a path γ from p to q - this is always possible as over the complex numbers C is connected. To a point q we associate the *complex* number $\int_\gamma \frac{dx}{y}$. Identifying the complex plane with $\mathbb{R}^2$ we obtain a map $f : C \to \mathbb{R}^2$. It turns out that $f(q)$ depends on our choice of γ. Indeed, let us choose p given by $z = 1$, $y = 0$ and $x = -1$. We may choose $q = p$ and γ equal to the oval depicted in Figure 3. The integral $\int_\gamma \frac{dx}{y}$ will be a nonzero real number λ. Thus $f(p)$ may be equal to any integral multiple of λ. Further, on the curve C there exists another loop γ' giving rise to the integral $\int_{\gamma'} \frac{dx}{y}$ that is a complex number τ. We may consider a *lattice* M, that is a subset of $\mathbb{C} \simeq \mathbb{R}^2$ given by all integral combinations $a\lambda + b\tau$ for $a, b \in \mathbb{Z}$. We know that $f(p)$ may be any point in M. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/M$ be the natural projection. The map $\pi \circ f : C \to \mathbb{R}^2/M$ is now well-defined!

As $\mathbb{R}^2/M$ may be identified with the torus, we indeed obtain a homeomorphism $C \simeq \mathbb{R}^2/M$. The real part of the curve C is mapped to two

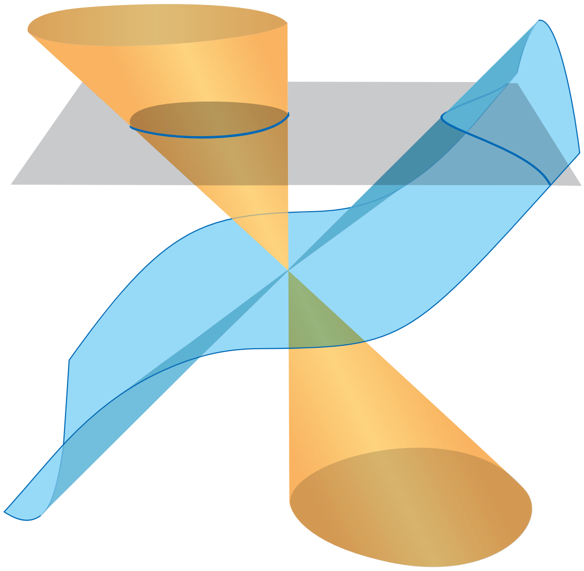


Figure 4. The cone $zy^2 - x^3 - xz^2 = 0$ over an elliptic curve. The *irreducible* variety is depicted in yellow and blue according to two connected components in the real projective space. We intersect the cone with the grey plane given by $z = 1$. This corresponds to the affine chart, and the blue curve C we obtain is exactly the same as in Figure 3.

disjoint circles, as shown in Figure 6. Indeed, both the oval and the pseudoline are circles - they are only distinguished by their embedding in the real projective plane.

Remark 2.27. We contrast the topological torus mentioned here with the *algebraic* torus $(\mathbb{C}^*)^n$ playing a central role in Chapters 8 and 10. Indeed, a variety with a dense *algebraic* torus action will be called *toric*. The elliptic curve C is the most basic example of a smooth, projective variety that is *not* toric.

Remark 2.28. Elliptic curves make probably their first appearance in third century. Diophantus of Alexandria, in modern terms, asked for a (positive) rational point on a specific elliptic curve $y(6 - y) = x^3 - x$. As we argued above, an elliptic curve has a structure of a group (torus). The geometric interpretation of this was already well-known in 19th century. Since early 20th century, elliptic curves play a central role in (modern) number theory (studied mainly over fields of finite characteristic or rational numbers). By the end of the 20th century the group structure (over fields with finite characteristic!) started to be intensively used in applied *cryptography*.

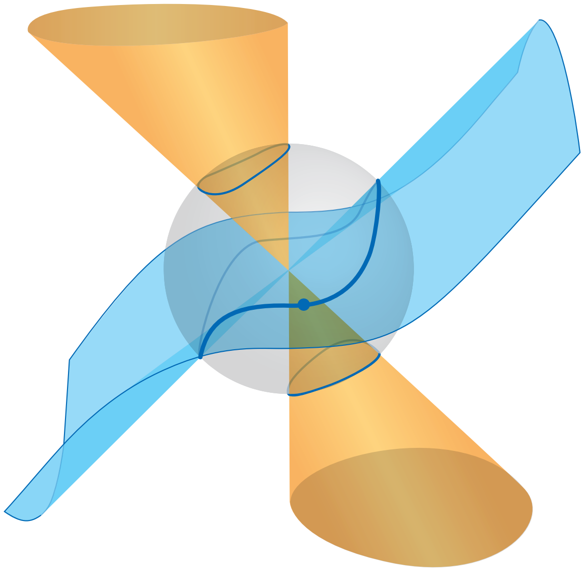


Figure 5. A cone $zy^2 - x^3 - xz^2 = 0$ over an elliptic curve, as in Figure 4. The grey sphere represents the projective space $\mathbb{P}^2_{\mathbb{R}}$, where we have to identify the antipodal points. The intersection of the surface with the sphere has three connected components. Two of them are identified, when we identify the antipodal points. These two components correspond to the oval – indeed cutting it out of the sphere, separates it into two pieces, even after identifying antipodal points. The other component corresponds to a pseudoline. It does not separate a sphere after identifying the antipodal points. The points on the blue curve C in Figure 4 correspond to pairs of antipodal points on the blue curve in this picture with one exception. This curve has one more pair of antipodal points - these are represented by a thickened blue point. Indeed, the line through that point is parallel to the grey plane in Figure 4. This point corresponds to the unique point of the projective curve that does not belong to the affine chart given by $z = 1$. It is precisely $z = 0$, $x = 0$, $y = 1$.

Example 2.29. Let us consider a *cuspidal curve* defined by $x^3 - y^2$. Over $\mathbb{R}$ it is presented in Figure 7. How can we draw it over $\mathbb{C}$? If we identify $\mathbb{C}$ with $\mathbb{R}^2$ we obtain *a surface* in $\mathbb{R}^4$. Indeed:

$$(x_1 + ix_2)^3 - (y_1 + iy_2)^2 = 0 \Leftrightarrow x_1^3 - 3x_1x_2^2 = y_1^2 - y_2^2 \text{ and } 3x_1^2x_2 - x_2^3 = 2y_1y_2.$$

Hence, interpreted as a surface in $\mathbb{R}^4$ the variety is cut out by two polynomials. Although we may not make a picture in $\mathbb{R}^4$ we can project the given surface to $\mathbb{R}^3$. The result is the surface presented in Figure 7, together with the black line, that is the real part.

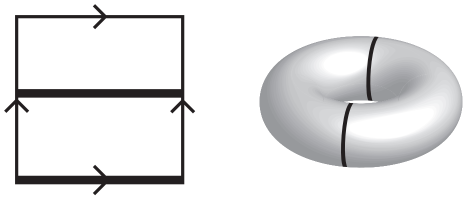


Figure 6. Two pictures showing a real torus, that is homeomorphic to the elliptic curve. The left picture presents the torus as $\mathbb{R}^2/\mathbb{Z}^2$. The right one is the familiar figure we know from topology. The two thickened circles, on both pictures, correspond to the real part of the curve.

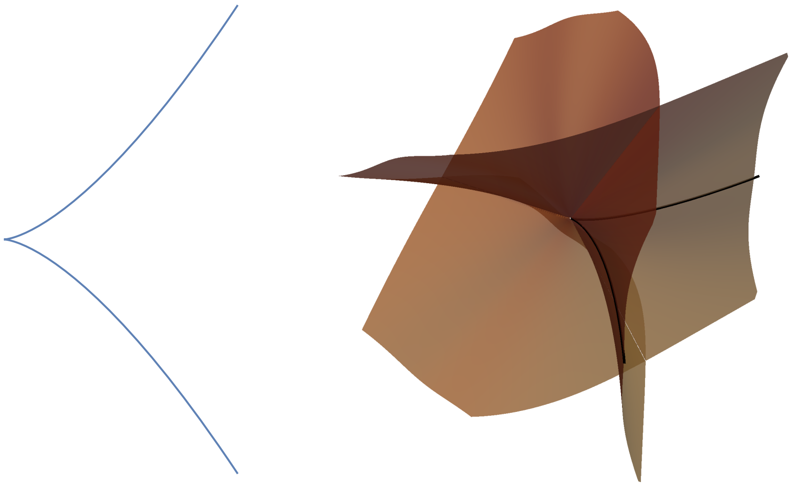


Figure 7. Real part of a cuspidal curve

The surface seems more singular - this is the result of projection. The original surface in $\mathbb{R}^4$ has just one singular point. In Chapter 4 methods allowing the computation of projections of algebraic varieties are presented.

Exercises

(1) Prove that the definition of $\mathcal{V}(I)$ does not depend on the choice of the generators of I.

(2) (a) Show that $J \subseteq I$ implies $\mathcal{V}(I) \subseteq \mathcal{V}(J)$.
(b) Show that for any subsets $A, B \subseteq K^n$ if $A \subset B$ then $\mathcal{I}(B) \subseteq \mathcal{I}(A)$.
(c) Give counterexamples to both opposite implications.

(3) Prove that varieties (in K^n) satisfy the axioms of closed sets.

(4) By identifying the point $(p_i) \in K^n$ with the prime ideal $\langle x_1 - p_1, \dots, x_n - p_n \rangle$ consider K^n as a subset of $\operatorname{Spec} K[\mathbf{x}]$. Show that the Zariski topology induced from $\operatorname{Spec} K[\mathbf{x}]$ to K^n is the Zariski topology on K^n.

(5) Show that a morphism of rings $f : R_1 \to R_2$ induces a map $f^* : \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$, by proving that a pull-back of a prime ideal is prime. Show that the induced map is continuous with respect to the Zariski topology.

(6) Describe the variety $\mathcal{V}(I)$ in the affine line K^1 for $I = \langle x^2 + 1 \rangle$ when $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}$. Also, describe $\mathcal{V}(I) \subset \operatorname{Spec}(K[x])$ for each of these three fields.

(7) Realize the set of $n \times n$ nilpotent matrices as an affine variety. What is its dimension?

(8) (a) Consider a polynomial $f \in K[\mathbf{x}]$ (e.g. $f = x$). Let D be the (open) set $D_f = \{p \in K^n : f(p) \neq 0\}$. Construct an affine variety V and a polynomial map inducing a bijection $V \to D$.
(b) Realize nondegenerate $n \times n$ matrices as an affine variety.

(9) (a) Use (or not) your favorite computer algebra system to determine the ideal of the image of the map given by formula (2.3). What is the meaning of the lowest degree polynomial in this ideal?
(b) Describe the ideal of the image of the map given by formula (2.2).
(c) Generalize the previous point to more (independent) variables possibly with different (but finite) number of states.

(10) Determine for which prime numbers p, the ideal $I_2 = \langle x^2 - 2y^2 \rangle \subset \mathbb{F}_p[x, y]$ is prime.

(11) For a polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} x^{\mathbf{a}}$ we call the degree k part of f the homogeneous polynomial $\sum_{\mathbf{a}:|\mathbf{a}|=\mathbf{k}} c_{\mathbf{a}} x^{\mathbf{a}}$.
(a) Provide an example of a homogeneous ideal generated by nonhomogeneous polynomials.
(b) Prove that an ideal $I = \langle f_1, \dots, f_j \rangle$ is homogeneous if and only if for any f_i and any k the degree k part of f_i belongs to I.
(c) Propose an algorithm that, given a set of generators of $I \subset K[\mathbf{x}]$, decides if I is a homogeneous ideal.

(12) (a) Let $I \subset K[\mathbf{x}]$ be a monomial ideal. Prove that $\mathcal{V}(I)$ is a union of (some) vector subspaces of K^n spanned by basis vectors.
(b) How do you characterize the sets of basis vectors that span a subspace belonging to $\mathcal{V}(I)$?

(13) Draw various pseudolines in $\mathbb{P}^2_{\mathbb{R}}$, in analogy to Figure 2. Topologically, what is the complement of a pseudoline?

(14) Can you solve the problem of Diophantus of Alexandria in Remark 2.28?

Hint: Consider a tangent line to the elliptic curve at the point $(-1, 0)$.

Chapter 3

Solving and Decomposing

Solving systems of polynomial equations is a key task in nonlinear algebra. But, what does it mean to solve such a system? How should the solutions be presented? The answer to this question depends on the dimension of the variety of solutions. If the variety is zero-dimensional then it consists of finitely many points in K^n and we aim to list each point explicitly. If $K = \mathbb{R}$ or $K = \mathbb{C}$ then this is usually done by displaying a floating point approximation to each of the n coordinates of a solution.

If the solution variety has positive dimension then it has infinitely many points and we cannot list them all. In that case, the answer consists of a description of each irreducible component. Algebraically, this leads us to the topic of primary decomposition. If the given ideal is not radical then its constituents are primary ideals and we distinguish between minimal primes and embedded primes. To some readers, these objects may seem unnatural at first. However, they become quite natural in the setting of linear partial differential equations with constant coefficients.

3.1. Zero-dimensional Ideals

Let K be a field and consider the polynomial ring $K[x]$ in one variable x. Every ideal in $K[x]$ is principal, so it has the form $I = \langle f \rangle$. The variety $\mathcal{V}(I)$ consists of the zeros of f and is zero-dimensional (unless $f = 0$). The polynomial f has a unique factorization $f = \prod_{i=1}^{k} g_i^{a_i}$, where each g_i is irreducible and the a_i are positive integers. The set of solutions decomposes as

$$\mathcal{V}(I) \;=\; \mathcal{V}(g_1) \cup \cdots \cup \mathcal{V}(g_k).$$

On the level of ideals we have the following decomposition as an intersection:

$$I \;=\; \langle g_1\rangle^{a_1} \,\cap\, \cdots \,\cap\, \langle g_k\rangle^{a_k}.$$

This primary decomposition remembers the multiplicity a_i of each factor g_i, so it contains more information than the irreducible decomposition of $\mathcal{V}(I)$.

The decomposition depends on the field K. If K is algebraically closed, such as $K = \mathbb{C}$, then each factor g_i is a linear polynomial $g_i(x) = x - u_i$, where $u_1, \ldots, u_k$ are the zeros of f. If $K = \mathbb{R}$ then each g_i is either linear or quadratic. If $K = \mathbb{Q}$ then g_i can have arbitrarily high degree. In each case, the quotient ring $K[x]/\langle g_i\rangle$ is a field. It is an algebraic extension of K.

Example 3.1. The polynomial $f = x^3 - 2x^2 + x - 2 \in \mathbb{R}[x]$ satisfies

$$\langle f\rangle \;=\; \langle x-2\rangle \,\cap\, \langle x^2+1\rangle.$$

The first ideal corresponds to the real zero 2, while the second to the *pair* of complex zeros i and $-i$. Such factorizations are easy to find in a computer algebra systems. What if we now replace the given polynomial by $g = x^3 - 2x^2 + x - 1$? How does the ideal $\langle g\rangle$ decompose in $\mathbb{R}[x]$? And in $\mathbb{C}[x]$?

Polynomials of degree m in one variable can have up to m zeros. The number m can be large. Often we are not interested in all the zeros, but only in specific ones. For instance, we might only be interested in solutions that are real and positive. This restriction is very important for many applications, e.g. in statistics where the solutions represent probabilities.

Example 3.2. Let $I = \langle x^m - x - 1\rangle$, where $m \geq 2$. The variety $\mathcal{V}(I)$ consists of m complex points but only one of them is real and positive. Thus, $\mathcal{V}(I) \cap \mathbb{R}_{>0}$ is a singleton. This follows from *Descartes' Rule of Signs*, which states that the number of positive real solutions is bounded above by the number of sign alternations in the coefficient sequence. If m is even then there is also one negative solution.

In many applications one encounters polynomials whose coefficients depend on parameters. For instance, let ϵ be an unknown that represents a small positive real number. Let $\mathbb{Q}(\epsilon)$ be the field of rational functions in that unknown and $K = \overline{\mathbb{Q}(\epsilon)}$ its algebraic closure. Elements in K can be expressed as series in ϵ with rational exponents. These are known as *Puiseux series*. This is analogous to the floating point expansion of numbers in $\mathbb{R}$.

Example 3.3. The polynomial $f = \epsilon^2 x^3 + x^2 + x - \epsilon$ is irreducible in $\mathbb{Q}(\epsilon)[x]$. It factors into three linear factors $f = (x-u_1)(x-u_2)(x-u_3)$ in $K[x]$, where

$$\begin{aligned}
u_1 \;&=\; -\epsilon^{-2} + 1 + \epsilon^2 + \epsilon^3 + 2\epsilon^4 + 3\epsilon^5 + 5\epsilon^6 + 10\epsilon^7 + \cdots \\
u_2 \;&=\; -1 - \epsilon - 3\epsilon^3 + 3\epsilon^4 - 16\epsilon^5 + 32\epsilon^6 - 121\epsilon^7 + \cdots \\
u_3 \;&=\; \epsilon - \epsilon^2 + 2\epsilon^3 - 5\epsilon^4 + 13\epsilon^5 - 37\epsilon^6 + 111\epsilon^7 + \cdots
\end{aligned}$$

Each of these three roots is an algebraic number over $\mathbb{Q}(\epsilon)$. We wrote them a Puiseux series. If we think of ϵ as a very small positive quantity then $u_1 \sim -\epsilon^{-2}$, $u_2 \sim -\epsilon^0$ and $u_3 \sim \epsilon^1$. The exponents -2, 0 and 1 tell us the asymptotic behavior. They are known as *tropical solutions*; cf. Chapter 7.

We have seen that *solving* a polynomial equation $f = 0$ amounts to *decomposing* the principal ideal $I = \langle f \rangle$, i.e. presenting it as an intersection of simpler ideals. The situation is analogous for systems of polynomials in $n \geq 2$ variables, i.e. ideals $I \subset K[\mathbf{x}]$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Suppose now that K is algebraically closed and assume that $\mathcal{V}(I)$ is zero-dimensional. This means that the quotient ring $K[\mathbf{x}]/I$ is a finite-dimensional vector space over K. A basis is given by the standard monomials for a given monomial order. The number of standard monomials is an upper bound for the cardinality of $\mathcal{V}(I)$. Equality holds if and only if I is radical.

In the next subsection we will decompose our zero-dimensional ideal I as

$$I \;=\; \bigcap_{i=1}^{k} q_i,$$

where $\mathrm{rad}(q_i)$ is a prime ideal. Every prime ideal of dimension 0 in $K[\mathbf{x}]$ is a maximal ideal, so each $\mathrm{rad}(q_i)$ is a maximal ideal. Since K is algebraically closed, $\mathcal{V}(q_i)$ is a point in K^n. These points are the solutions to our system.

Example 3.4. Let $n = 2$ and $I = \langle xy, x^2 - x, y^2 - y \rangle$. This ideal is radical:

$$I \;=\; \langle x, y \rangle \,\cap\, \langle x-1, y \rangle \,\cap\, \langle x, y-1 \rangle.$$

The variety of this ideal consists of three points: $\mathcal{V}(I) = \{(0,0), (1,0), (0,1)\}$.

If the given ideal is not radical then we cannot express it as an intersection of maximal ideals. This should not be surprising: already in the case of one variable, if a root had a multiplicity we needed powers of linear forms.

Example 3.5. Let $I = \langle xy, y^2 - y, x^2y - x^2 \rangle$. We have the decomposition

$$I \;=\; \langle y-1, x \rangle \,\cap\, \langle y, x^2 \rangle.$$

The varieties of both ideals are points: $(0,1)$ and $(0,0)$ respectively. However, the second ideal remembers additional data. It is not just $\langle x, y \rangle$, but indicates a 'multiplicity' of the solution $(0,0)$. We are now equipped to measure this multiplicity! The degree of I equals 3. The first ideal in the decomposition contributes with degree one, while the second with degree 2.

We now discuss an example that was seen in Exercise 8 of Chapter 1.

Example 3.6. Fix the rationals $K = \mathbb{Q}$ and $I = \langle x^3 - yz, y^3 - xz, z^3 - xy\rangle$ in $K[x, y, z]$. This ideal is an irredundant intersection of 11 distinct ideals:

$$\begin{aligned} I \;=\; Q &\cap \langle z-1, y-1, x-1\rangle \,\cap\, \langle z-1, y+1, x+1\rangle \,\cap\, \langle z-1, x+y, y^2+1\rangle \\ &\cap \langle z+1, y-1, x+1\rangle \,\cap\, \langle z+1, y+1, x-1\rangle \,\cap\, \langle z+1, x-y, y^2+1\rangle \\ &\cap \langle y-1, x+z, z^2+1\rangle \,\cap\, \langle y+1, x-z, z^2+1\rangle \\ &\cap \langle y-z, x+1, z^2+1\rangle \,\cap\, \langle x-1, y+z, z^2+1\rangle. \end{aligned}$$

The first intersectand is a primary ideal with radical $\mathrm{rad}(Q) = (x, y, z)$:

$$Q \;=\; \big\langle\, x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, x^3 - yz, y^3 - xz, z^3 - xy \,\big\rangle,$$

Each of the other 10 intersectands is a prime ideal. If we were to replace K by the complex numbers $\mathbb{C}$ then six of the prime ideals decompose further:

$$\langle x-1, y+z, z^2+1\rangle \;=\; \langle x-1, y-i, z+i\rangle \,\cap\, \langle x-1, y+i, z-i\rangle.$$

We learn that $\mathcal{V}(I)$ consists of 17 complex points. Only five are real.

3.2. Primary Decomposition

The idea of decomposing a mathematical object into simpler pieces is important. In this section we present a theory of decomposing ideals. We shall express them as intersections of simpler ideals. Our point of departure is the following proposition. It shows how algebraic varieties may be decomposed.

Proposition 3.7. *Any variety in K^n can be uniquely represented as a finite union of irreducible varieties (pairwise not contained in each other).*

Proof. We start by proving the existence of such a decomposition. Any variety W is either irreducible it is a union $W_1 \cup V_1$. We next write W_1 as a union $W_2 \cup V_2$ etc. We obtain an ascending chain of ideals, $\mathcal{I}(W_1) \subseteq \mathcal{I}(W_2) \subseteq \ldots$. This chain stabilizes by Hilbert Basis Theorem. Thus the decomposition procedure finishes with finitely many irreducible varieties.

Suppose we have two irreducible decompositions of the same variety:

$$V_1 \cup \cdots \cup V_k \;=\; W_1 \cup \cdots \cup W_s.$$

As each W_{i_0} is irreducible and covered by $\bigcup_j (V_j \cap W_{i_0})$ we have $W_{i_0} \subset V_{j_0}$ for some j_0. But similarly $V_{j_0} \subset W_{i_1}$ for some i_1. As we cannot have $W_{i_0} \subsetneq W_{i_1}$ it follows that $W_{i_0} = V_{j_0}$. Hence, for every component W_{i_0} there is a unique component V_{j_0} equal to it. The uniqueness of the decomposition follows. □

In what follows we present a vast generalization of these two basic facts:

(1) Every integer $n > 1$ can be uniquely decomposed as a product of powers of prime numbers:

$$n = p_1^{a_1} \cdots p_k^{a_k}.$$

(2) Any variety can be uniquely decomposed as a union of irreducible varieties - Proposition 3.7.

The algebraic notion of an ideal connects the first (number-theoretic) fact and the second (geometric) fact. Indeed, any integer n can be identified with the ideal $\langle n \rangle$ in the ring $\mathbb{Z}$. The elements of $\langle n \rangle$ are the integer multiples of n. The ideal $\langle n \rangle$ is prime in $\mathbb{Z}$ if and only if n is a prime number. We can restate fact (1) in terms of intersections of powers of prime ideals as follows:

(1') Every nonzero ideal $I \subset \mathbb{Z}$ has a unique decomposition

$$I \;=\; (I_1)^{a_1} \cap \cdots \cap (I_k)^{a_k},$$

where the I_i are prime ideals.

Over an algebraically closed field, we have an identification of varieties with radical ideals (cf. Chapter 6). This yields the following restatement of (2):

(2') Every radical ideal $I \subset \mathbb{C}[\mathbf{x}]$ has a unique decomposition as an intersection of prime ideals, pairwise not contained in each other:

$$I \;=\; p_1 \cap \cdots \cap p_k.$$

These examples suggest that our aim should be to decompose ideals I in a ring R. Here, a decomposition of I is a presentation as an intersection of other ideals. At this point, we need to answer the following questions:

(1) What kind of ideals should be allowed in the intersection?

(2) What restrictions should be put on the ring R?

(3) Can we expect the decomposition to be unique?

We start with the first question. The number-theoretic example suggests all ideals might be intersections of powers of prime ideals. But this is not true.

Example 3.8. The ideal $I = \langle x^2, y \rangle$ is not an intersection of powers of prime ideals in $\mathbb{C}[x, y]$. Indeed, suppose $I = \bigcap_i p_i^{a_i}$. For all i, we have $p_i \supset I$. Hence $p_i = \langle x, y \rangle$ as this is the only prime ideal containing I. The ideal $\bigcap_i \langle x, y \rangle^{a_i}$ would be a power of $\langle x, y \rangle$, whereas I is no such power.

The right constituents are *primary* ideals. Recall that I is primary if and only if $ab \in I$ and $a \notin I$ implies $b^n \in I$ for some n, given any $a, b \in R$.

Next consider question (2): which rings R to take? Clearly, $\mathbb{Z}$ and $K[\mathbf{x}]$ share a lot of nice properties. But there is a larger class of rings that works.

Definition 3.9. A ring R is *Noetherian* if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

stabilizes, i.e. there exists k such that $I_k = I_{k+1} = I_{k+2} = \ldots.$

Noetherian rings are named after the German algebraist Emmy Noether. A hint how important they are is given in Exercises 4 and 5. Note that $\mathbb{Z}$ and $K[\mathbf{x}]$ are Noetherian rings because their ideals are finitely generated. Before stating our main existence theorem, let us introduce a technical definition.

Definition 3.10. An ideal I in a ring R is *irreducible* if and only if whenever $I = J_1 \cap J_2$ for some ideals J_1, J_2 in R then $I = J_1$ or $I = J_2$.

Theorem 3.11. *Let I be an ideal in a Noetherian ring R. Then there exist primary ideals $q_1, q_2, \ldots, q_k$ in R such that*

$$I \;=\; q_1 \cap q_2 \cap \cdots \cap q_k.$$

Proof. First we show that every ideal in R is a finite intersection of irreducible ideals. Suppose not, and let I_1 be an ideal that cannot be presented in this way. In particular, it is not irreducible. Thus, $I_1 = J_1 \cap J_2$ and each J_i strictly contains I_1. If J_1 and J_2 are finite intersections of irreducible ideals, then so is I_1. Hence, we may assume J_1 cannot be presented as such a finite intersection. Let $I_2 := J_1$. We have $I_1 \subsetneq I_2$. We repeat the construction starting with I_2 and get an ideal I_3 with $I_1 \subsetneq I_2 \subsetneq I_3$, where I_3 is not a finite intersection of irreducible ideals. Continuing, we get a chain of strictly ascending ideals. However, this is not possible in a Noetherian ring.

We next prove that every irreducible ideal q is primary. By replacing the ring R with R/q, we may assume $q = \{0\}$. Suppose $ab = 0$ and $a \neq 0$. We must prove that b is nilpotent. Consider the following ascending chain:

$$\{x \in R : bx = 0\} \;=: \operatorname{Ann}(b) \;\subseteq\; \operatorname{Ann}(b^2) \;\subseteq\; \operatorname{Ann}(b^3) \;\subseteq\; \ldots$$

Since R is Noetherian, ascending chains of ideals become stationary. Hence $\operatorname{Ann}(b^n) = \operatorname{Ann}(b^{n+1})$ for some n. We claim that $\langle a \rangle \cap \langle b^n \rangle = \{0\}$. Indeed, suppose $\lambda a = \mu b^n \in \langle a \rangle \cap \langle b^n \rangle$ for some $\lambda, \mu \in R$. Clearly,

$$0 \;=\; \lambda a b \;=\; \mu b^{n+1}.$$

Hence, $\mu \in \operatorname{Ann}(b^{n+1}) = \operatorname{Ann}(b^n)$. Thus, $\mu b^n = 0$. As $\{0\}$ was assumed to be irreducible, and $\langle a \rangle \supsetneq \{0\}$, we have $b^n = 0$. This completes the proof. □

We now pass to the third question, concerning uniqueness. We need not assume that R is Noetherian, as long as the ideal I in question is an intersection of finitely many primary ideals: $I = \bigcap_{i=1}^{k} q_i$. Here it is assumed that each q_i is necessary, i.e. $\bigcap_{j \neq i_0} q_j \not\subset q_{i_0}$ for all $1 \leq i_0 \leq k$. The next two lemmas suggests grouping the primary ideals q_i by their radical.

Lemma 3.12. *The radical of a primary ideal q is the unique smallest prime ideal containing it.*

The proof is left as Exercise 6 for the reader. A primary ideal q with radical equal to p is called p-primary. An easy way to create a primary ideal is to take a power of a prime ideal. However, it is not true that the power of a prime ideal is always a primary ideal, even in the polynomial ring $\mathbb{C}[\mathbf{x}]$.

Example 3.13. Let P be the ideal generated by the nine 2×2 minors of a 3×3 matrix $X = (x_{ij})$ of unknowns. This ideal is prime and it contains none of the x_{ij}. We claim that the ideal P^2 is not primary. To see this, we verify (using Gröbner bases) that $x_{ij} \cdot \det(X)$ lies in P^2 for all $1 \leq i, j \leq 3$. However, P^2 is generated by quartics and contains no cubics. Thus, neither $\det(P)$ nor any power of x_{ij} is in P^2. We conclude that P^2 is not primary.

In what follows, our standing assumption is that R is a Noetherian ring. We focus on p-primary ideals for a fixed prime ideal p.

Lemma 3.14. *If $q_1, \ldots, q_k$ are p-primary ideals, then so is $q_1 \cap \cdots \cap q_k$.*

Proof. The following shows that the radical of $I := \bigcap_{i=1}^k q_i$ equals p:

$$\begin{aligned} a \in \operatorname{rad}(I) &\iff \exists n : a^n \in I \iff \exists n \, \forall i : a^n \in q_i \\ &\iff \forall i : a \in \operatorname{rad}(q_i) = p \iff a \in p. \end{aligned}$$

To see that I is primary, we assume that $ab \in I$ and $a \notin I$. Then $a \notin q_{i_0}$ for some i_0. Since $ab \in q_{i_0}$ and q_{i_0} is primary, $b \in \operatorname{rad}(q_{i_0}) = p = \operatorname{rad}(I)$. Hence $b^n \in I$ for some n. □

Lemma 3.14 suggests that, given any primary decomposition $I = \bigcap_{i=1}^k q_i$, we aggregate the q_i's with the same radical and replace them by their intersection. The result is still a primary decomposition of I. This motivates the following definition. A *minimal primary decomposition* is a representation

$$I = q_1 \cap q_2 \cap \cdots \cap q_k. \tag{3.1}$$

where the q_i's are primary ideals that have pairwise distinct radicals and the intersection is irredundant, meaning $\bigcap_{j \neq i_0} q_j \not\subset q_{i_0}$ for all $1 \leq i_0 \leq k$.

To sum up, we have proved the following result for Noetherian rings R:

(1) every ideal has a (finite) primary decomposition, and

(2) every(finite) primary decomposition of an ideal can be changed to a minimal one (apply Lemma 3.14 and remove unnecessary ideals).

We next show that minimal primary decompositions may still be not unique.

Example 3.15. The following are two minimal primary decompositions:

$$\langle x^2, xy \rangle = \langle x \rangle \cap \langle x, y \rangle^2 = \langle x \rangle \cap \langle x^2, y \rangle \quad \subset \quad \mathbb{C}[x, y]. \tag{3.2}$$

It turns out that, while the primary ideals q_i in the decomposition (3.1) need not be unique, their radicals are unique. Recall that the quotient of an ideal I by a ring element a is the ideal $(I : a) = \{b \in R : ab \in I\}$.

Theorem 3.16. *For any ideal I in a ring R, the set of k prime ideals $\operatorname{rad}(q_i)$ arising in a minimal primary decomposition (3.1) does not depend on the choice of that decomposition. They are precisely the prime ideals of the form $\operatorname{rad}(I : a)$ for some $a \in R$. If R is Noetherian then the last radical is not needed: they are precisely the prime ideals $(I : a)$ for some $a \in R$.*

Proof. Fix a minimal primary decomposition $I = \bigcap_{i=1}^{k} q_i$. Intersection commutes with ideal quotients, so $(I : a) = \bigcap_{i=1}^{k}(q_i : a) = \bigcap_{a \notin q_j}(q_j : a)$. It also commutes with radicals. Hence $\operatorname{rad}(I : a) = \bigcap_{i=1}^{k} \operatorname{rad}(q_i : a) = \bigcap_{a \notin q_j} \operatorname{rad}(q_j : a)$. We next argue that $a \notin q_i$ implies $\operatorname{rad}(q_i : a) = \operatorname{rad}(q_i)$. Suppose $b \in \operatorname{rad}(q_i : a)$, i.e. $b^n a \in q_i$. As q_i is primary and $a \notin q_i$, we have $(b^n)^m \in q_i$, i.e. $b \in \operatorname{rad}(q_i)$. Hence, $\operatorname{rad}(q_i : a) \subset \operatorname{rad}(q_i)$ and the other inclusion is obvious. At this point we conclude that $\operatorname{rad}(I : a)$ equals the intersection of the prime ideals $\operatorname{rad}(q_j)$ satisfying $a \notin q_j$.

By Exercise 8, if $\operatorname{rad}(I : a)$ is prime then it equals $\operatorname{rad}(q_j)$ for some j. Next, consider any $\operatorname{rad}(q_{i_0})$. As the primary decomposition is minimal, there exists $a \in \bigcap_{j \neq i_0} q_j \backslash q_{i_0}$. The conclusion above shows $\operatorname{rad}(I : a) = \operatorname{rad}(q_{i_0})$.

It remains to prove the last assertion. If $(I : a)$ is prime, then it equals its radical. Thus, we must consider a prime ideal of the form $\operatorname{rad}(I : a)$ and show that it equals $(I : a')$ for some $a' \in I$. We already know that $\operatorname{rad}(I : a) = \operatorname{rad}(q_{i_0})$ for some i_0. By Exercise 9, $\operatorname{rad}(q_{i_0})^n \subset q_{i_0}$ for some positive integer n. Hence, there exists n such that $(\bigcap_{j \neq i_0} q_j) \cdot (\operatorname{rad}(q_{i_0}))^n \subseteq I$. We fix the smallest n with this property. Then we pick

$$a' \in \left((\bigcap_{j \neq i_0} q_j) \cdot (\operatorname{rad}(q_i))^{n-1} \right) \setminus I.$$

(Here, if $n = 1$ then $\operatorname{rad}(q_i)^{n-1}$ equals the ring R.) By definition, $a' \cdot \operatorname{rad}(q_i) \subseteq I$, and thus $\operatorname{rad}(q_i) \subseteq (I : a')$. However, $a' \in (\bigcap_{j \neq i_0} q_j) \backslash I$, thus $a' \notin q_{i_0}$. We have the inclusions $\operatorname{rad}(q_i) \subseteq (I : a') \subseteq \operatorname{rad}(I : a') = \operatorname{rad}(q_i)$, which are in fact equalities. The last equation follows from the previous paragraph. □

Definition 3.17. The *associated primes* of an ideal I are the radicals of the primary ideals appearing in a minimal primary decomposition. Equivalently, these are the prime ideals of the form $\operatorname{rad}(I : a)$ for some element a of the ring. If the ring is Noetherian, these are the prime ideals of the form $(I : a)$.

Before going further, let us discuss the geometric meaning of the associated primes. If $I = \bigcap_{i=1}^{k} q_i$ then $\operatorname{rad}(I) = \bigcap_{i=1}^{k} \operatorname{rad}(q_i)$. Thus, every component in the irreducible decomposition of the variety $\mathcal{V}(I)$ corresponds to one of the associated primes of I. However, the converse is not true.

Example 3.18. Let $I = \langle x^2, xy \rangle$ as in Example 3.15. We have $\mathrm{rad}(I) = \langle x \rangle$, i.e. the variety $\mathcal{V}(I)$ is irreducible - a line in a plane. However, the minimal primary decompositions (3.2) reveal that I has two associated primes. The expected prime $\langle x \rangle$ and the unexpected prime $\langle x, y \rangle$ - a point on the line. Thus, the associated primes remember more information than just the variety. There is a 'hidden' - embedded - point on that line whose ideal is I.

The formal replacement of varieties (corresponding to radical ideals) by arbitrary ideals allowed a tremendous advance of 20th century algebraic geometry. One is now able to work with 'functions' that are nonzero, but their square is zero, using basic, well-understood algebra. This advance should be compared to the introduction of complex numbers in 18th and 19th century, where (basically in the same way) instead of answering the question 'does there exist a square root of -1?' one introduces imaginary numbers and shows how to use them in an efficient way. Still, we should not forget the classical geometry we started from. The line from Example 3.18 is of a different nature than the point, and these two should be distinguished.

Definition 3.19. For an ideal I, let $\mathrm{Ass}(I)$ be the set of associated primes. The minimal (with respect to inclusion) elements of $\mathrm{Ass}(I)$ are the *minimal primes* of I. Associated primes that are not minimal are called *embedded*.

An embedded prime p of an ideal I must contain a minimal prime p'. This means that the irreducible component $\mathcal{V}(p')$ of $\mathcal{V}(I)$ strictly contains the irreducible variety $\mathcal{V}(p)$. We do not see $\mathcal{V}(p)$ geometrically inside $\mathcal{V}(I)$: it is *embedded* in $\mathcal{V}(p')$. Further the minimal primes correspond exactly to irreducible components of $\mathcal{V}(I)$. They are the irredundant intersectands in

$$\mathrm{rad}(I) \;=\; \bigcap_{i=1}^{k} \mathrm{rad}(q_i).$$

The next lemma offers an another explanation for the name minimal primes.

Lemma 3.20. *A prime ideal is a minimal prime of I if and only if it is a minimal element (with respect to inclusion) among the primes that contain I.*

Proof. It is enough to prove that every prime p containing I contains also a prime in $\mathrm{Ass}(I)$. Then p also contains a minimal prime. They are equal if p is minimal with respect to inclusion. Thus, suppose p contains $I = \bigcap_{i=1}^{k} q_i$. By Exercise 8, $p \supseteq q_{i_0}$ for some some i_0. Hence, $p = \mathrm{rad}(p) \supset \mathrm{rad}(q_{i_0})$. □

The geometry that distinguishes embedded and minimal primes suggests an idea how to get additional uniqueness properties in primary decompositions. Indeed, in Example 3.15 it is the ideal corresponding to the embedded component that changes, while the minimal prime remains the same.

Theorem 3.21. *Let $I = \bigcap_{i=1}^k q_i$ be a minimal primary decomposition. The primary ideals q_i corresponding to minimal primes are determined by I.*

Proof. Let q_{i_0} be such that $\mathrm{rad}(q_{i_0})$ is a minimal prime. We claim that

$$q_{i_0} \;=\; \bigl\{a \,:\, ab \in I \text{ for some } b \notin \mathrm{rad}(q_{i_0})\bigr\}.$$

We already saw that the right hand side does not depend on the decomposition of I. Thus the equation implies the theorem. We prove both inclusions.

Let $a \in q_{i_0}$. For every $i \neq i_0$ we have $q_i \not\subset \mathrm{rad}(q_{i_0})$. Otherwise, $\mathrm{rad}(q_i) \subset \mathrm{rad}(q_{i_0})$, which would contradict the hypothesis that $\mathrm{rad}(q_{i_0})$ is minimal. Hence, there exists $b_i \in q_i \backslash \mathrm{rad}(q_{i_0})$. We define $b := \prod_{j \neq i_0} b_j$. As $\mathrm{rad}(q_{i_0})$ is prime we have $b \notin \mathrm{rad}\, q_{i_0}$. However, $ab \in q_j$ for $j \neq i_0$, as $b \in q_j$. Furthermore, $ab \in q_{i_0}$, as $a \in q_{i_0}$. This implies $ab \in I = \bigcap_{i=1}^k q_i$. This means that a is contained in the right hand side.

Now we pick a and $b \notin \mathrm{rad}(q_{i_0})$ such that $ab \in I$. In particular, $ab \in q_{i_0}$. If $a \notin q_{i_0}$ we get a contradiction to the fact that q_{i_0} is primary. This shows that the right hand side is contained in the left hand side. □

Primary decomposition for monomial ideals is easier than for general polynomial ideals. The associated primes are generated by subsets of the variables and they can be characterized combinatorially. We here just show this for one example. For more information we refer to the textbook [**40**].

Example 3.22. Let $n = 3$ and $I = \langle xy^2z^3, x^2yz^3, xy^3z^2, x^3yz^2, x^2y^3z, x^3y^2z\rangle$. This has seven associated primes. A minimal primary decomposition equals

$$I \;=\; \langle x\rangle \cap \langle y\rangle \cap \langle z\rangle \cap \langle x^2, y^2\rangle \cap \langle x^2, z^2\rangle \cap \langle y^2, z^2\rangle \cap \langle x^3, y^3, z^3\rangle.$$

This example generalizes to $n \geq 4$ as follows. The ideal I is generated by the $n!$ monomials $\prod_{i=1}^n x_i^{\pi_i}$, indexed by permutations $\pi \in S_n$, and $\mathrm{Ass}(I)$ consists of all $2^n - 1$ ideals generated by nonempty subsets of $\{x_1, \ldots, x_n\}$.

There are many algorithms and implementations for computing primary decompositions. The input is an ideal I in a polynomial ring $K[\mathbf{x}]$ and the output is the set $\mathrm{Ass}(I)$ and primary ideals $q_1, \ldots, q_k$ satisfying (3.1). Traditionally, these are symbolic methods built upon Gröbner bases. In recent years, numerical tools for decomposing ideals and varieties have received much attention. Solving polynomial systems means running such software.

3.3. Linear PDE with Constant Coefficients

In this section we offer an alternative perspective on the problem of solving systems of polynomial equations. This is aimed at highlighting the role of embedded primes and primary ideals in a context of practical importance.

Every polynomial with real or complex coefficients can be interpreted as a linear differential operator with constant coefficients. This operator is obtained by simply replacing x_i by the differential operator $\frac{\partial}{\partial x_i}$. Every ideal I in $\mathbb{R}[x_1, x_2, \ldots, x_n]$ can thus be interpreted as a system of linear partial differential equations (PDE) with constant coefficients. Suppose we are interested in the solutions to these PDE within some nice class of functions, like polynomial functions, real analytic functions $\mathbb{R}^n \to \mathbb{R}$, or complex holomorphic functions $\mathbb{C}^n \to \mathbb{C}$. Then the set of solutions to our PDE is a linear space over $\mathbb{R}$ or $\mathbb{C}$. We are interested in computing a basis for that solution space. This computation rests on the primary decomposition of the ideal I. Both minimal primes and embedded primes will play a role, and all primary components will contribute to our basis for the solution space. But, first of all, let us start by interpreting the usual points of $\mathcal{V}(I)$ in terms of PDE.

Lemma 3.23. *Let I be an ideal in $\mathbb{C}[\mathbf{x}]$. A point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ lies in the variety $\mathcal{V}(I)$ if and only if the exponential function* $\exp(a_1x_1+\cdots+a_nx_n)$ *is a solution of the partial differential equations given by I.*

Proof. Let $f(\mathbf{x}) = \exp(a_1x_1+\cdots+a_nx_n)$. Then $\frac{\partial f}{\partial x_i} = a_i \cdot f$ for $i = 1, \ldots, n$. Let g be any polynomial and $g\big(\frac{\partial}{\partial x}\big)$ the corresponding differential operator. By induction on the degree of g, with degree one as base case, we find that the application of the operator $g\big(\frac{\partial}{\partial x}\big)$ to the function $\mathbf{f}(\mathbf{x})$ equals $g(a_1, \ldots, a_n)$ times $\mathbf{f}(\mathbf{x})$. This is zero for all $g \in I$ if and only if $(a_1, \ldots, a_n) \in \mathcal{V}(I)$. □

Lemma 3.23 embeds the classical solutions of a polynomial system into the solution space of the associated linear PDE. But, if the ideal is not radical then there are more solutions, governed by the primary decomposition. We shall explain this for the ideal in Example 3.6 and in Exercise 8 of Chapter 1.

Example 3.24. Let $n = 3$ and $I = \langle x^3 - yz, y^3 - xz, z^3 - xy\rangle$. The corresponding system of PDE asks for all functions $f = f(x, y, z)$ that satisfy

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2 f}{\partial y \partial z} \quad \text{and} \quad \frac{\partial^3 f}{\partial y^3} = \frac{\partial^2 f}{\partial x \partial z} \quad \text{and} \quad \frac{\partial^3 f}{\partial z^3} = \frac{\partial^2 f}{\partial x \partial y}. \tag{3.3}$$

To make this problem precise, we would need to specify the class of functions f that are allowed. For instance, we might take all holomorphic functions $f : \mathbb{C}^3 \to \mathbb{C}$. Or we might seek real analytic solutions $f : \mathbb{R}^3 \to \mathbb{R}$, or, among these, all polynomial solutions. Let's leave this unspecified for now.

The degree of our ideal I is $27 = 3 \times 3 \times 3$, which comes from the degrees of the three generators of I. The number 27 is also the dimension of the space of holomorphic solutions f to (3.3). A basis of that solution space is

$$\begin{array}{c} 1\,,\,x\,,\,y\,,\,z\,,\,x^2\,,\,y^2\,,\,z^2\,,\,x^3{+}6yz\,,y^3{+}6xz,\;z^3{+}6xy,\;x^4{+}y^4{+}z^4{+}24xyz, \\ \exp(x-y-z)\,,\;\;\exp(x+y+z)\,,\;\;\exp(-x-y+z)\,,\;\;\exp(-x+y-z)\,, \\ \exp(x{-}iy{+}iz)\,,\exp(x{+}iy{-}iz)\,,\exp(-x{-}iy{-}iz)\,,\exp(-x{+}iy{+}iz)\,, \\ \exp(ix-y+iz)\,,\exp(ix+y-iz)\,,\exp(ix-iy+z)\,,\exp(ix+iy-z)\,, \\ \exp(-ix{-}y{-}iz)\,,\exp(-ix{+}y{+}iz)\,,\exp(-ix{-}iy{-}z)\,,\exp(-ix{+}iy{+}z). \end{array} \tag{3.4}$$

The subspace of polynomial solutions has dimension 11 and is spanned by the first row. The larger subspace of real analytic solutions has dimension 15 and is spanned by the first two rows. All other basis functions are exponentials of linear forms that have $i = \sqrt{-1}$ among its coefficients. The 16 basis solutions in the last four rows, along with the solution $1 = \exp(0x + 0y + 0z)$, are explained by Lemma 3.23. They are the exponentials corresponding to the 17 distinct points in $\mathcal{V}(I) \subset \mathbb{C}^3$.

This basis in (3.4) was derived from the minimal primary decomposition

$$I \quad = \quad Q \;\cap \bigcap_{\substack{a+b+c\,\equiv\,0 \\ \text{mod } 4}} \big\langle\, x-i^a,\; y-i^b,\; z-i^c \,\big\rangle \qquad \text{in } \mathbb{C}[x,y,z].$$

This decomposition is obtained by refining the primary decomposition over the rational numbers $\mathbb{Q}$ shown in Example 3.6. The 16 ideals in the intersection on the right hand side are maximal and hence prime. They correspond to the 16 exponential solutions in (3.4). The ideal Q is primary to the maximal ideal $\text{rad}(Q) = \langle x, y, z\rangle$. Since all associated primes are minimal, by Theorem 3.21, this primary ideal is uniquely determined from I:

$$Q \;=\; \big\langle\, x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, x^3-yz, y^3-xz, z^3-xy \,\big\rangle.$$

This zero-dimensional primary ideal has degree 11. It contributes the 11 polynomial solutions to the three partial differential equations in (3.3).

Here is a general result explaining our observations from Example 3.24.

Theorem 3.25. *Let I be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$, interpreted as a system of linear PDE. The space of holomorphic solutions has dimension equal to the degree of I. There exist non-zero polynomial solutions if and only if the maximal ideal $M = \langle x_1, \ldots, x_n\rangle$ is associated to I. In that case, the polynomial solutions are precisely the solutions to the system of PDE given by the M-primary component $(I : (I : M^\infty))$.*

Proof. Fix a degree compatible monomial order and let $\text{in}(I)$ be the initial monomial ideal of I for that order. The set $\mathcal{S}$ of standard monomials is finite. It consists of monomials of degree $< D$ for some D. These form a basis for the solution space of the PDE associated with $\text{in}(I)$.

Now, for any $d \geq D$, we have the following isomorphism of vector spaces

$$\mathbb{C}[\mathbf{x}]/I \quad \simeq \quad \mathbb{C}[\mathbf{x}]_{\leq d}/I_{\leq d}.$$

The pairing between monomials in the $\partial/\partial x_i$ and the monomials in the x_j defines a nondegenerate inner product on this vector space. If we interpret each element of $I_{\leq d}$ as a differential operator, then its solution space in $\mathbb{C}[\mathbf{x}]_{\leq d}$ is the orthogonal complement with respect to that inner product.

Our Gröbner basis for I translates into a triangular vector space basis for $I_{\leq d}$. By solving that triangular system of linear equations, we construct a unique solution basis whose elements have the form

$$(3.5) \quad p_{\mathbf{u}}(x_1,\ldots,x_n) \;=\; \mathbf{x}^{\mathbf{u}} + \text{higher order terms}, \;\; \text{where } \mathbf{x}^{\mathbf{u}} \text{ runs over } \mathcal{S}.$$

By increasing the value of d, we obtain a formal series (3.5) that solves our PDE. This formal series terminates as a polynomial if and only if it is annihilated by the operators $(\partial/\partial x_i)^d$ for $i = 1,2,\ldots,n$. This is equivalent to saying that the series is a solution to any M-primary ideal containing I. If I is M-primary then all solutions $p_{\mathbf{u}}$ are polynomials. The solution space is then spanned by polynomials and its dimension equals $|\mathcal{S}| = \text{degree}(I)$.

Suppose now that I is primary in $\mathbb{C}[\mathbf{x}]$. Since I is zero-dimensional, its radical is the maximal ideal $\text{rad}(I) = \langle x_1 - a_1,\ldots,x_n - a_n\rangle$, where $\mathcal{V}(I) = \{(a_1,\ldots,a_n)\} \subset \mathbb{C}^n$. By translating $(a_1,\ldots,a_n)$ to the origin $(0,\ldots,0)$, we can apply the analysis in the previous paragraph. From this and Lemma 3.23, we obtain degree(I) many polynomials $p_{\mathbf{u}}$, each with its lowest term a standard monomial $\mathbf{x}^u \in \mathcal{S}$, such that

$$(3.6) \qquad p_{\mathbf{u}}(x_1,\ldots,x_n)\cdot\exp(a_1x_1 + \cdots + a_nx_n)$$

solves the PDE given by I. These functions form a basis for the holomorphic solutions to I. None of them is a polynomial unless $(a_1,\ldots,a_n) = (0,\ldots,0)$.

Next, let I be an arbitrary zero-dimensional ideal. Its minimal primary decomposition (3.1) is unique, by Theorem 3.21. The solution space to I, regarded as PDE, is spanned by the solution spaces of its primary components $q_1, q_2, \ldots, q_k$. For each of these, we constructed a basis of holomorphic functions (3.6). The union of these bases is a basis for the solution space of I, and its cardinality equals degree(I).

Finally, we argue that if $M \in \text{Ass}(I)$ then the M-primary component of I is the double quotient $(I : (I : M^\infty))$. In the primary decomposition (3.1), suppose that q_1 is M-primary. Then $(I : M^\infty) = q_2 \cap \cdots \cap q_k$. Taking the ideal quotient of I by $q_2 \cap \cdots \cap q_k$ recovers the ideal q_1. Hence $q_1 = (I : (I : M^\infty))$ as desired. This completes the proof. □

In the preceding discussion, we studied the solutions to zero-dimensional polynomial systems in the guise of linear PDE with constant coefficients. We saw that the solution space of such an ideal I is always a vector space of the same dimension, namely the degree of I. This is different from the situation for solving polynomial equations. The variety $\mathcal{V}(I)$ of classical solutions in

$\mathbb{C}^n$ changes its cardinality depending on whether I is radical or not. The solution space to the PDE, on the other hand, always has the expected dimension $\deg(I)$, independently of whether the ideal I is radical or not.

The solution spaces to our PDE vary gracefully under parameter changes. This underscores the utility of primary decompositions in the context of solving equations. We demonstrate this perspective in a simple example.

Example 3.26 ($n = 2$)**.** Consider the ideal $I = \langle x^2 - \delta^2, y^2 - \epsilon^2 \rangle \subset \mathbb{R}[x, y]$, where δ, ϵ are small real parameters. As before, we view I as a PDE system:

$$\frac{\partial^2 f}{\partial x^2} = \delta^2 f \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \epsilon^2 f.$$

For $\delta, \epsilon \neq 0$, the solution space is spanned by the four exponential functions

$$f_{ij} \;:=\; \exp\big((-1)^i \delta x + (-1)^j \epsilon y\big) \quad \text{where } i, j \in \{0, 1\}.$$

However these four functions become linearly dependent when $\delta\epsilon = 0$. We therefore change the basis of our four-dimensional solution space as follows:

$$\begin{array}{lclcl}
g_{00} & = & \frac{1}{4}(f_{00} + f_{01} + f_{10} + f_{11}) & = & 1 + \frac{\delta^2}{2}x^2 + \frac{\epsilon^2}{2}y^2 + \cdots \\
g_{01} & = & \frac{1}{4\epsilon}(f_{00} - f_{01} + f_{10} - f_{11}) & = & y + \frac{\delta^2}{2}x^2 y + \frac{\epsilon^2}{6}y^3 + \cdots \\
g_{10} & = & \frac{1}{4\delta}(f_{00} + f_{01} - f_{10} - f_{11}) & = & x + \frac{\delta^2}{6}x^3 + \frac{\epsilon^2}{2}xy^2 + \cdots \\
g_{11} & = & \frac{1}{4\epsilon\delta}(f_{00} - f_{01} - f_{10} + f_{11}) & = & xy + \frac{\delta^2}{6}x^3 y + \frac{\epsilon^2}{6}xy^3 + \cdots
\end{array}$$

This family remains linearly independent for all values of δ and ϵ. In particular, for $\delta = \epsilon = 0$, we obtain the standard basis $\mathcal{S} = \{1, x, y, xy\}$ modulo the ideal $\operatorname{in}(I) = \langle x^2, y^2 \rangle$. This is a basis for the solutions to $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$.

We next discuss briefly the PDE from polynomial ideals I that are not zero-dimensional. It is still true that the primary decomposition of I reveals the solution space of these PDE. The precise statement is an important result in analysis known as *Ehrenpreis' Fundamental Principle* or as *Palamodov-Ehrenpreis Theorem*. The details of this theorem are outside our scope. For the statement of this result see [**53**, §10.5] and the references given there.

We here illustrate the role of primary decomposition in one example. The key observation is that embedded primes reveal spurious solutions spaces.

Example 3.27. Let $n = 4$ and consider the ideal

$$J \;=\; \langle\, xw,\, xz + yw,\, yz \rangle.$$

Somewhat surprisingly, this is not radical. Its radical is the monomial ideal

$$\sqrt{J} \;=\; \langle x, y \rangle \,\cap\, \langle z, w \rangle \;=\; \langle xw, xz, yw, yz \rangle.$$

The given ideal J has three associated primes. The primes $\langle x, y \rangle$ and $\langle z, w \rangle$ are minimal primes, and the maximal ideal $\langle x, y, z, w \rangle$ is an embedded prime.

A minimal primary decomposition of the given ideal equals

$$J \;=\; \langle x,y\rangle \;\cap\; \langle z,w\rangle \;\cap\; (J+\langle x,y,z,w\rangle^3).$$

The third primary ideal is not unique. If we replace the third power of the maximal ideal by any higher power, then the intersection remains the same.

As before, we interpret the generators of J as a system of linear PDE:

$$\frac{\partial^2 f}{\partial x \partial w} \;=\; \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial w} \;=\; \frac{\partial^2 f}{\partial y \partial z} \;=\; 0.$$

The linear space of solutions $f(x,y,z,w)$ is infinite-dimensional. It is spanned by all functions of the form $g(y,z)$ and $h(x,y)$, together with the one special function $xz - yw$. The former correspond to the two minimal primes. The latter spurious solution arises from the embedded primary component.

Whenever one encounters a system of polynomial equations with special structure, and one is curious about the variety of solutions, it pays to explore the primary decomposition and to ponder the solutions to the associated PDE. Students who struggle with *schemes* in an algebraic geometry class will find our PDE interpretation a useful way to understand their structure.

Given a system of polynomial equations, the primary decomposition often reveals interesting structures. Most importantly, it tells us how to break up the solutions into meaningful pieces. As an illustration, we examine the following question from linear algebra: *Let A,B,C be 2×2-matrices. In which ways is it possible that the triple product ABC is the zero matrix?*

We approach this problem as follows. We set $n=12$ and we fix the polynomial ring $\mathbb{R}[a_{ij},b_{ij},c_{ij}]$ whose variables are the 12 entries of the matrices A,B,C. Let I be the ideal in $\mathbb{R}[a_{ij},b_{ij},c_{ij}]$ that is generated by the four entries of the matrix product ABC. For example, one of the four generators of I is the upper left entry of ABC. This is the trilinear form

$$a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21}.$$

In the back of our minds, we think of this as a partial differential equation:

$$\frac{\partial^3 f}{\partial a_{11}\partial b_{11}\partial c_{11}} + \frac{\partial^3 f}{\partial a_{12}\partial b_{21}\partial c_{11}} + \frac{\partial^3 f}{\partial a_{11}\partial b_{12}\partial c_{21}} + \frac{\partial^3 f}{\partial a_{12}\partial b_{22}\partial c_{21}} \;=\; 0. \tag{3.7}$$

The scheme-theoretic version of our linear algebra question is this: *Which functions on matrix triples satisfy these four partial differential equations?*

A computation with a computer algebra system reveals that the ideal I is radical. It is the intersection of six prime ideals. Three of them are ideals generated by the entries of A or B or C respectively. The next two associated primes are generated respectively by the 2×2 minors of the matrices

$$\begin{pmatrix} a_{11} & a_{21} & -b_{21} & -b_{22} \\ a_{12} & a_{22} & b_{11} & b_{12} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{11} & b_{21} & -c_{21} & -c_{22} \\ b_{12} & b_{22} & c_{11} & c_{12} \end{pmatrix}.$$

Finally, the last associated prime of I is the ideal $I + \langle \det(A), \det(C)\rangle$. Thus $\mathrm{Ass}(I)$ consists of six primes, and all are minimal. Using the computer algebra system of their choice, the reader now ought to check that our ideal I is indeed equal to the intersection of these six prime ideals.

Geometrically, we are studying a variety $\mathcal{V}(I)$ in the affine space $\mathbb{C}^{12}$. It is the solution set of four cubic equations. We found that $\mathcal{V}(I)$ is the union of six irreducible components. Three of them are linear spaces of dimension 8. The other three irreducible components have dimension 9 and they are not linear spaces. Their degrees are $4, 4$ and 8 respectively. In response to the original linear algebra question, the six irreducible components of $\mathcal{V}(I)$ correspond to the following six scenarios for a triple of 2×2-matrices:

$$\begin{gathered}\mathrm{rank}(A) = 0 \quad \text{or} \quad \mathrm{rank}(B) = 0 \quad \text{or} \quad \mathrm{rank}(C) = 0 \quad \text{or} \\ \mathrm{rank}(A) = \mathrm{rank}(B) = 1 \quad \text{or} \quad \mathrm{rank}(B) = \mathrm{rank}(C) = 1 \\ \text{or} \quad \mathrm{rank}(A) = \mathrm{rank}(C) = 1.\end{gathered}$$

Each of the six irreducible components $\mathcal{V}(I)$ is a rational variety, and it admits a nice polynomial parametrization. Using Lemma 3.23, we can then write down all exponential solutions to the four partial differential equations, like (3.7), that are given by I. The solutions come in six families.

The solutions contributed by the first irreducible component $\{\mathrm{rank}(A) = 0\}$ are the functions $f(B, C)$ that do not depend on the matrix A. The solutions contributed by the last irreducible component have the form

$$\begin{aligned} f(A,B,C) \;=\; \exp\big[\, & r_1 s_1 a_{11} + r_1 s_2 a_{12} + r_2 s_1 a_{21} + r_2 s_2 a_{22} + (t_{11}u_2 - s_2 t_{12}) b_{11} \\ & + (s_2 t_{21} - t_{11} u_1) b_{12} + (s_1 t_{12} - t_{22} u_2) b_{21} + (t_{22} u_1 - s_1 t_{21}) b_{22} \\ & + u_1 v_1 c_{11} + u_1 v_2 c_{12} + u_2 v_1 c_{21} + u_2 v_2 c_{22} \,\big], \end{aligned}$$

where $r_i, s_j, t_{ij}, u_i, v_j$ are arbitrary complex numbers. The functions f above satisfy the PDE because the coefficients of $a_{11}, a_{22}, \ldots, c_{22}$ furnish a parametrization of the irreducible variety $\{ABC = 0, \mathrm{rank}(A) = \mathrm{rank}(C) = 1\}$.

Here is our conclusion for this example, valid for the entire book: taking a fresh look at linear algebra offers a point of entry to nonlinear algebra.

Exercises

(1) Let $R = \mathbb{C}[x, y]/\langle x^2, xy, y^2\rangle$. Is $\{0\}$ an irreducible ideal? Is it primary?

(2) Prove that an ideal $I \subsetneq \mathbb{Z}$ is a power of a prime ideal if and only if it is primary.

(3) Prove that a ring is Noetherian if and only if every ideal is finitely generated.

(4) (a) Prove that if R is Noetherian, then so is R/I for any ideal I.

(b) Prove Hilbert Basis Theorem: If R is Noetherian, then so is $R[x]$.

(5) Prove Lemma 3.12

(6) Check that Example 3.15 provides two distinct minimal primary decompositions.

(7) a) Prove that a prime ideal p cannot be equal to an intersection of (finitely many, more than one, incomparable) ideals.
b) More generally prove that if a prime ideal contains an intersection of finitely many ideals, then it contains one of them.

(8) Prove that in a Noetherian ring every ideal contains a power of its radical. Give a counterexample in case of a non-Noetherian ring.

(9) Find three polynomials in three unknowns, each having degree precisely five, whose variety in $\mathbb{C}^3$ consists of precisely 37 complex solutions.

(10) Find all solutions (x, y) of the two equations $x^2 + y = \epsilon$ and $y^2 + x = \epsilon$ over the algebraic closure of the field $\mathbb{Q}(\epsilon)$. Write down series solutions.

(11) Let I be the ideal generated by the 2×2-subpermanents $x_i y_j + x_j x_i$ of a 2×5-matrix of unknowns. Find a minimal primary decomposition of I. Interpret your result in terms of solving partial differential equations.

(12) Which 2×3 matrices A and B satisfy $AB^T = BA^T$? How about 3×2?

(13) Let $K = \mathbb{F}_2$ be the field with two elements. Find an ideal I in $K[x, y]$ that has precisely ten associated primes, of which five are embedded.

Chapter 4

Mapping and Projecting

A frequently encountered challenge is to compute the image of a polynomial map. Such an image need not be an algebraic variety. However, a natural outer approximation of the image is given by its Zariski closure. The Zariski closure of the image is a variety, described by the polynomials that vanish on it. In this chapter we show how this variety can be found by eliminating variables. Gröbner bases and resultants serve as our primary tools. Further, we provide theorems that allow us to understand the difference between the image and its closure. The answer we obtain depends heavily on the setting, whether we work over the complex numbers $\mathbb{C}$ or over the real numbers $\mathbb{R}$, and whether the given polynomials are homogeneous or nonhomogeneous.

4.1. Elimination

In this section we introduce elimination of variables for polynomial ideals. This is our main tool for computing the closure of the image of a polynomial map. We show how to carry it out in practise using Gröbner bases.

We fix an algebraically closed field K and the polynomial ring $K[\mathbf{x}] = K[x_1, \ldots, x_n]$. Every ideal $I \subset K[\mathbf{x}]$ has an associated affine variety:

$$\mathcal{V}(I) \,=\, \{\, \mathbf{p} \in K^n \,:\, f(\mathbf{p}) = 0 \text{ for all } f \in I \,\}.$$

We consider the projection from K^n onto the subspace given by the first m coordinates:

$$\pi \,:\, K^n \to K^m,\; (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) \mapsto (p_1, \ldots, p_m).$$

If V is a variety in K^m then its image $\pi(V)$ need not be a variety.

Example 4.1 ($n = 2, m = 1$)**.** The image of the hyperbola $V = \mathcal{V}(xy - 1)$ under the projection $K^2 \to K^1$ from the plane to the x-axis equals $\pi(V) = K^1 \backslash \{0\}$. This is not a variety in K^1. Note that the image will be closed if, prior to projecting, we first perform a change of coordinates. For instance if we replace V by the hyperbola $V' = \mathcal{V}\big((x{+}y)(x{-}y) - 1\big)$ then $\pi(V') = K^1$.

By definition, the Zariski closure $\overline{\pi(V)}$ of the image is a variety in K^m. It is the smallest variety containing $\pi(V)$. We call the variety $\overline{\pi(V)}$ the *closed image* of V under the map π. The next theorem characterizes its ideal.

Theorem 4.2. *Let $I \subset K[\mathbf{x}]$ be an ideal and $V = \mathcal{V}(I)$ its variety in K^n, where K is an algebraically closed field. Then its closed image in K^m is the variety $\overline{\pi(V)} = \mathcal{V}(J)$ defined by the* elimination ideal

$$J \;=\; I \cap K[x_1, \ldots, x_m]. \tag{4.1}$$

If I is radical or prime then the elimination ideal J has the same property.

Proof. If J is not a prime ideal then there exist polynomials f and g in $K[x_1, \ldots, x_m]$ such that $fg \in J$ but $f, g \notin J$. The same polynomials show that I is not prime. Similarly if J is not radical then there exists f in $K[x_1, \ldots, x_m]$ and $r \geq 2$ such that $f^r \in J$ but $f \notin J$. The same f shows that J is not radical. Similar reasoning shows that all ideal $I \subset K[\mathbf{x}]$ satisfy

$$\mathrm{Rad}(I) \cap K[x_1, \ldots, x_m] \quad = \quad \mathrm{Rad}\big(I \cap K[x_1, \ldots, x_m]\big).$$

Since passing to the radical does not change the variety of a given ideal, we may assume that I and J are radical ideals. We shall now make a forward reference and use the Nullstellensatz (Chapter 6). A polynomial belongs to I if and only if it vanishes on $V = \mathcal{V}(I)$. This holds, in particular, for polynomials f in the subring $K[x_1, \ldots, x_m]$. Such an f belongs to J if and only if it vanishes on $\pi(V)$ if and only if it vanishes on $\overline{\pi(V)}$. The latter condition means that f lies in the radical ideal of $\overline{\pi(V)}$. We conclude that the the radical ideal of the closed image $\overline{\pi(V)}$ is precisely the elimination ideal J. For further details we refer to [**10**, §2.2, Theorem 3]. □

Theorem 4.2 says that the algebraic operation of elimination corresponds to the geometric operation of projection. This holds in many settings, not just in algebraic geometry. For instance, Gaussian elimination in linear algebra corresponds to projection of linear subspaces, and Fourier-Motzkin elimination in convex geometry corresponds to projection of polyhedra. Alternatively, from the perspective of logic, we can think of our projection as quantifier elimination. We are eliminating the $n - m$ existentially quantified variables from the first-order logic statement $\exists\, x_{m+1}, \ldots, x_n \,:\, \mathbf{x} \in V$.

Elimination and projection are fundamental operations in many applications. One good example is the problem of matrix completion or tensor completion which arises frequently in data science. Here is an illustration.

Example 4.3 (Matrix Completion)**.** Fix $n = 15$ and let V be the irreducible variety of symmetric 5×5-matrices $X = (x_{ij})$ of rank ≤ 2. Its prime ideal $I = \mathcal{I}(V)$ is minimally generated by 50 homogeneous cubic polynomials, namely the 3×3-minors of the matrix X. In fact, these 50 cubics from a Gröbner basis for the degree reverse lexicographic order.

Now let $m = 10$ and order the variables so that the five diagonal entries $x_{11}, x_{22}, x_{33}, x_{44}, x_{55}$ come last. Then the elimination ideal is principal:

$$\begin{aligned} J \,=\, \langle\, & x_{14}x_{15}x_{23}x_{25}x_{34} - x_{13}x_{15}x_{24}x_{25}x_{34} - x_{14}x_{15}x_{23}x_{24}x_{35} \\ & + x_{13}x_{14}x_{24}x_{25}x_{35} + x_{12}x_{15}x_{24}x_{34}x_{35} - x_{12}x_{14}x_{25}x_{34}x_{35} \\ & + x_{13}x_{15}x_{23}x_{24}x_{45} - x_{13}x_{14}x_{23}x_{25}x_{45} - x_{12}x_{15}x_{23}x_{34}x_{45} \\ & + x_{12}x_{13}x_{25}x_{34}x_{45} + x_{12}x_{14}x_{23}x_{35}x_{45} - x_{12}x_{13}x_{24}x_{35}x_{45} \,\rangle. \end{aligned}$$

The ideal generator is known as the *pentad* in algebraic statistics [**18**, Example 4.2.8]. The 15 terms correspond to the 15 maximal matchings in the complete graph K_5. The hypersurface $\mathcal{V}(J)$ equals the image $\pi(V)$ of the determinantal variety V under the projection from K^{15} onto the subspace K^{10} whose coordinates are the off-diagonal entries.

Our result has the following interpretation in terms of matrix completion. If the 10 off-diagonal entries of a symmetric 5×5-matrix are given then this can be completed to a matrix of rank ≤ 2 if and only if the pentad vanishes. This constraint appears in the statistical theory of *factor analysis* [**18**].

Our next example shows how find algebraic relations via elimination.

Example 4.4. The first four power sums in three variables are the polynomials $x^i + y^i + z^i$ for $i = 1, 2, 3, 4$. These four must be algebraically dependent since they involve only three variables. But, what is the algebraic relation satisfied by these four power sums?

We approach this question by setting $n = 7, m = 4$ and fixing the ideal

$$I \,=\, \langle\, x + y + z - p_1,\; x^2 + y^2 + z^2 - p_2,\; x^3 + y^3 + z^3 - p_3,\; x^4 + y^4 + z^4 - p_4 \,\rangle.$$

This ideal lives in a polynomial ring in seven variables. We wish to eliminate the three original variables x, y, z. Thus, we ask for the elimination ideal

$$J \,=\, I \,\cap\, K[p_1, p_2, p_3, p_4].$$

This is a principal prime ideal. Its generator is a polynomial of degree four:

$$J \,=\, \langle\, p_1^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4 \,\rangle.$$

This is the desired relation. Please check by plugging in the power sums.

The computations in our two examples were carried out using Gröbner bases. Here is how this works. We first fix the lexicographic monomial order $\prec$ on $K[\mathbf{x}]$ with $x_1 \prec x_2 \prec \cdots \prec x_n$. We then compute the reduced Gröbner basis for the ideal generated by the given polynomials. And, finally, we select those polynomials from the output that use only the first m variables.

Theorem 4.5. *If $\mathcal{G}$ is a lexicographic Gröbner basis for an ideal I in $K[\mathbf{x}]$ then its elimination ideal J in (4.1) has the Gröbner basis $\mathcal{G}' = \mathcal{G} \cap K[x_1, \ldots, x_m]$. If $\mathcal{G}$ is the reduced Gröbner basis of I then $\mathcal{G}'$ is the reduced Gröbner basis of J.*

Proof. Clearly, the set $\mathcal{G}'$ is contained in $J = I \cap K[x_1, \ldots, x_m]$. Consider any nonzero polynomial $f \in J$. The initial monomial $\text{in}_\prec(f)$ is divisible by $\text{in}_\prec(g)$ for some $g \in \mathcal{G}$. None of the variables $x_{m+1}, \ldots, x_n$ appears in the monomial $\text{in}_\prec(g)$. Every trailing term of g is lexicographically smaller, so it cannot use any of the last $n - m$ variables. Hence g lies in $\mathcal{G}'$. We have shown that some initial monomial from $\mathcal{G}'$ divides $\text{in}_\prec(f)$. Since f was chosen arbitrarily from $J \backslash \{0\}$, this means that $\mathcal{G}'$ is a Gröbner basis for J. If the given Gröbner basis $\mathcal{G}$ is reduced then $\mathcal{G}'$ also satisfies the two requirements for being a reduced Gröbner basis. □

This result shows that the lexicographic Gröbner basis $\mathcal{G}$ solves the elimination problem simultaneously for all m. Thus computing $\mathcal{G}$ means triangularizing a given system of polynomial equations. We saw in Chapter 1 that it can be quite costly to compute a lexicographic Gröbner basis. One therefore often uses different strategies to carry out the elimination process. But Theorem 4.5 represents the main idea that underlies these strategies. Lexicographic elimination is a key tool for solving systems of polynomial equations. The Gröbner basis in Theorem 4.5 triangularizes the given system. It is instructive to try this for some zero-dimensional varieties.

Example 4.6. Here is a simple question: can you find three real numbers x, y, z whose i-th power sum equals i for $i = 1, 2, 3$? To answer this question, we compute the lexicographic Gröbner basis of the following ideal:

$$I \;=\; \langle x + y + z - 1,\; x^2 + y^2 + z^2 - 2,\; x^3 + y^3 + z^3 - 3 \,\rangle.$$

This Gröbner basis equals

$$\mathcal{G} \;=\; \{\, 6\underline{z^3} - 6z^2 - 3z - 1,\; 2\underline{y^2} + 2yz - 2y + 2z^2 - 2z - 1,\; \underline{x} + y + z - 1 \,\}.$$

Theorem 4.2 says that we can solve our equations by back-substitution. Indeed, the equations have six complex zeros. We first compute the three roots of the cubic in z, we substitute them into the second equation and solve for y, and then we set $x = 1 - y - z$. The cubic has one real root and two complex conjugate roots:

$$z \in \{1.4308, -0.21542 - 0.26471i, -0.21542 + 0.26471i\}.$$

By symmetry, the zeros of I are precisely the six points in $\mathbb{C}^3$ whose coordinates are permutations of the three complex numbers above. Hence the answer to our question is "no". The variety $\mathcal{V}(I)$ has no real points.

4.2. Implicitization

Implicitization is a special instance of elimination. Here, the problem is to compute the image of a polynomial map between two affine spaces. This can be done by forming the graph of the map and then projecting onto the image coordinates. To be precise, we consider a map of the form

$$f : K^m \to K^n, \ \mathbf{p} = (p_1, \ldots, p_m) \mapsto \big(f_1(\mathbf{p}), \ldots, f_n(\mathbf{p})\big), \tag{4.2}$$

where $f_1, \ldots, f_n$ are polynomials in $K[z_1, \ldots, z_m]$, and K is an algebraically closed field. We write image(f) for the image of K^m under this map. This need not be a variety, as the following example shows:

Example 4.7. Let $m = 2, n = 3$ and consider the map $f = (z_1, z_1z_2, z_1z_2^2)$. The Zariski closure of the image is the surface $V = \mathcal{V}(x_1x_3 - x_2^2)$ in K^3. The point $(0, 0, 1)$ is in the surface but not in image(f). For $K = \mathbb{C}$ we can approximate $(0, 0, 1)$ by a sequence of points in the image, e.g. by taking $z_1 = \epsilon^2$ and $z_2 = \epsilon^{-1}$ for $\epsilon \to 0$.

The *closed image* of the map $f : K^m \to K^n$ is the Zariski closure of the set-theoretic image image(f). The closed image is denoted $\overline{\text{image}}(f) \subset K^n$.

Corollary 4.8. *Given the map f in (4.2), let I be the ideal in the polynomial ring $K[\mathbf{x}, \mathbf{z}]$ in $n+m$ variables which is generated by $f_i(z_1, \ldots, z_m) - x_i$ for $i = 1, 2, \ldots, n$. The closed image of $f : K^m \to K^n$ is the variety defined the elimination ideal $J = I \cap K[\mathbf{x}]$. In symbols, $\overline{\text{image}}(f) = \mathcal{V}(J)$.*

Proof. The graph of f is Zariski closed in K^{n+m}, and I is the ideal that defines it. The image of f is the projection of the graph onto K^n. With this, the claim follows from Theorem 4.2. □

Example 4.9 (Plücker relations)**.** What are the algebraic relations among the 2×2-minors of a 2×5-matrix? We answer this question by setting $m = n = 10$ and considering the map $f : K^{10} \to K^{10}$ that takes a matrix $\begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \end{pmatrix}$ to the vector $(x_{12}, x_{13}, \ldots, x_{45})$ where $x_{ij} = z_{1i}z_{2j} - z_{1j}z_{2i}$ for $1 \le i < j \le 5$. The graph of f is described by an ideal I in the polynomial ring $K[\mathbf{x},\mathbf{z}]$ in 20 variables. Note that I is generated by 10 polynomials. The desired elimination ideal equals

$$\begin{aligned} I \cap K[\mathbf{x}] \;=\; \langle\, & x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}\,,\; x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}\,, \\ & x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24}\,,\; x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}\,, \\ & x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} \,\rangle. \end{aligned}$$

These five quadrics are the Plücker relations among the maximal minors. They play a key role in our study of Grassmannians in Chapter 5. The ten variables in $K[\mathbf{x}]$ can be written as the entries of a skew-symmetric matrix

$$X \;=\; \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}.$$

The Plücker relations are the *pfaffians* of size 4×4, that is, the square roots of the principal 4×4 minors of X. Thus $\mathcal{V}(I \cap K[\mathbf{x}])$ is the variety of skew-symmetric 5×5 matrices of rank ≤ 2. We shall see that, as a projective variety in $\mathbb{P}^9$, this is the Grassmannian of lines in $\mathbb{P}^4$. Each such line is written in Plücker coordinates as the image of the rank 2 matrix X.

The notion of the determinant is central to linear algebra. In nonlinear algebra, there is an analogous notion of a hyperdeterminant for tensors.

Example 4.10 (Hyperdeterminant)**.** Let $X = (x_{ijk})$ be a tensor of format $2 \times 2 \times 2$, where the $n = 8$ tensor entries are variables. The tensor represents an affine-trilinear polynomial in $m = 3$ variables:

$$f \;=\; x_{000}+x_{100}z_1+x_{010}z_2+x_{001}z_3+x_{110}z_1z_2+x_{101}z_1z_3+x_{011}z_2z_3+x_{111}z_1z_2z_3.$$

For any fixed X, this polynomial defines a surface $\mathcal{V}(f)$ in K^3. We are interested in the condition under which this surface is singular. It is singular at the point $\mathbf{z}$ if and only if the pair $(X, \mathbf{z}) \in K^{11}$ lies in the variety of

$$I \;=\; \Big\langle\, f,\, \frac{\partial f}{\partial z_1}\,,\, \frac{\partial f}{\partial z_2}\,,\, \frac{\partial f}{\partial z_3} \,\Big\rangle.$$

The elimination ideal $I \cap K[\mathbf{x}]$ is principal. We find that its generator is

$$\begin{aligned} & x_{110}^2x_{001}^2 + x_{100}^2x_{011}^2 + x_{010}^2x_{101}^2 + x_{000}^2x_{111}^2 + 4x_{000}x_{110}x_{011}x_{101} + 4x_{010}x_{100}x_{001}x_{111} \\ & \quad -2x_{100}x_{110}x_{001}x_{011} - 2x_{010}x_{110}x_{001}x_{101} - 2x_{010}x_{100}x_{011}x_{101} \\ & \quad -2x_{000}x_{110}x_{001}x_{111} - 2x_{000}x_{100}x_{011}x_{111} - 2x_{000}x_{010}x_{101}x_{111}. \end{aligned}$$

This quartic is the $2 \times 2 \times 2$ *hyperdeterminant.* It vanishes whenever the surface $V(f)$ fails to be smooth in K^3. Hyperdeterminants exist for tensors of many larger formats. Their study is a fascinating topic in nonlinear algebra. A standard reference is the book by Gel'fand, Kapranov and Zelevinsky [**24**]

The most basic scenario in elimination arises when m variables are eliminated from a system of $m+1$ equations. One expects the result to be a single equation in the coefficients of that system. We saw this for $m=3$ in Examples 4.4 and 4.10. The theory of *resultants* is custom-taylored to predict the eliminant in such cases. We set this up over the field $\mathbb{Q}$ as follows.

Let $i \in \{1,2,\ldots,m+1\}$ and fix a general inhomogeneous polynomial f_i of degree d_i in $z_1,\ldots,z_m$. This polynomial has $\binom{d_i+m}{m}$ unknown coefficients $x_{i,\mathbf{u}}$, one for each monomial $\mathbf{z}^{\mathbf{u}}$ of degree $\leq d_i$. The total number of unknown coefficients equals $n = \sum_{i=1}^{m+1}\binom{d_i+m}{m}$. We write $\mathbb{Q}[\mathbf{x},\mathbf{z}]$ for the resulting polynomial ring in $n+m$ variables. Inside this ring we consider the ideal

$$I \;=\; \langle\, f_1, f_2, \ldots, f_m, f_{m+1} \,\rangle \;\subset\; \mathbb{Q}[\mathbf{x},\mathbf{z}].$$

We are interested in the ideal in $\mathbb{Q}[\mathbf{x}]$ found by eliminating the m variables z_i.

Theorem 4.11. *The elimination ideal $I \cap \mathbb{Q}[\mathbf{x}]$ is principal. Its generator is an irreducible polynomial in the entries of the coefficient vector $\mathbf{x}$. This is denoted* $\mathrm{Res}(f_1,\ldots,f_{m+1})$ *and called the* resultant. *The degree of the resultant in the coefficients of f_i equals $d_1\cdots d_{i-1}d_{i+1}\cdots d_{m+1}$ for $i=1,2,\ldots,m+1$.*

Proof. We refer to [**11**, Chapter 3] for the proof. In that source, and many others, the f_i are taken to be homogeneous polynomials in $m+1$ variables. We here prefer the inhomogeneous case, which allows for a simpler formulation as an elimination ideal. The two versions are equivalent. □

Example 4.12 (Determinants)**.** Let $d_1 = \cdots = d_{m+1} = 1$. The $m+1$ polynomials f_i are affine-linear. They can be written as a matrix-vector product

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \\ f_{m+1} \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} & x_{1,m+1} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} & x_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} & x_{m,m+1} \\ x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,m} & x_{m+1,m+1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \\ 1 \end{pmatrix}.$$

The resultant $\det(f_1,\ldots,f_{m+1})$ is the determinant of the coefficient matrix $(x_{i,j})$. This is a homogeneous polynomial of degree $m+1$ in $n=(m+1)^2$ unknowns with $(m+1)!$ terms. It has degree one in the coefficients of each f_i.

Example 4.13 (Eliminating one variable from two quadratic polynomials)**.** Let $m=1$ and $d_1=d_2=2$ and abbreviate $z=z_1$. Our system consists of two univariate polynomials of degree two with six unspecified coefficients:

$$f_1 = x_{11}z^2 + x_{12}z + x_{13} \quad \text{and} \quad f_2 = x_{21}z^2 + x_{22}z + x_{23}.$$

The generator of the elimination ideal $\langle f_1, f_2\rangle \cap \mathbb{Q}[\mathbf{x}]$ is the *Sylvester resultant*

$$(4.3) \qquad \operatorname{Res}(f_1,f_2) \;=\; \det\begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ 0 & x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} & 0 \\ 0 & x_{21} & x_{22} & x_{23} \end{pmatrix}.$$

This is a bihomogeneous polynomial of bidegree $(d_1, d_2) = (2, 2)$. Its expansion has 7 terms. It vanishes if the two quadrics have a common zero.

The formula (4.3) generalizes to two polynomials in z of arbitrary degrees d_1, d_2. The following is the *Sylvester matrix* of format $(d_2{+}d_1) \times (d_2{+}d_1)$:

$$\mathrm{Syl}_{d_1,d_2} \;=\; \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,d_1+1} & 0 & \cdots & 0 & 0 \\ 0 & x_{11} & x_{12} & \ddots & x_{1,d_1+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1,d_1+1} & 0 \\ 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1,d_1+1} \\ x_{21} & x_{22} & \cdots & x_{2,d_2+1} & 0 & \cdots & 0 & 0 \\ 0 & x_{21} & x_{22} & \ddots & x_{2,d_2+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2,d_2+1} & 0 \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2,d_2+1} \end{pmatrix}$$

For $d_1 = d_2 = 2$ this is the 4×4 matrix seen in (4.3).

Theorem 4.14. *The determinant of the Sylvester matrix* Syl_{d_1,d_2} *is equal to the resultant* $\operatorname{Res}(f_1, f_2)$ *of the two univariate polynomials*

$$\begin{aligned} f_1(z) \;&=\; x_{11}z^{d_1} + \cdots + x_{1,d_1}z + x_{1,d_1+1} \\ \textit{and} \quad f_2(z) \;&=\; x_{21}z^{d_2} + \cdots + x_{2,d_2}z + x_{2,d_2+1}. \end{aligned}$$

Proof. We first note that $\det(\mathrm{Syl}_{d_1,d_2})$ is a non-zero polynomial. We can see this by specializing $f_1 = z^{d_1}$ and $f_2 = 1$. Here the Sylvester matrix Syl_{d_1,d_2} specializes to the identity matrix, so its determinant is non-zero.

Let Z denote the column vector with entries $z^{d_1+d_2-1}, z^{d_1+d_2-2}, \ldots, z^2, z, 1$, and F the column vector with entries $z^{d_2-1}f_1, \ldots, zf_1, f_1,\ z^{d_1-1}f_2, \ldots, zf_2, f_2$. Both vectors have length $d_1 + d_2$. They are related by the Sylvester matrix:

$$\mathrm{Syl}_{d_1,d_2} \cdot Z \;=\; F.$$

Multiplying on the left by the adjoint of the Sylvester matrix, we obtain

$$\det(\mathrm{Syl}_{d_1,d_2}) \cdot Z \;=\; \operatorname{adj}(\mathrm{Syl}_{d_1,d_2}) \cdot F.$$

The last coordinate of the column vector Z equals 1. Hence the last coordinate in this equation shows that $\det(\mathrm{Syl}_{d_1,d_2})$ is a polynomial linear combination of the entries of F, and hence it lies in the ideal $\langle f_1, f_2\rangle$. The Sylvester determinant is a non-zero homogeneous polynomial of degree $d_1 + d_2$ that lies in the ideal $\langle f_1, f_2\rangle \cap \mathbb{Q}[\mathbf{x}]$. We know from Theorem 4.11 that this ideal is principal, and its generator $\mathrm{Res}(f_1, f_2)$ also has degree $d_1 + d_2$. This implies that the resultant $\mathrm{Res}(f_1, f_2)$ is equal to the Sylvester determinant $\det(\mathrm{Syl}_{d_1,d_2})$, up to a non-zero multiplicative constant. □

Example 4.15. Let $f_1(z)$ and $f_2(z)$ be univariate polynomials of degree d_1 and d_2 in $\mathbb{Q}[z]$. This defines a map $f : \mathbb{C} \to \mathbb{C}^2$ whose closed image is an algebraic curve in the plane $\mathbb{C}^2$ with coordinates x_1, x_2. The implicit equation of this curve is the resultant $\mathrm{Res}_z\big(x_1 - f(z), x_2 - g(z)\big)$, taken with respect to the variable z. For a concrete example consider the plane cubic curve given parametrically by $f = (z^3 + 4z, z^2 - 3)$. Its equation equals

$$\det \begin{bmatrix} -1 & 0 & -4 & x_1 & 0 \\ 0 & -1 & 0 & -4 & x_1 \\ -1 & 0 & x_2+3 & 0 & 0 \\ 0 & -1 & 0 & x_2+3 & 0 \\ 0 & 0 & -1 & 0 & x_2+3 \end{bmatrix} \;=\; x_2^3 - x_1^2 + 17x_2^2 + 91x_2 + 147.$$

If $m \geq 2$ then the resultant $\mathrm{Res}(f_1, f_2, \ldots, f_{m+1})$ is more difficult to compute, and there does not always exists a formula as the determinant whose entries are linear expressions in the coefficients of $f_1, f_2, \ldots, f_{m+1}$. In some cases, however, such formulas are available in the literature. For instance, Sylvester already gave such a formula for $m = 2$ and $d_1 = d_2 = d_3$. A considerable body of information on matrix formulas for resultants can be found in the excellent book by Gel'fand, Kapranov and Zelevinsky [**24**]

4.3. The Image of a Polynomial Map

We discussed methods for computing the Zariski closure of the image of a polynomial map. Can we say something about the image itself? The answer is 'yes', but the situation very much depends on the field K and whether we are in the projective case or the affine case. In this section we discuss tools for computing such images. We begin by highlighting the difference between the real numbers and complex numbers with regard to this problem.

We start with a *real, affine* variety $X \subset \mathbb{R}^m$. We would like to understand the image of X under a polynomial map:

$$f = (f_1, \ldots, f_n) : \mathbb{R}^m \to \mathbb{R}^n.$$

Easy examples show that the Zariski closure of the image and the image itself can differ a lot. For instance, this happens for $n = m = 1$, $X = \mathbb{R}$

and $f(z) = z^2$. Is there a chance in general of describe the image using polynomials? The following theorem provides a positive answer.

Theorem 4.16 (Tarski-Seidenberg)**.** *Over the field of real numbers the image of a variety is a semi-algebraic set (recall Definition 2.12).*

Proof. See [**5**, Section 1.4]. □

Thus to provide a description of the image over $\mathbb{R}$ we need two ingredients: polynomial equations and polynomial inequalities, suitably combined.

Example 4.17. Let $m = 9$, $n = 6$ and f the map that multiplies a 3×2-matrix A with its transpose to get a symmetric 3×3 matrix $Z = (z_{ij})$:

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{bmatrix} \mapsto X = \begin{bmatrix} z_{11}^2 + z_{12}^2 & z_{11}z_{21} + z_{12}z_{22} & z_{11}z_{31} + z_{12}z_{32} \\ z_{11}z_{21} + z_{12}z_{22} & z_{21}^2 + z_{22}^2 & z_{21}z_{31} + z_{22}z_{32} \\ z_{11}z_{31} + z_{12}z_{32} & z_{21}z_{31} + z_{22}z_{32} & z_{31}^2 + z_{32}^2 \end{bmatrix}$$

The image of f is the set of positive semidefinite symmetric 3×3 matrices of rank ≤ 2. This is a 5-dimensional semialgebraic set in $\mathbb{R}^6$. Its polynomial description consists of one equation $\det(X) = 0$ and six inequalities

$$x_{11} \geq 0,\ x_{22} \geq 0,\ x_{33} \geq 0,\ x_{11}x_{22} \geq x_{12}x_{21},\ x_{11}x_{33} \geq x_{13}x_{31},\ x_{22}x_{33} \geq x_{22}x_{31}.$$

If we pass to an algebraically closed field, the situation is much simpler. In Example 4.17 the image of $f : \mathbb{C}^9 \to \mathbb{C}^6$ is closed: it is precisely the hypersurface $\{\det(Z) = 0\}$. In general, we have the following result.

Theorem 4.18 (Chevalley)**.** *If K is algebraically closed, then the image of a variety under a polynomial map is a constructible set. Hence, if $K = \mathbb{C}$ then the Euclidean closure and the Zariski closure of the image coincide.*

Proof sketch. The image is a projection of the graph of the map. One can apply the Nullstellensatz - see Chapter 6 - to turn the problem into one from linear algebra. Resultants and their matrices play a central role. Details can be found in e.g. [**59**, Sections 7.4.6–7.4.8]. □

Suppose we want to check if a random (in a reasonable sense) point belongs to the image and we work over $\mathbb{C}$. It is enough to check polynomial *equations*, which can be obtained as described in previous two subsection.

Obtaining the whole description of the image is slightly more complicated. We may proceed as follows:

- Compute the closed image X_0.
- Subtract from it a proper subvariety X_1.
- Add back X_2 - a proper subvariety of X_1, etc.

This procedure must finish in a finite number of steps, by Hilbert's Basis Theorem. For a recent algorithm and its implementation we refer to [**26**].

Example 4.19. Let $m = n = 3$ and consider the map

$$f : \mathbb{C}^3 \to \mathbb{C}^3,\ (z_1, z_2, z_3) \mapsto (z_1 z_2, z_1 z_3, z_2 z_3).$$

Its image is Zariski dense in $\mathbb{C}^3$. It equals

$$\text{image}(f) \ = \ (\mathbb{C}^3 \backslash \mathcal{V}(x_1 x_2 x_2)) \cup \mathcal{V}(x_1 x_2, x_1 x_3, x_2 x_3).$$

Over the real numbers, we would also need the inquality $x_1 x_2 x_3 \geq 0$.

The images are nicest when we work with projective varieties over a field like $\mathbb{C}$. The following theorem may be regarded as an algebraic analog of the fact that images of compact sets under continuous maps are compact.

Theorem 4.20. *Let X be a projective variety over an algebraically closed field. Then the image of X under a regular map is (Zariski) closed.*

Proof. See [**46**, Section 5.2] □

This is a very powerful theorem. For instance, it implies the following. Consider a map $f = (f_1, \ldots, f_n)$ given by homogeneous polynomials of the same degree. Assume that the affine variety $\mathcal{V}(f_1, \ldots, f_n)$ equals $\{0\}$. Then the image of f is Zariski closed - we can compute it using elimination. For an application see Exercise 18. In general, the following theorem holds.

Theorem 4.21. *Consider a map $f = (f_1, \ldots, f_n) : \mathbb{C}^{m+1} \to \mathbb{C}^n$ given by homogeneous polynomials of the same degree in $\mathbb{C}[z_0, z_1, \ldots, z_m]$. Suppose that $\dim \mathcal{V}(f_1, \ldots, f_n) = b+1$ and the closed image of f has affine dimension $d+1$, i.e. projective dimension d. If $d+b < m$ then the image of f is closed.*

Proof. We consider f as a map from $\mathbb{P}^m \backslash \mathcal{V}(f_1, \ldots, f_n)$. Let $\mathbb{P}^d \subset \mathbb{P}^m$ be a generic linear subspace. It is disjoint from $\mathcal{V}(f_1, \ldots, f_n)$. Thus we may assume that f is well-defined on $\mathbb{P}^d$. The image of $\mathbb{P}^d$ under f is closed by Theorem 4.20. It is contained in the image of $\mathbb{P}^m$ under f. Both have the same dimension. So, the images coincide, and the image of f is closed. □

Exercises

(1) Eliminate the variable z from the equations $x^3y^3z^3 - x - y - z = 1$ and $x^5 + y^5 + z^5 = 2$.

(2) Prove: If an ideal I is prime then so are its elimination ideals, and same for radical. Give the examples when the coverse does not hold. What is the geometric meaning of these statements?

(3) Compute the determinants of the Sylvester matrices $\mathrm{Syl}_{1,5}$, $\mathrm{Syl}_{2,4}$ and $\mathrm{Syl}_{3,3}$. Each of them is a polynomial of degree 6 in 8 unknowns. Which of them has the most terms?

(4) A plane curve has the parametrization $z \mapsto \big(f(z), g(z)\big)$ where f and g are polynomials of degree 10. At most how many terms do you expect the implicit equation to have?

(5) Can you find an invertible 5×5-matrix that is skewsymmetric?

(6) You are given all entries of a skewsymmetric 5×5 matrix $X = (x_{ij})$ except for x_{12} and x_{45}. Under which condition on the 8 visible entries can you complete with $\mathrm{rank}(X) \leq 2$?

(7) Let π be the linear map from $\mathbb{C}^3$ to $\mathbb{C}^2$ given by the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Given an algebraic curve V in $\mathbb{C}^3$, explain how one can compute the plane curve $\overline{\pi(V)} \subset \mathbb{C}^2$.

(8) Consider the Fermat curve $V = \mathcal{V}(x^3 + y^3 + z^3)$ in the projective plane $\mathbb{P}^2$. Compute the ideal in 6 variables whose variety is the image of V under the *Veronese map*

$$\mathbb{P}^2 \to \mathbb{P}^5,\ (x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2).$$

(9) Determine the prime ideal of relations among the 3×3-minors of a 3×6-matrix.

(10) Let V_1 and V_2 be curves in $\mathbb{C}^3$ and $V_1 + V_2$ their pointwise sum. The Zariski closure $\overline{V_1 + V_2}$ is an algebraic variety in $\mathbb{C}^3$. Explain how one can compute its ideal $\mathcal{I}(V_1 + V_2)$.

(11) Compute the hyperdeterminant of a $2 \times 2 \times 3$ tensor whose 12 entries are unknowns.

(12) Apply the resultant method in Example 4.15 to compute the implicit equation of the plane cubic curve that has the parametrization

$$z \mapsto \big(2z^3 + 3z^2 + 5z + 7,\ 11z^3 + 13z^2 + 17z + 19\big).$$

(13) Let $m = 2$, $d_1 = 1$, $d_2 = d_3 = 2$. The total number of coefficients is $n = 15 = 3+6+6$. Compute the resultant $\mathrm{Res}(f_1, f_2, f_3)$ explicitly, as a polynomial in all 15 unknowns.

(14) Which constraints hold for off-diagonal entries of a rank one 3×3-matrix?

(15) Which constraints hold for the off-diagonal entries of a nilpotent 3×3-matrix? Answer this question over the complex numbers $\mathbb{C}$.

(16) Which constraints hold for the off-diagonal entries of an orthogonal 3×3-matrix? Answer this question over the complex numbers $\mathbb{R}$.

(17) Let $m = 2$ and $d_1 = d_2 = d_3 = 2$. Then $\mathrm{Res}(f_1, f_2, f_3)$ is the resultant of three quadrics in the plane. This is a polynomial in $18 = 6+6+6$ variables of degree $12 = 4+4+4$. How many terms does it have? Find an explicit matrix formula for $\mathrm{Res}(f_1, f_2, f_3)$.

(18) Let V be the complex vector space of homogeneous polynomials in n variables of degree d. The d-th powers of linear forms form a subset of V. Is it Zariski closed for any n and d? What happens if we change the field to the real numbers?

(19) Consider a degree two polynomial ax^2+bx+c, where a, b, c are unknown. When does it have a double root and how is this related to resultants? What happens for higher degree polynomials?

Chapter 5

Linear Spaces and Grassmannians

In previous chapters we saw the construction of projective space. We argued that in many applications projective varieties are preferable to affine varieties. Points in a projective space correspond to lines through the origin in the underlying vector space. In this chapter we replace lines with higher-dimensional linear subspaces. The role of the projective space is now played by a *Grassmannian.* This is a smooth projective variety whose points correspond to linear subspaces of a fixed dimension. For instance, the Grassmannian of lines in projective 3-space is 4-dimensional variety. Its subvarieties represent families of lines. Counting lines that satisfy a certain property (e.g. lying on a cubic surface) leads us to *enumerative algebraic geometry*, a subject in which Grassmannians play a fundamental role.

5.1. Coordinates for Linear Spaces

Let V be a vector space of dimension n over a field K. In Chapter 2 we constructed the projective space $\mathbb{P}(V)$. Its points are the 1-dimensional subspaces of V. We note that $\mathbb{P}(V)$ is the key example of a compact algebraic variety when $K = \mathbb{C}$. Our aim is to generalize this construction from lines to subspaces of arbitrary dimension k. We will construct a projective variety $G(k, V)$ whose points correspond bijectively to k-dimensional subspaces of V. This variety is called the Grassmannian, after the 19th century mathematician Hermann Grassmann. If $V = K^n$ then we use the notations $\mathbb{P}^{n-1}$ for the projective space $\mathbb{P}(V)$, and $G(k, n)$ for the Grassmannian $G(k, V)$.

We start with an explicit construction in coordinates, by fixing a basis $e_1, \dots, e_n$ of V. Consider any k linearly independent vectors $v_1, \dots, v_k \in V$. We represent them in a form of a $k \times n$ matrix M_W of rank k. To these vectors, or equivalently to a full rank matrix, we associate the linear subspace $\langle v_1, \dots, v_k \rangle$ in V. This association is surjective, but not injective, as we may replace the v_i's by linear combinations. In other words, the group $GL(k)$ of invertible $k \times k$ matrices acts on the set of $k \times n$ matrices by left multiplication, and this does not change the linear span of the rows.

We know some polynomial functions that do not change (up to scaling) under taking linear combinations of the rows: these are the $k \times k$ minors of the $k \times n$ matrix. Suppose that W is a k-dimensional subspace of V. Pick any basis and express W as the row space of a $k \times n$-matrix. We then write $\mathfrak{i}(W)$ for the vector of all $k \times k$-minor of that matrix, up to scale. This construction defines a map

$$\mathfrak{i} : \{k\text{-dimensional subspaces of } V\} \to \mathbb{P}(K^{\binom{n}{k}}).$$

This map is well-defined since $\mathfrak{i}(W)$ does not depend on chosen basis of W.

Lemma 5.1. *The map* $\mathfrak{i}$ *is injective.*

Proof. Consider two k-dimensional subspaces $W_1, W_2 \subset V$. Assume $\mathfrak{i}(W_1) = \mathfrak{i}(W_2)$. The matrices M_{W_1} and M_{W_2} that represent W_1 and W_2 have rank k. Without loss of generality we may assume that the first k columns are linearly independent. By performing linear operations on the rows of both matrices, we transform M_{W_i} to a matrix $\tilde{M}_{W_i}$ whose left-most $k \times k$ submatrix is the identity. We observe that any entry of $\tilde{M}_{W_i}$ not in the first k columns, is equal to some maximal minor or its negation. Thus, if $\mathfrak{i}(W_1) = \mathfrak{i}(W_2)$ then the two matrices $\tilde{M}_{W_1}$ and $\tilde{M}_{W_2}$ must be equal. This implies $W_1 = W_2$. □

The image of $\mathfrak{i}$ is the *Grassmannian* $G(k, n)$. Its inclusion in $\mathbb{P}(K^{\binom{n}{k}})$ is the *Plücker embedding*. For readers familiar with the exterior power of a vector space, here is a more invariant description of the Grassmannian:

$$G(k, n) = \{[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V) : v_1, \dots, v_k \in V \text{ are linearly independent}\}.$$

Indeed, first we may identify $\mathbb{P}(K^{\binom{n}{k}})$ with $\mathbb{P}(\bigwedge^k V)$ by fixing a basis of V and an induced basis of $\bigwedge^k V$. Expanding $v_1 \wedge \cdots \wedge v_k$ in that basis, we indeed obtain the $k \times k$ minors of the $n \times k$ matrix $[v_1, \dots, v_k]$. The group $GL(V)$ acts naturally on V, taking subspaces to subspaces. This induces an action on $\mathbb{P}(\bigwedge^k V)$ which restricts to the Grassmannian. A matrix $g \in GL(V)$ transforms $v_1 \wedge \cdots \wedge v_k$ to $g(v_1) \wedge \cdots \wedge g(v_k)$. We note that the action is *transitive*: for any $p_1, p_2 \in G(k, V)$ there exists a (non-unique) automorphism $g \in GL(V)$ such that $g(p_1) = p_2$. This holds because any set of k linearly independent vectors may be transformed by an invertible linear map to any other such set. Hence, $G(k,V)$ is an *orbit* under the action of $GL(V)$ on $\mathbb{P}(\bigwedge^k V)$. In fact, $G(k,V)$ is the unique closed orbit in this space.

Projective algebraic varieties that are orbits of linear algebraic groups are called *homogeneous*, the Grassmannians being prominent examples. Homogeneous varieties are always smooth. Indeed, any algebraic variety always contains a smooth point and an action of a group must take a smooth point to a smooth point - a version of this statement is given in Exercise 2.

5.2. Plücker Relations

Our aim is now to demonstrate that the Grassmannian is a projective variety. Equivalently, we need to express the fact that $\binom{n}{k}$ numbers are minors of a matrix, by vanishing of (homogeneous) polynomials.

Theorem 5.2. *The Grassmannian $G(k,n) \subset \mathbb{P}(K^{\binom{n}{k}})$ is Zariski closed and irreducible.*

Proof. Lemma 5.1 gives us an idea how to proceed. Namely, first let us assume that the matrix M_W representing W is of the form:

$$\left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & \\ \vdots & & \ddots & \vdots & \mathrm{A} \\ 0 & \dots & 0 & 1 & \end{array}\right], \tag{5.1}$$

where A is a $k \times (n-k)$ matrix. Each maximal minor of M_W is now, up to sign, a minor of A of some size. Further, by Laplace expansion, a $q \times q$ minor of A for $q > 1$ may be expressed, as a quadratic polynomial, in terms of smaller minors. This provides with $\sum_{q=2}^{\min(k,n-k)} \binom{k}{q}\binom{n-k}{q}$ inhomogeneous quadratic equations in the entries of the $k \times (n-k)$-matrix A. These equations define the part of the image of our map $\mathfrak{i}$ that lies in the affine open set $K^{\binom{n}{k}-1} \subset \mathbb{P}(K^{\binom{n}{k}})$ given by the nonvanishing of the first Plücker coordinate.

If $\mathfrak{i}(W)$ has its first coordinate zero then some other coordinate will be nonzero. In other words, the matrix M_W must have some invertible $k \times k$ submatrix. If we multiply M_W on the left by the inverse of that matrix then we obtain a matrix that looks like (5.1) but with its columns permuted. The same construction as before gives us a system of $\sum_{q=2}^{\min(k,n-k)} \binom{k}{q}\binom{n-k}{q}$ inhomogeneous quadratic equations in the $k(n-k)$ entries of the new matrix A.

Each of the quadratic equations in $k(n-k)$ variables obtained above can be written as a homogeneous quadric in the $\binom{n}{k}$ coordinates on $\mathbb{P}(K^{\binom{n}{k}})$. Namely, a minor of A is replaced by the corresponding maximal minor of

M_W, and then the quadric is homogenized by the special minor that corresponds to the identity matrix in (5.1). The collection of all these homogeneous quadratic equations gives a full polynomial description of $G(k,n)$.

The Grassmannian $G(k,n)$ is an irreducible subvariety of $\mathbb{P}(K^{\binom{n}{k}})$ because it is the image of a polynomial map $\mathfrak{i}$, namely the image of the space $K^{k\times n}$ of all $k \times n$ matrices under taking all maximal minors. □

We have proved that the set $G(k,n)$ is cut out by quadratic equations. In fact, with slightly more effort one can show that $I(G(k,n))$ may be generated by quadratic polynomials. These are known as *Plücker relations* [**38**, Chapter 3]. Further below we will discuss the Plücker relations for $k = 2$. From a more algebraic perspective the equations vanishing on $G(k,n)$ are exactly the *polynomial relations among maximal minors.* We point out that finding the ideal of polynomial relations among the nonmaximal minors of a fixed size is an open problem in commutative algebra.

Another fact that follows from the proof, is that the intersection $G(k,n) \cap K^{\binom{n}{k}-1}$ of the Grassmannian with the open affine is the affine space $K^{k\times(n-k)}$.

Corollary 5.3. *The dimension of the Grassmannian $G(k,n)$ equals $k(n-k)$.*

Remark 5.4. *The Grassmannian $G(k,n)$ parametrizes k-dimensional vector subspaces of an n-dimensional vector space, or equivalently $k-1$ dimensional projective subspaces of an $n-1$ dimensional projective space.*

Example 5.5 (k=2, n=4)**.** The Grassmannian $G(2,4)$ is the image of the map

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \mapsto (af{-}be : ag{-}ce : ah{-}de : bg{-}cf : bh{-}df : ch{-}dg) \in \mathbb{P}^5.$$

Alternatively, fixing a basis (v_1, v_2, v_3, v_4) of $V \simeq K^4$, we may write:

$$(av_1 + bv_2 + cv_3 + dv_4) \wedge (ev_1 + fv_2 + gv_3 + hv_4) =$$

$$(af - be)v_1 \wedge v_2 + (ag - ce)v_1 \wedge v_3 + (ah - de)v_1 \wedge v_4$$

$$+(bg - cf)v_2 \wedge v_3 + (bh - df)v_2 \wedge v_4 + (ch - dg)v_3 \wedge v_4.$$

This Grassmannian has dimension 4, i.e. it is a hypersurface in $\mathbb{P}^5$. We write the coordinates on $\mathbb{P}^5$ as $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$. The indices refer to the minors of a 2×4-matrix. Following the proof of Theorem 5.2, we look at matrices (5.1). They take the form

$$\begin{bmatrix} 1 & 0 & c & d \\ 0 & 1 & g & h \end{bmatrix}.$$

The expansion of the rightmost 2×2-minor yields the inhomogeneous quadratic equation $p_{34} = ch - dg = (-p_{23})p_{14} - (-p_{24})p_{13}$. We homogenize this equation with the extra variable p_{12}. We conclude that the Grassmannian $G(2,4)$ is the hypersurface in $\mathbb{P}^5$ that is defined by the *Plücker quadric*

$$p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}. \tag{5.2}$$

We now discuss the homogeneous prime ideal $I(G(k,n))$ of the Grassmannian $G(k,n)$. A complete description is known, in terms of certain quadratic relations that form a Gröbner basis. These are known as *straightening relations*. For a derivation and explanation see e.g. [**51**, Chapter 3].

We here present the answer in the special case $k = 2$. The corresponding Grassmannian $G(2,n)$ is the space of lines in $\mathbb{P}^{n-1}$. It is convenient to write the $\binom{n}{2}$ Plücker coordinates as the entries of a skew-symmetric $n \times n$-matrix $P = (p_{ij})$. We are interested in the principal submatrices of P having size 4×4. One such submatrix is given by taking the first four rows and first four columns. The determinant of that matrix is the square of the Plücker quadric (5.2). One refers to the square root of the determinant of a skew-symmetric matrix of even order as its *pfaffian*. Thus the 4×4 pfaffians of our matrix P are the $\binom{n}{4}$ quadrics

$$\underline{p_{il}p_{jk}} - p_{ik}p_{jl} + p_{ij}p_{kl} \qquad \text{for} \quad 1 \leq i < j < k < l \leq n. \tag{5.3}$$

Theorem 5.6. *The $\binom{n}{4}$ quadrics in (5.3) form the reduced Gröbner basis of the Plücker ideal $I(G(2,n))$, for any monomial ordering on the polynomial ring in the $\binom{n}{2}$ variables p_{ij} that selects the underlined leading terms.*

Proof. The argument in the proof of Theorem 5.2 shows that the quadrics (5.3) cut out $G(2,n)$ as a subset in $\mathbb{P}^{\binom{n}{2}-1}$. In other words, our Grassmannian is given as the set of skew-symmetric $n \times n$-matrices whose 4×4 pfaffians vanish. These are skew-symmetric matrices of rank 2.

By Hilbert's Nullstellensatz (proved in the next chapter), we can conclude that the radical of the ideal $I(G(2,n))$ is generated by (5.3). We need to argue that this ideal is radical. However, this follows from the assertion that (5.3) form a Gröbner basis. Indeed, the leading monomials $p_{il}p_{jk}$ are square-free, so they generate a radical monomial ideal. However, if the initial ideal $\mathrm{in}(J)$ of some ideal J is radical then also J itself is radical. So, all we need to do is to verify the Gröbner basis property for our quadrics. That Gröbner basis is then automatically a reduced Gröbner basis because none of the two trailing terms in (5.3) is a multiple of some other leading term.

To verify the Gröbner basis property, we reason as follows. For $n = 4$, it is trivial because there is only one generator. For $n = 5, 6, 7$, this is a direct computation, e.g. using `Macaulay2`. One checks that the S-polynomial of any two quadrics in (5.3) reduces to zero. Suppose that $n \geq 8$ and consider two Plücker quadrics. These involve at most 8 distinct indices. If the number of distinct indices is 7 or less then we are done by the aforementioned computation, which verified the claim for $n \leq 7$. Hence we may assume that all eight indices occurring in the two Plücker quadrics are distinct. In that case, the two underlined leading monomials are relatively prime. Here, Buchberger's Second Criterion applies, and we can conclude that the S-polynomial automatically reduces to zero. In conclusion, all S-polynomials formed by pairs from (5.3) reduce to zero. This completes the proof. □

The Plücker relations for arbitrary Grassmannians are hard-wired in the computer algebra system `Macaulay2`. One finds generators for the ideal $I(G(k,n))$ with the convenient command `Grassmannian(k-1,n-1)`. Here, the parameters k and n are decreased by one because `Macaulay2` refers to the projective geometry interpretation: points in $G(k,n)$ correspond to linear spaces of dimension $k-1$ in an ambient projective space of dimension $n-1$. Another thing that is tricky about using the command `Grassmannian` is the ordering of the Plücker coordinates in `Macaulay2`. Here is how it works.

Example 5.7 $(k = 3, n = 6)$**.** The following two command lines yield equations defining the Grassmannian of 3-dimensional vector subspaces in K^6.

```
R = QQ[p123,p124,p134,p234,p125,p135,p235,p145,p245,
       p345,p126,p136,p236,p146,p246,p346,p156,p256,p356,p456];
I = Grassmannian(2,5,R)
```

This produces 35 quadratic relations. Note that $G(3,6)$ is a smooth projective variety of dimension 9 and degree 42 in $\mathbb{P}^{19}$, as is seen by also typing

```
dim I, degree I, betti mingens I
```

Among the 35 minimal generators of $I(G(3,6))$, there are 30 trinomials, like $p_{134}p_{125} - p_{124}p_{135} + p_{123}p_{145}$, plus five additional relations that involve all six indices, like $p_{345}p_{126} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456}$. What is the combinatorial pattern? Can you generalize it to larger values of k and n?

5.3. Schubert Calculus

In this section we show how Grassmannians can help us answer enumerative questions. We would like to know how many lines or planes in space satisfy some properties. This general subject area is known as *enumerative geometry*, and the specific answers we provide are based on *Schubert calculus.* Thus, we introduce an intersection theory for subvarieties of a Grassmannian.

We shall illustrate the concepts and questions for the special case of $G(2,4)$. Here is the simplest question of Schubert calculus. *How many lines L intersect four general lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$?* The answer to this question is: *two.* To see this, we represent the line L by its corresponding point $p = (p_{12} : p_{13} : \cdots : p_{34})$ in $G(2,4)$ in $\mathbb{P}^3$. The condition that L

intersects a fixed L_i is a linear condition in p. We must solve four such linear equations, and these have general coefficients since L_1, L_2, L_3, L_4 are general lines. In addition, there is the quadratic equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. Thus, we are solving five equations in $\mathbb{P}^5$ of degrees $1, 1, 1, 1, 2$. By Bézout's Theorem, this system has two solutions p, representing the two lines L.

To study such intersection problems more systematically, one introduces some special subvarieties of Grassmannians. We continue with the example of $G(2,4)$, the manifold of all lines in $\mathbb{P}^3$. We fix a complete flag in $\mathbb{P}^3$, consisting of a point in a line in a plane: $f_0 = \mathbb{P}^0 \subset f_1 = \mathbb{P}^1 \subset f_2 = \mathbb{P}^2 \subset \mathbb{P}^3$. Our aim is to group families of projective lines according to how they intersect that flag. These will be subvarieties X_i of dimension i in $G(2,4)$.

First, the flag distinguishes a point in $G(2,4)$, namely $X_0 := f_1 \in G(2,4)$. There is also a distinguished curve X_1 in $G(2,4)$. The points of X_1 are the lines l such that $f_0 \in l \subset f_2$. The most interesting is the case of surfaces in our family. There are two types of those in $G(2,4)$:

(1) the surface X_2 consisting of all lines l such that $f_0 \in l$ and

(2) the surface $X_{2'}$ consisting of all lines l such that $l \subset f_2$.

Finally, there is also one three-dimensional variety X_3, consisting of all lines that intersect the given line f_1. The varieties $X_1, X_2, X_{2'}$ and X_3 we described are called the *Schubert varieties* in $G(2,4)$. In Exercise 1 you will generalize the construction of Schubert varieties to larger Grassmannians.

Let us now fix a basis v_1, v_2, v_3, v_4 for K^4 and take f_i to be the linear subspace in $\mathbb{P}^3$ spanned by $v_1, \ldots, v_{i+1}$. The point $f_1 \in G(2,4)$ is given in $\mathbb{P}^5$ by the vanishing of all coordinates p_{ij} apart from p_{12}. We have $f_0 \in l \subset f_2$ if and only if the line l is spanned by the rows of a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & 0 \end{bmatrix}.$$

Hence, the curve X_1 is a line $\mathbb{P}^1$ in $G(2,4) \subset \mathbb{P}^5$, defined by the vanishing of all p_{ij} apart from p_{12} and p_{13}. Similarly, X_2 is the $\mathbb{P}^2$ with coordinates p_{12}, p_{13}, p_{14}, and $X_{2'}$ is a different $\mathbb{P}^2$ with coordinates p_{12}, p_{13}, p_{23}. The common span of these two planes is the 3-dimensional variety $X_3 = \mathcal{V}(p_{34})$.

The relationship between X_2, $X_{2'}$ and X_1 inside $G(2,4)$ can also be understood as follows. For any integer $k \leq 1$, let Q be a nonsingular quadratic hypersurface in $\mathbb{P}^{2k+1}$ and consider a linear subspace $L = \mathbb{P}^{k-1}$ that is contained in Q. Then there exist precisely two k-dimensional subspaces that contain L and are contained in Q. For $k = 2$ and $Q = G(2,4)$ containing the line $L = X_1$ in $\mathbb{P}^5$, the two subspaces are the planes X_2 and $X_{2'}$ in $\mathbb{P}^5$.

For $k = 1$, our statement is a classical fact of projective geometry. A quadratic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For point $p = L$ in the

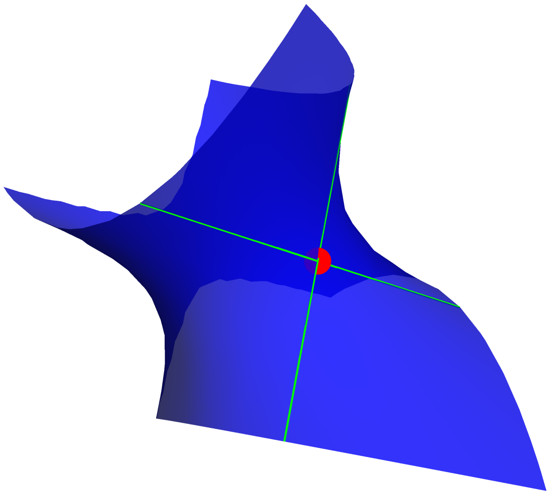


Figure 1. Every (red) point on a smooth quadratic surface lies on two (green) lines that are contained in the surface. This picture illustrates the relationship among the Schubert subvarieties $X_1, X_2, X_{2'}$ of $G(2,4)$.

quadric, there are precisely two lines that contain p and lie on the quadric. In Figure 1, the blue quadric contains the red point p, and the two lines are green. This 3-dimensional picture arises by intersecting $G(2,4)$ with the subspace $H = \mathbb{P}^3$ that is defined by $p_{12} = p_{34} = 0$ inside $\mathbb{P}^5$. The red point p equals $H \cap X_1$. The quadric is simply $H \cap G(2,4) = \mathcal{V}(p_{23}p_{14} - p_{13}p_{24})$, and the two green lines are the intersections $X_2 \cap H$ and $X_{2'} \cap H$.

The Schubert subvarieties of a Grassmannian represent a basis for the cohomology ring of the Grassmannian when the underlying field is $K = \mathbb{C}$. One learns in *algebraic topology* that the multiplication in a cohomology ring corresponds to intersections of submanifolds. This can then be used to answer enumerative questions, namely for counting the number of points in an intersection that turns out to be 0-dimensional. *Schubert calculus* is the art of making this precise when the ambient manifold is a Grassmannian.

When intersecting Schubert varieties X_i as cohomology classes $[X_i]$, one should consider them coming from *different, general* flags. For instance, consider the two surfaces X_2 and $X_{2'}$. They intersect in the line X_1. However, when thinking of the classes, the former represents all lines through some arbitrary point in $\mathbb{P}^3$ and the latter represents all lines contained in some entirely unrelated plane in $\mathbb{P}^3$. There are no lines satisfying both condition, so the intersection of the $[X_2]$ and $[X_{2'}]$ is the class of the empty set. We write this as $[X_2] \cdot [X_{2'}] = 0$. On the other hand, if we ask for lines going through two distinct points, or for lines contained in two distinct planes, then there is one solution. The selfintersections of the classes $[X_2]$ and $[X_{2'}]$

give one point. That one point is represented by the element 1 in the cohomology ring. Our discussion is summarized by the following relations that hold in the cohomology ring of the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$:

$$[X_3][X_3] = [X_2]+[X_{2'}],\ [X_2][X_2] = [X_{2'}][X_{2'}] = [X_3][X_1] = 1,\ [X_2][X_{2'}] = 0.$$

Recall that multiplication represents intersection and sum represents union. The fact that the Schubert classes $[X_i]$ form a basis for the cohomology ring means that the class $[Z]$ of any subvariety Z is an $\mathbb{N}$-linear combination of the $[X_i]$. Finding that linear combination for a given Z is similar to computing the degree of a subvariety in $\mathbb{P}^n$, as discussed at the end of Chapter 1.

We now have a conceptual framework for studying enumerative questions like *How many lines pass through four general lines in $\mathbb{P}^3$?* The set of such lines is a finite subset in $G(2,4)$. It is the intersection of four hypersurfaces, all of the form X_3. Since intersections are represented by multiplication in the cohomology ring, the following formal computation reveals the answer:

$$\begin{aligned}[X_3]^4 \;=\; ([X_3][X_3])^2 \;=\; ([X_2]+[X_{2'}])^2 \;=\\ [X_2]^2 + 2[X_2][X_{2'}] + [X_{2'}]^2 \;=\; 1+2\cdot 0+1 \;=\; \mathbf{2}.\end{aligned}$$

Here is another question that can be answered by Schubert calculus: *How many lines are simultaneously tangent to four general quadratic surfaces in $\mathbb{P}^3$?* Suppose a given quadric Q in $\mathbb{P}^3$ is represented as a symmetric 4×4-matrix. Let $\wedge^2 Q$ be the symmetric 6×6 matrix with entries given by the 2×2 minors of the matrix representing Q. The condition for a line to be tangent to Q is expressed by the vanishing of the quadratic form $P(\wedge^2 Q)P^T$ in the Plücker coordinates $P = (p_{12}, p_{13}, \ldots, p_{34})$ of that line. This defines a threefold in $G(2,4)$ and the cohomology class of that threefold is $2[X_3]$.

We conclude that the number of lines that are tangent to four given general quadrics in $\mathbb{P}^3$ is equal to

$$(2[X_3])^4 \;=\; 16[X_3]^4 \;=\; 16\cdot 2 \;=\; 32.$$

In order to actually compute the 32 lines over $\mathbb{C}$, given four concrete quadrics in $\mathbb{P}^3$, we would need to carry out some serious Gröbner basis computations.

Remark 5.8. Grassmannians are named after Hermann Grassmann. However, it was Julius Plücker who first noted that lines in 3-space may be studied as a four-dimensional object [**43**]. Yet, Grassmann's earlier discoveries were fundamental: he was the one to realize that the algebraic setting of geometry allows to consider objects not only in 3-dimensional space, but in any dimension. Can you imagine data science without Grassmann's insight?

Exercises

(1) Fix a complete flag in $\mathbb{P}^n$. Construct a bijection between:
- subvarieties of $G(k,n)$ that consist of $l \in G(k,n)$ that intersect each element of the flag in at most the given dimension, and
- Young diagrams contained in a $k \times (n-k)$ rectangle.

Either the codimension or the dimension of the subvariety in $G(k,n)$ should equal the number of boxes in the corresponding Young diagram.

(2) Let G be a subgroup of the general linear group $GL(V)$, and let $X \subset V$ be a variety such that the action of G on V restricts to X. Prove that if x is a smooth point of X and $g \in G$, then gx is also a smooth point.

(3) Consider the inclusion $G(2,4) \times G(2,4) \subset \mathbb{P}^5 \times \mathbb{P}^5$. Using Plücker coordinates, describe the locus of pairs of lines $(l_1, l_2) \in G(2,4) \times G(2,4)$ such that l_1 intersects l_2 in $\mathbb{P}^3$. Hint: Present both lines as 2×4 matrices. Note that two lines in $\mathbb{P}^3$ intersect if and only if they do not span the whole ambient space. Apply Laplace expansion of the determinant.

(4) For a variety $X \subset \mathbb{P}^n$, one considers a subset of $G(k+1, n+1)$ of $\mathbb{P}^k \subset X$. This is known as the *Fano variety* of k-dimensional subspaces of X. Fix a nondegenerate quadric $Q \subset \mathbb{P}^3$. Describe the Fano variety of lines in it. Hint: One may solve this exercise either theoretically or using algebra software. Also Figure 1 gives a hint about the answer.

(5) How many real lines in 3-space can be simultaneously tangent to four given spheres?

(6) The two lines incident to four given real lines in $\mathbb{P}^3$ can be either real or complex. In the latter case they form a complex conjugate pair. Write down a polynomial in the $24 = 4 \cdot 6$ Plücker coordinates of four given lines whose sign distinguishes the two cases.

(7) How many lines in $\mathbb{P}^3$ are simultaneously incident to two given lines and tangent to two given quadratic surfaces?

(8) Consider the set of all lines in $\mathbb{P}^3$ that are tangent to the cubic Fermat surface $\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}$. This set is an irreducible hypersurface in the Grassmannian $G(2,4)$. Compute a polynomial in $p_{12}, p_{13}, \ldots, p_{34}$ that defines this hypersurface.

(9) Find a minimal generating set for the ideal of the Grassmannian $G(3,7)$.

(10) Prove that the determinant of a skew-symmetric $n \times n$-matrix is zero if n is odd, and it is the square of a polynomial when n is even.

(11) Examine the monomial ideal that is generated by the underlined initial monomials in (5.3). Express this ideal as the intersection of prime ideals. How many primes occur?

(12) Fix six general planes $\mathbb{P}^2$ in $\mathbb{P}^4$. How many lines in $\mathbb{P}^4$ intersect all six planes? Embark towards Schubert calculus in the Grassmannian $G(2,5)$.

(13) Let $n = 2k$ and suppose that the $n \times n$-matrix $A = (a_{ij})$ in (5.1) is symmetric, i.e. its entries satisfy the equations $a_{ij} = a_{ji}$. Express these equations in terms of the $\binom{2n}{n}$ Plücker coordinates. The resulting subvariety of $G(n, 2n)$ is known as the *Lagrangian Grassmannian.*

Chapter 6

Nullstellensätze

The German noun *Nullstellensatz* refers to a theorem that characterizes the existence of a zero (= Nullstelle) for a system of polynomials. The classical version, due to Hilbert, works over algebraically closed fields. It says that the nonexistence of zeros is equivalent to the existence of a partition of unity for the given polynomials. A more general version furnishes a bijection between varieties and radical ideals. In this chapter we also discuss the analogous results over the field of real numbers. Here the main results are the real Nullstellensatz and the Positivstellensatz. These furnish criteria for polynomial equations and inequalities to have no real solutions. This leads us to *real radical ideals* and their characterization via *sums of squares*.

6.1. Certificates for Infeasibility

In Chapter 3 we discussed how to find and represent solutions to a system of polynomial equations. But what if such a solution does not exist? In this chapter we present methods to prove that a given system has no solution.

Throughout this section, we fix an algebraically closed field K, such as the complex numbers $K = \mathbb{C}$. We write $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ for its polynomial ring in n variables. For an ideal $I \subset K[\mathbf{x}]$ we denote the associated variety in K^n by $\mathcal{V}(I)$, as in Chapter 2. We begin with the following weak version of the Nullstellensatz. This result appears as Theorem 1 in [**10**, §4.1].

Theorem 6.1. *If I is a proper ideal in $K[\mathbf{x}]$, i.e. $1 \notin I$, then its variety $\mathcal{V}(I)$ in K^n is non-empty.*

Proof. Our proof follows [**10**, §4.1]. We use induction on n. For $n = 1$, the statement of the theorem holds because every non-constant polynomial in one variable has a zero in the algebraically closed field K.

Let now $n \geq 2$. For $a \in K$, we write $I_{x_n=a}$ for the ideal in $K[x_1, \ldots, x_{n-1}]$ that is obtained by setting $x_n = a$ in each element of I. One easily checks that this is indeed an ideal. We claim that there exists a scalar $a \in K$ such that $1 \notin I_{x_n=a}$. In such a case, by induction, there is a point $(a_1, \ldots, a_{n-1})$ in $\mathcal{V}(I_{x_n=a})$. This implies that $(a_1, \ldots, a_{n-1}, a)$ is a point in the variety $\mathcal{V}(I)$.

Consider the elimination ideal $I \cap K[x_n]$. To prove the claim, we distinguish two cases. First suppose that this ideal is not the zero ideal. Since $1 \notin I$, the principal ideal $I \cap K[x_n]$ is generated by a nonconstant polynomial

$$f(x_n) \;=\; \prod_{i=1}^{r} (x_n - b_i)^{m_i}.$$

Suppose that $1 \in I_{x_n=b_i}$ for $i = 1, 2, \ldots, r$. If this is not the case then we are done. Hence there exist $B_1, \ldots, B_r \in I$ such that $B_i(x_1, \ldots, x_{n-1}, b_i) = 1$ for all i. Note that B_i is congruent to 1 modulo $\langle x_i - b_i \rangle$ in $K[\mathbf{x}]$. This implies that the product $\prod_{i=1}^{r}(B_i - 1)^{m_i}$ belongs to the ideal $\langle f \rangle$. Since $f \in I$ and $B_i \in I$, the following identify holds modulo the ideal I:

$$0 = \prod_{i=1}^{r} (B_i - 1)^{m_i} = \prod_{i=1}^{r} (-1)^{m_i} = \pm 1, \quad \text{ i.e. } 1 \in I.$$

Next suppose $I \cap K[x_n] = \{0\}$. Let $\{g_1, \ldots, g_t\}$ be a Gröbner basis for I with respect to the lexicographic order with $x_1 > \cdots > x_n$. Write $g_i = c_i(x_n)x^{\alpha_i} +$ lower order terms, where x^{α_i} is a monomial in $x_1, \ldots, x_{n-1}$. Since the field K is infinite, we can choose $a \in K$ such that $c_i(a) \neq 0$ for all i. The polynomials $\overline{g}_i = g_i(x_1, \ldots, x_{n-1}, a)$ form a Gröbner basis for $I_{x_n=a}$, for the lexicographic monomial order, with leading monomials x^{α_i} for $i = 1, \ldots, r$. None of these monomials is 1, since $I \cap K[x_n] = \{0\}$. This implies that 1 is not in the ideal $I_{x_n=a}$. □

Theorem 6.1 gives a certificate for the non-existence of solutions to a system of polynomial equations, namely the partition of unity we had promised.

Corollary 6.2. *A collection of polynomials $f_1, \ldots, f_r \in K[\mathbf{x}]$ either has a common zero in K^n or there exists an identity $g_1 f_1 + \cdots + g_r f_r = 1$ with polynomial multipliers $g_1, \ldots, g_r \in K[\mathbf{x}]$. This is the desired certificate.*

Proof. Let $I = \langle f_1, \ldots, f_r \rangle$. The either $\mathcal{V}(I) \neq \emptyset$ or $\mathcal{V}(I) = \emptyset$. In the latter case, $1 \in I$, meaning that 1 is a polynomial linear combination of the f_i. □

Example 6.3. Let $n = 2$ and consider the following three polynomials

$$f_1 = (x+y-1)(x+y-2), \;\; f_2 = (x-y+3)(x+2y-5), \;\; f_3 = (2x-y)(3x+y-4).$$

These do not have a common zero. This is proved by the certificate

$$g_1 f_1 \;+\; g_2 f_2 \;+\; g_3 f_3 \;\;=\;\; 1, \tag{6.1}$$

$$\text{where}\quad g_1 \;=\; \tfrac{895}{756}x^2 - \tfrac{6263}{2160}x - \tfrac{2617}{2520}y + \tfrac{4327}{1008}\,,$$

$$g_2 \;=\; \tfrac{5191}{3780}x^2 + \tfrac{358}{945}xy - \tfrac{6907}{3024}x - \tfrac{2123}{15120}y + \tfrac{3823}{7560}\,,$$

$$\text{and}\quad g_3 \;=\; -\tfrac{179}{420}x^2 - \tfrac{716}{945}xy + \tfrac{1453}{1080}x - \tfrac{716}{945}y + \tfrac{13771}{7560}\,.$$

The reader is invited to verify the identity (6.1). The multipliers g_1, g_2, g_3 do not look so pretty, or? But, remember: beauty is in the eye of the beholder.

There are two possible methods for computing the multipliers $(g_1, \ldots, g_r)$ for the Nulllstellensatz certification, as in Corollary 6.2. The first method is to use the *Extended Buchberger Algorithm*. This is analogous to the Extended Euclidean Algorithm for the ring of integers or the ring of polynomials in one variable. For instance, given any collection of relatively prime integers, this method writes 1 as a $\mathbb{Z}$-linear combination of these integers.

In the Extended Buchberger Algorithm one keeps track of the polynomial multipliers that are used to generate new S-polynomials from current basis polynomials. In the end, each element in the final Gröbner basis is written explicitly as a polynomial linear combination of the input polynomials. If $\mathcal{V}(I) = \emptyset$ then that final Gröbner basis is the singleton $\{1\}$.

The second method for computing Nullestellensatz certificates is to use degree bounds plus linear algebra. Let d be any integer that exceeds the degree of each f_i. Let g_i be a polynomial of degree $d - \deg(f_i)$ with coefficients that are unknowns, for $i = 1, 2, \ldots, r$. The desired identity $\sum_{i=1}^r g_i f_i = 1$ translates into a system of linear equations in all of these unknowns. We solve this system. If a solution is found then this gives a certificate. If not then there is no certificate in degree d, and we try a higher degree.

The two methods, in complete generality, can be very complicated to carry out in practice. The computation of Gröbner bases does not run in polynomial time. Worst-case complexity bounds for Gröbner bases are quite horrible. Furthermore, the degree of the multipliers g_i above are not polynomial in the input degrees either. Many mathematicians and computer scientists believe that there is no polynomial-time algorithm to decide if a given polynomial system has a complex solution. The situation is even worse if we want solutions with coordinates in $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$. In the last case, it is known there exists no algorithm at all – irrespective of complexity – to decide if a system has an integral solution. This was Hilbert's 10-th problem.

6.2. Hilbert's Nullstellensatz

Hilbert's Nullstellensatz offers a characterization of the set of all polynomials that vanish on a given variety. This classical result from 1890 works over any algebraically closed field K, such as the complex numbers $K = \mathbb{C}$. In this section we present this theorem and we discuss some of its ramifications.

Recall that the *radical* of an ideal I in $K[\mathbf{x}]$ is the (possibly larger) ideal

$$\sqrt{I} \;=\; \bigl\{\, f \in K[\mathbf{x}] \,:\, f^m \in I \text{ for some } m \in \mathbb{N} \,\bigr\}.$$

This is a radical ideal, hence it is an intersection of prime ideals.

Example 6.4. Consider the ideal $I \;=\; \langle\, x_1x_3,\; x_1x_4 + x_2x_3,\; x_2x_4 \,\rangle$ in the polynomial ring in four variables. This is not radical. To see this, note that the monomial $f = x_1x_4$ is not in I but f^2 is in I. The radical of I equals

$$\sqrt{I} \;=\; \langle\, x_1x_3,\, x_1x_4,\, x_2x_3,\, x_2x_4 \,\rangle \;=\; \langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle.$$

How many associated primes does the ideal I have? Do Gröbner bases of I give any hints? We refer to Example 3.27 for the answer.

We now show that $\sqrt{I}$ comprises all polynomials that vanish on $\mathcal{V}(I)$.

Theorem 6.5 (Hilbert's Nullstellensatz)**.** *For any ideal I in the polynomial ring $K[\mathbf{x}]$ in n variables over an algebraically closed field K, we have*

$$\mathcal{I}\bigl(\,\mathcal{V}(I)\,\bigr) \;=\; \sqrt{I}. \tag{6.2}$$

Proof. The radical $\sqrt{I}$ is contained in the vanishing ideal $\mathcal{I}\bigl(\,\mathcal{V}(I)\,\bigr)$, because $f^m(\mathbf{a}) = 0$ implies $f(\mathbf{a}) = 0$ for all $\mathbf{a} \in K^n$. We must show the left hand side is a subset of the right hand side in (6.2). Let $I = \langle f_1, \ldots, f_r\rangle$ and suppose that f is a polynomial which vanishes on $\mathcal{V}(I)$. Let y be a new variable and consider the ideal $J = \langle f_1, \ldots, f_r, yf - 1\rangle$ in the enlarged polynomial ring $K[\mathbf{x}, y] = K[x_1, \ldots, x_n, y]$. The variety $\mathcal{V}(J)$ in K^{n+1} is the empty set because $f = 0$ on every zero of $f_1, \ldots, f_r$ and $f \neq 0$ on every zero of $yf - 1$. By Theorem 6.1, there exist multipliers $g_1, \ldots, g_r, h$ in $K[\mathbf{x}, y]$ such that

$$\sum_{i=1}^{r} g_i(\mathbf{x}, y) \cdot f_i(\mathbf{x}) \;+\; h(\mathbf{x}, y) \cdot (yf(\mathbf{x}) - 1) \;=\; 1.$$

We now substitute $y = 1/f(\mathbf{x})$ into this identity. This yields the following identify of rational functions in n variables:

$$\sum_{i=1}^{r} g_i\bigl(\mathbf{x}, \frac{1}{f(\mathbf{x})}\bigr) \cdot f_i(\mathbf{x}) \;=\; 1.$$

The common denominator is $f(x)^m$ for some $m \in \mathbb{N}$. Multiplying both sides with this common denominator, we obtain a polynomial identity of the form

$$\sum_{i=1}^{r} p_i(\mathbf{x}) \cdot f_i(\mathbf{x}) \;=\; f(\mathbf{x})^m.$$

This shows that f^m lies in I, and hence f lies in the radical $\sqrt{I}$. $\square$

Example 6.6. Which polynomial functions vanish on all nilpotent 3 × 3-matrices? We set $n = 9$ and take I to be ideal generated by the entries of X^3, where $X = (x_{ij})$ is a 3 × 3-matrix with variables as entries. These are nine homogeneous cubic polynomials in nine unknowns x_{ij}. One of them is

$$x_{11}^3+2x_{11}x_{12}x_{21}+x_{12}x_{21}x_{22}+2x_{11}x_{13}x_{31}+x_{12}x_{23}x_{31}+x_{13}x_{21}x_{32}+x_{13}x_{31}x_{33}.$$

But what are all polynomials that vanish on nilpotent matrices? Can they all be written as linear combinations of these nine cubics? The answer is no. The ideal $\mathcal{I}\big(\mathcal{V}(I)\big) = \sqrt{I}$ is much larger than I. The radical of I equals

$$\big\langle x_{11}+x_{22}+x_{33}\,,\; x_{11}x_{22}+x_{11}x_{33}-x_{12}x_{21}-x_{13}x_{31}+x_{22}x_{33}-x_{23}x_{32}\,,\; \det(X)\,\big\rangle.$$

In words, the ideal $\sqrt{I}$ is generated by the three trailing coefficients of the characteristic polynomial of X. This reflects the familiar fact that a square matrix is nilpotent if and only if it has no eigenvalues other than zero. Theorem 6.5 implies that every polynomial that vanishes on nilpotent 3 × 3-matrices is a polynomial linear combination of the three generators above.

The Nullstellensatz implies a one-to-one correspondence between varieties in affine n-space and radical ideals in the polynomial ring in n variables.

Corollary 6.7. *The map $V \mapsto \mathcal{I}(V)$ defines a bijection between varieties in K^n and radical ideals in $K[\mathbf{x}]$. The inverse map that takes radical ideals to varieties is given by $I \mapsto \mathcal{V}(I)$.*

Proof. A variety V is Zariski closed and hence satisfies $V = \mathcal{V}(\mathcal{I}(V))$. The Nullstellensatz yields $I = \mathcal{I}(\mathcal{V}(I))$. These identities implies that both maps are one-to-one and onto, and that they are the inverses of each other. □

Corollary 6.8. *The map $V \mapsto \mathcal{I}(V)$ defines a bijection between irreducible varieties in the affine space K^n and prime ideals in the polynomial ring $K[\mathbf{x}]$. As before, the inverse map is given by $I \mapsto \mathcal{V}(I)$.*

Proof. By Proposition 2.3, a variety V is irreducible if and only if its associated radical ideal $\mathcal{I}(V)$ is prime. □

Example 6.9 ($n = 2$)**.** There are only two kinds of proper irreducible subvarieties in the affine plane K^2. First, we have the points (a, b), corresponding to maximal ideals $I = \langle x - a, y - b\rangle$. Second, there are irreducible curves, one for each principal ideal $I = \langle f\rangle$ where f is an irreducible polynomial in $K[x, y]$. Arbitrary varieties are unions of these. For instance, consider

$$J \;=\; \langle x^4 + 2x^2 + y^2 + 1\rangle \;\cap\; \langle y^3 - 4, 2x - y^2\rangle \quad \text{in } \mathbb{C}[x, y].$$

This ideal is radical. Its variety $\mathcal{V}(J)$ in $\mathbb{C}^2$ has five irreducible components, namely two quadratic curves and three points. Check that $\mathcal{I}(\mathcal{V}(J)) = J$.

In many applications one is interested in solving polynomial equations over the real numbers, and one cares less about non-real complex solutions. This raises the following important question: Does there exist an analog of Hilbert's Nullstellensatz over an ordered field, such as the real numbers $K = \mathbb{R}$? We shall see that the answer is affirmative. In the next section we discuss the real Nullstellensatz and the Positivstellensatz. These concern systems of polynomial equations and inequalities over the real numbers. They generalize Linear Programming Duality, for systems of linear equations and linear inequalities over $\mathbb{R}$. Moreover, as we shall see in Chapter 12, the Positivstellensatz plays an important role in Nonlinear Optimization.

The theorems above are false when $K = \mathbb{R}$ is the field of real numbers. To see this, let $n = 2$ and consider varieties in the plane $\mathbb{R}^2$. Theorem 6.1 fails for $I = \langle x^2 + y^2 + 1 \rangle$. This is a proper ideal in $\mathbb{R}[x, y]$ but $\mathcal{V}_{\mathbb{R}}(I) = \emptyset$. Theorem 6.5 is false also for $I = \langle x^2 + y^2 \rangle$. This is a radical ideal, but

$$\mathcal{I}\big(\mathcal{V}_{\mathbb{R}}(I)\big) \;=\; \langle x, y\rangle \quad \text{strictly contains} \quad \sqrt{I} = I.$$

We ask these two questions about ideals I in $\mathbb{R}[\mathbf{x}]$ and their varieties in $\mathbb{R}^n$:

- How to best certify that the real variety $\mathcal{V}_{\mathbb{R}}(I)$ is empty?
- How to compute the ideal $\mathcal{I}\big(\mathcal{V}_{\mathbb{R}}(I)\big)$ from generators of I ?

The goal of the next section is to give affirmative answers to these questions.

6.3. Let's Get Real

We here present the *real Nullstellensatz*. Our point of departure is the fact that polynomial f in $\mathbb{R}[\mathbf{x}]$ that is a sum of squares must be nonnegative, i.e. the inequality $f(\mathbf{u}) \geq 0$ holds for all $\mathbf{u} \in \mathbb{R}^n$. A natural question is whether the converse holds: can every nonnegative polynomial be written as a sum of squares? The answer depends on the nature of the summands.

Hilbert showed in 1893 that the answer is no if one asks for squares of polynomials. However, it is yes if one allows squares of rational functions. This was the 17th problem in Hilbert's famous list from the International Congress of Mathematicians in 1900. It was solved by Emil Artin in 1927.

Theorem 6.10 (Artin's Theorem)**.** *If $f \in \mathbb{R}[\mathbf{x}]$ is nonnegative on $\mathbb{R}^n$ then there exist polynomials $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_r \in \mathbb{R}[\mathbf{x}]$ such that*

$$f = \left(\frac{p_1}{q_1}\right)^2 + \left(\frac{p_2}{q_2}\right)^2 + \cdots + \left(\frac{p_r}{q_r}\right)^2.$$

Example 6.11 ($n = 2$)**.** The Motzkin polynomial $M(x, y)$ equals

$$x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \;=\; \frac{[x^2{+}y^2{+}1]\cdot x^2y^2(x^2+y^2-2)^2 \;+\; (x^2-y^2)^2}{(x^2+y^2)^2}.$$

Distributing the three terms of the factor $[x^2 + y^2 + 1]$, we see that the right hand side is a sum of four squares of rational functions. This shows that $M(x, y)$ is nonnegative. However, it is not a sum of squares in $\mathbb{R}[x, y]$. Suppose it were equal to $\sum_i f_i^2$ where f_i are polynomials. None of the f_i's may contain a monomial x^d or y^d for $d > 0$ — otherwise the largest such d contributes positively to the coefficient of x^{2d} in $M(x, y)$. We have $f_i = \alpha_i + \beta_i xy + \tilde{f}_i$, where $\tilde{f}_i$ have all terms of degree ≥ 3 and $\alpha_i, \beta_i \in \mathbb{R}$. The coefficient -3 of x^2y^2 in $M(x, y)$ would then be equal to $\sum_i \beta_i^2 \geq 0$.

We shall derive Artin's Theorem 6.10 as a special case from the following more general statement. Theorem 6.12 is the real number analogue to the weak form of the Nullstellensatz which was established in Theorem 6.1.

Theorem 6.12. *Let I be an ideal in $\mathbb{R}[\mathbf{x}]$ whose variety $\mathcal{V}_{\mathbb{R}}(I)$ is empty. Then -1 is a sum of squares of polynomials modulo I, i.e. we have*

$$(6.3) \qquad 1 + p_1^2 + p_2^2 + \cdots + p_r^2 \;\in\; I \qquad \textit{for some } p_1, p_2, \ldots, p_r \in \mathbb{R}[\mathbf{x}].$$

For the proof of Theorem 6.12 see Murray Marshall's book [**39**, §2.3].

Proof of Theorem 6.10. Let y be a new variable. Consider the polynomial $g = f(\mathbf{x})y^2 + 1$ in $\mathbb{R}[\mathbf{x}, y]$. Since f is nonnegative, the real variety $\mathcal{V}_{\mathbb{R}}(g)$ is empty in $\mathbb{R}^{n+1}$. By Theorem 6.12, there exists a polynomial identity

$$(6.4) \qquad 1 + p_1(\mathbf{x}, y)^2 + p_2(\mathbf{x}, y)^2 + \cdots + p_r(\mathbf{x}, y)^2 + h(\mathbf{x}, y)g(\mathbf{x}, y) \;=\; 0.$$

We substitute $y = \pm\frac{1}{\sqrt{-f(\mathbf{x})}}$ into (6.4), which makes the last term cancel in both substitutions. Thereafter we multiply the two resulting expressions. The result no longer contains any radicals. We obtain an identity

$$1 \;+\; \frac{1}{(-f(\mathbf{x}))^d} \cdot \Big(g_1(\mathbf{x})^2 + g_2(\mathbf{x})^2 + \cdots + g_r(\mathbf{x})^2 \Big) \;=\; 0,$$

where $g_1, g_2, \ldots, g_r$ are polynomials, and d is a positive integer, necessarily odd. We subtract the constant 1 on both sides of this identity, and we multiply by $-f(\mathbf{x})$ to obtain a representation of $f(\mathbf{x})$ as a sum of squares of rational functions. This gives Artin's Theorem 6.10. □

For systems involving both equations and inequalities, there is the *Positivstellensatz*. To motivate this, we review the corresponding result for linear polynomials. Known as *Farkas' Lemma*, this is at the heart of *Linear Programming Duality*. Informally, Farkas' Lemma states that a system of linear equations and inequalities either has a solution in $\mathbb{R}^n$, or it has a dual solution which certifies that the original system has no solution. The precise statement can be stated in many equivalent versions. Here is one of them, selected to make the extension to higher-degree polynomials transparent.

Let $f_1, \ldots, f_r, g_1, \ldots, g_s$ be polynomials of degree 1 in $\mathbb{R}[\mathbf{x}]$, and consider

$$(6.5) \qquad f_1(\mathbf{u}) = 0,\ \ldots,\ f_r(\mathbf{u}) = 0,\ g_1(\mathbf{u}) \geq 0,\ \ldots,\ g_s(\mathbf{u}) \geq 0.$$

In the dual problem, we seek numbers $a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R}$ such that

$$(6.6) \quad a_1 \cdot f_1 + \cdots + a_r \cdot f_r + b_1^2 \cdot g_1 + \cdots + b_s^2 \cdot f_s = -1 \quad \text{in } \mathbb{R}[\mathbf{x}].$$

At most one of these two can have a solution. Indeed, since $b_1^2, \ldots, b_s^2 \geq 0$, the left hand side of (6.6) is nonnegative for every vector $\mathbf{x}$ that solves (6.5).

Theorem 6.13 (Farkas' Lemma)**.** *Given any choice of polynomials $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ in $\mathbb{R}[\mathbf{x}]$, exactly one of the following two statements is true:*

(P) *There exists a point $\mathbf{u} \in \mathbb{R}^n$ such that (6.5) holds.*

(D) *There exist real numbers $a_1, \ldots, a_r, b_1, \ldots, b_s$ such that (6.6) holds.*

Consider the system (6.5) where the f_i and g_j are now arbitrary polynomials. In the dual problem, we seek polynomials a_i and $b_{j\nu}$ in $\mathbb{R}[\mathbf{x}]$ such that

$$(6.7) \quad a_1 \cdot f_1 + \cdots + a_r \cdot f_r + \sum_{\nu \in \{0,1\}^s} \bigl(\sum_j b_{j\nu}\bigr)^2 \cdot g_1^{\nu_1} \cdots g_s^{\nu_s} = -1.$$

In the double sum on the right, we see linear combinations of squarefree monomials in $g_1, \ldots, g_s$ whose coefficients are sums of squares. The set of polynomials that admit such a representation is the *quadratic module* generated by $g_1, \ldots, g_s$. Quadratic modules associated with inequality constraints are fundamental in the study of semi-algebraic sets [**39**, §2.1].

Theorem 6.14 (Positivstellensatz)**.** *Given any polynomials $f_1, \ldots, f_r$ and $g_1, \ldots, g_s$ in $\mathbb{R}[\mathbf{x}]$, exactly one of the following two statements is true:*

(P) *There exists a point $\mathbf{u} \in \mathbb{R}^n$ such that (6.5) holds.*

(D) *There exist polynomials a_i and $b_{j\nu}$ in $\mathbb{R}[\mathbf{x}]$ such that (6.7) holds.*

Proof. See [**39**, §2.3]. □

The dual solution (D) in Theorem 6.14 is similar to that in Farkas' Lemma. One extra complication is that we now need products of the g_i. The result can be rephrased as follows: if a system of polynomial equations and inequalities is infeasible then -1 lies in the sum of the ideal of equations and the quadratic module of inequalities. There is a more general version of the Positivstellensatz which also incorporates strict inequalities $h_1 > 0, \ldots, h_t > 0$. This is stated in [**53**, Theorem 7.5] and it is also proved in [**39**, §2.3].

The radical $\sqrt{I}$ of a polynomial ideal I was the main player in the strong form of Hilbert's Nullstellensatz (Theorem 6.5). It offers an algebraic representation for polynomials that vanish on a given complex variety. We now come to the analogous result for varieties over the real numbers.

Given an ideal I in $\mathbb{R}[\mathbf{x}]$, we define its *real radical* $\sqrt[\mathbb{R}]{I}$ to be the set

$$\{\, f \in \mathbb{R}[\mathbf{x}] \,:\, f^{2m}+g_1^2+\cdots+g_s^2 \,\in\, I \text{ for some } m \in \mathbb{N} \text{ and } g_1,\ldots,g_s \in \mathbb{R}[\mathbf{x}] \,\}.$$

One checks that $\sqrt[\mathbb{R}]{I}$ is an ideal in $\mathbb{R}[\mathbf{x}]$. Here is the analogue to Theorem 6.5:

Theorem 6.15 (Real Nullstellensatz)**.** *For any ideal in* $\mathbb{R}[\mathbf{x}]$*, we have*

$$\mathcal{I}\big(\mathcal{V}_{\mathbb{R}}(I)\big) \;=\; \sqrt[\mathbb{R}]{I}. \tag{6.8}$$

Proof. The argument is similar to the proof of Theorem 6.5. Clearly, $\sqrt[\mathbb{R}]{I}$ is contained in $\mathcal{I}\big(\mathcal{V}_{\mathbb{R}}(I)\big)$. We need to show the reverse inclusion. Suppose that f vanishes on the real variety of $I = \langle f_1,\ldots,f_r\rangle \subset \mathbb{R}[\mathbf{x}]$. We introduce a new variable y and consider ideal $J = \langle f_1,\ldots,f_r, yf-1\rangle$ in $\mathbb{R}[\mathbf{x},y]$. It satisfies $\mathcal{V}_{\mathbb{R}}(J) = \emptyset$. By Theorem 6.12, there exists an identity of the form (6.3) for the ideal J. Substituting $y = 1/f(\mathbf{x})$ into that identity and clearing denominators, we find that some even power of f plus a sum of squares lies in I. This means that the polynomial f is in the real radical $\sqrt[\mathbb{R}]{I}$. □

Example 6.16. Fix the ideal generated by the Motzkin polynomial

$$I \;=\; \langle\, M(x,y)\,\rangle \;=\; \langle\, x^4y^2 + x^2y^4 + 1 - 3x^2y^2\,\rangle.$$

Building on Example 6.11, we wish to compute the real radical $\sqrt[\mathbb{R}]{I}$. It must contain the numerators of the four summands in the rational sum of squares representation of M. This leads us to consider the ideal

$$J \;=\; \big\langle\, M, xy(x^2+y^2-2)\,,\, x^2-y^2\,\big\rangle.$$

We find that the radical of J is the Jacobian ideal of the Motzkin polynomial:

$$\sqrt{J} \;=\; \Big\langle\, M,\, \frac{\partial M}{\partial x}\,,\frac{\partial M}{\partial y}\,\Big\rangle.$$

Furthermore, this radical ideal is precisely the real radical we are looking for:

$$\sqrt[\mathbb{R}]{I} \;=\sqrt{J} \;=\; \langle x-1,y-1\rangle \cap \langle x-1,y+1\rangle \cap \langle x+1,y-1\rangle \cap \langle x+1,y+1\rangle.$$

The real variety $\mathcal{V}_{\mathbb{R}}(M)$ defined by the Motzkin polynomial consists of the four points $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ in $\mathbb{R}^2$. Since M is nonnegative, these zeros are singular points of the complex curve $\mathcal{V}(M) \subset \mathbb{C}^2$.

Exercises

(1) Find univariate polynomials g_1, g_2, g_3, g_4 in $\mathbb{Q}[x]$ such that

$$\begin{aligned} & g_1(x-2)(x-3)(x-4) \;+\; g_2(x-1)(x-3)(x-4) \\ + \quad & g_3(x-1)(x-2)(x-4) \;+\; g_4(x-1)(x-2)(x-3) \;=\; 1. \end{aligned}$$

(2) An ideal I in $\mathbb{C}[\mathbf{x}]$ contains a monomial if and only if each point in its variety $\mathcal{V}(I)$ has at least one zero coordinate. Prove this fact, and describe an algorithm for testing whether an ideal contains a monomial.

(3) Let M be an ideal generated by monomials in $K[\mathbf{x}]$. How to compute the radical $\sqrt{M}$? How to compute the real radical $\sqrt[\mathbb{R}]{M}$?

(4) Let I be the ideal generated by the two cubics $x_1^2x_2 - x_3^2x_4$ and $x_1x_2^3 - x_4^3$. Describe the projective variety $\mathcal{V}(I)$ in $\mathbb{P}^3$. Find the radical ideal $\sqrt{I}$. How many minimal generators does $\sqrt{I}$ have and what are their degrees?

(5) Let $V \subset \mathbb{R}^7$ be the variety of orthogonal Hankel matrices of format 4×4. Describe the ideal $\mathcal{I}(V)$. What are the irreducible components of V?

(6) Let I be the ideal generated by the two quartics $x_1^4 - x_1^2x_2^2$ and $x_2^4 - x_3^4$ in $\mathbb{R}[x_1, x_2, x_3]$. Determine the radical $\sqrt{I}$ and the real radical $\sqrt[\mathbb{R}]{I}$. Write each of these two radical ideals as an intersection of prime ideals.

(7) Let $f_1, \ldots, f_r$ and f be polynomials in $\mathbb{Q}[\mathbf{x}]$. Explain how Gröbner bases can be used to test whether f lies in the radical of $I = \langle f_1, \ldots, f_r \rangle$.

(8) The circle given by $f = x^2 + y^2 - 4$ does not intersect the hyperbola given by $g = xy - 10$ in the plane $\mathbb{R}^2$. Find a real Nullstellensatz certificate for this, i.e. write -1 as a sum of squares modulo the ideal $\langle f, g \rangle$ in $\mathbb{R}[x, y]$.

(9) For $d \in \mathbb{N}$, exhibit a polynomial f and an ideal I in $K[\mathbf{x}]$ with $f^d \notin I$ but $f^{d+1} \in I$. How small can the degrees of the generators of I be?

(10) Let I be the ideal in $\mathbb{R}[x, y, z]$ generated by the *Robinson polynomial*

$$x^6 + y^6 + z^6 \;+\; 3x^2y^2z^2 \;-\; x^4y^2 - x^4z^2 - x^2y^4 - x^2z^4 - y^4z^2 - y^2z^4.$$

Determine the real radical $\sqrt[\mathbb{R}]{I}$ and the real variety $\mathcal{V}_{\mathbb{R}}(I)$ in $\mathbb{P}^2_{\mathbb{R}}$.

(11) Show that Theorem 6.15 implies Theorem 6.10.

(12) What is the Effective Nullstellensatz?

(13) Find the radical and the real radical of the ideal $I = \langle x^7 - y^7,\, x^8 - z^8 \rangle$ in $\mathbb{R}[x, y, z]$. Explain the difference between these two radical ideals.

Chapter 7

Tropical Algebra

The operations of addition and multiplication are familiar from primary school. We here redefine them we introducing tropical arithmetic. This new arithmetic may at first seem unnatural to the reader, we justified it with several applications, e.g. in the design of dynamic programming algorithms. A big part this chapter is dedicated to tropical linear algebra. The point is that the piecewise linear structures of tropical mathematics offer yet another transition point between linear and nonlinear algebra. On the fully nonlinear side lies *tropical algebraic geometry* [**37**]. We offer a welcome to this subject with a brief discussion of tropical varieties and their geometric properties.

7.1. Arithmetic and Valuations

The *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ consists of the real numbers $\mathbb{R}$, together with an extra element ∞ that represents plus-infinity. The arithmetic operations of addition and multiplication are

$$x \oplus y \; := \; \min(x, y) \qquad \text{and} \qquad x \odot y \; := \; x + y.$$

The *tropical sum* of two numbers is their minimum, and the *tropical product* of two numbers is their usual sum. It takes some practise to carry out arithmetic in the tropical world. Here is an example with numbers:

$$3 \oplus 7 \; = \; 3 \qquad \text{and} \qquad 3 \odot 7 \; = \; 10.$$

Tropical addition and tropical multiplication are both *commutative*:

$$x \oplus y \; = \; y \oplus x \qquad \text{and} \qquad x \odot y \; = \; y \odot x.$$

These two arithmetic operations are also associative, and the times operator $\odot$ takes precedence when plus $\oplus$ and times $\odot$ occur in the same expression.

The *distributive law* holds:

$$x \odot (y \oplus z) \quad = \quad x \odot y \;\oplus\; x \odot z.$$

Here is a numerical example to show distributivity:

$$\begin{aligned} 3 \odot (7 \oplus 11) \quad &= \quad 3 \odot 7 \quad = \quad 10, \\ 3 \odot 7 \;\oplus\; 3 \odot 11 \quad &= \quad 10 \oplus 14 \quad = \quad 10. \end{aligned}$$

Both arithmetic operations have an identity element. Infinity is the *identity element* for addition and zero is the *identity element* for multiplication:

$$x \;\oplus\; \infty \;=\; x \qquad \text{and} \qquad x \;\odot\; 0 \;=\; x.$$

We also note the following identities involving the two identity elements:

$$x \;\odot\; \infty \;=\; \infty \qquad \text{and} \qquad x \;\oplus\; 0 \;=\; \begin{cases} 0 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

There is no subtraction in tropical arithmetic. There is no real number x to be called "17 minus 8" because the equation $8 \oplus x \;=\; 17$ has no solution x. Tropical division is defined to be classical subtraction, so $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ satisfies all ring axioms except for the existence of an additive inverse.

Such algebraic structures without division are called *semirings*, whence the name tropical semiring. It is essential to remember that "0" is the multiplicative identity element. If we write a term without explicit coefficient then that coefficient is zero. For instance, $x \oplus y$ means $0 \odot x \;\oplus\; 0 \odot y$.

Example 7.1 (Binomial Theorem)**.** We consider the third tropical power of a tropical sum. The following identities hold for all real numbers $x, y \in \mathbb{R}$:

$$\begin{aligned} (x \oplus y)^3 \quad &= \quad (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= \quad 0 \odot x^3 \;\oplus\; 0 \odot x^2 y \;\oplus\; 0 \odot xy^2 \;\oplus\; 0 \odot y^3. \end{aligned}$$

Of course, the zero coefficients can here be dropped:

$$(x \oplus y)^3 \quad = \quad x^3 \;\oplus\; x^2 y \;\oplus\; xy^2 \;\oplus\; y^3 \quad = \quad x^3 \;\oplus\; y^3.$$

What is the relationship between classical arithmetic and tropical arithmetic? An informal answer is that the latter is the image of the former under taking logarithms. Indeed, if u and v are positive real numbers then $\log(u \cdot v)$ equals $\log(u) \odot \log(v)$, and $\log(u + v)$ is approximately the same as $\log(u) \oplus \log(v)$. Thus tropical geometry arises naturally when one draws a log-log-plot of figures in $\mathbb{R}^2_{>0}$. We refer to [**37**, Chapter 1] for a discussion. A more formal way of understanding this is to introduce fields with valuations.

Definition 7.2. A *valuation* on a field K is a function $\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}$ that satisfies the following three axioms for all $a, b \in K$:

(1) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,

(2) $\operatorname{val}(a+b) \geq \min\{\operatorname{val}(a), \operatorname{val}(b)\}$, and

(3) $\operatorname{val}(a) = \infty$ if and only if $a = 0$.

We often identify a valution with its restriction $K^* \to \mathbb{R}$ to $K^* = K \backslash \{0\}$. The image of val is an additive subgroup of $\mathbb{R}$, known as the *value group.*

A field K with valuation is a metric space. Namely, the valuation induces a *norm* $|\cdot| : K \to \mathbb{R}$ by setting $|a| = \exp(-\operatorname{val}(a))$ for $a \in K^*$ and $|0| = 0$. The field K is a metric space with distance $|a-b|$ between two elements $a, b \in K$. In fact, the metric is an ultrametric since $|a+b| \leq \max(|a|, |b|) \leq |a|+|b|$. This allows the use of analytical and topological method for studying K.

An important example is the field of *Puiseux series* in a variable t with complex coefficients. This field is denoted $K = \mathbb{C}\{\{t\}\}$. It contains the field $\mathbb{C}(t)$ of rational functions and its algebraic closure $\overline{\mathbb{C}(t)}$. Indeed, every element in $\overline{\mathbb{C}(t)}$ can be expanded into a Puiseux series with integer exponents.

The *valuation* of a scalar c in K is the smallest exponent a of any term $c_a t^a$ with $c_a \neq 0$ that appears in the series expansion of c. We write $a = \operatorname{val}(c)$. This is an element in the value group $(\mathbb{Q}, +)$ of K. Here are two examples of scalars in the Puiseux series field K and their valuations:

$$c \;=\; \frac{1}{t^2+2t^3+t^5} \;=\; t^{-2}-2t^{-1}+4-9t+20t^2-44t^3+97t^4-214t^5+472t^6-\cdots$$

has $\operatorname{val}(c) = -2$, while the following scalar has $\operatorname{val}(c') = \frac{2}{7}$:

$$c' \;=\; t^{2/7}\sqrt{1-t^{2/3}} \;=\; t^{2/7} - \frac{1}{2}t^{20/21} - \frac{1}{8}t^{34/21} - \frac{1}{16}t^{16/7} - \frac{5}{128}t^{62/21} - \cdots$$

It is known that the field K is algebraically closed [**37**, Theorem 2.1.5]. So, every polynomial of degree d in $K[x]$ has d roots, counting multiplicities.

Example 7.3 (Puiseux series)**.** Every cubic polynomial in $K[x]$ has three roots. For instance, the three roots of $f(x) = tx^3 - x^2 + 3tx - 2t^5$ are

$$\begin{gathered}
t^{-1} - 3t - 9t^3 - 54t^5 + 2t^6 - 405t^7 + 18t^8 - 3402t^9 + 180t^{10} - 30618t^{11} + \cdots \\
3t + 9t^3 - \tfrac{2}{3}t^4 + 54t^5 - 2t^6 + \tfrac{10931}{27}t^7 - 18t^8 + 3402t^9 - \tfrac{43756}{243}t^{10} + 30618t^{11} + \cdots \\
\tfrac{2}{3}t^4 + \tfrac{4}{27}t^7 + \tfrac{16}{243}t^{10} - \tfrac{8}{81}t^{12} + \tfrac{80}{2187}t^{13} - \tfrac{80}{729}t^{15} + \tfrac{448}{19683}t^{16} - \tfrac{224}{2187}t^{18} + \cdots
\end{gathered}$$

Such Puiseux series can be computed in a computer algebra system. The valuations of the three roots are -1, 1 and 4. These characterize the asymptotic behavior of the roots when t is a real number that is close to zero.

We define the *tropicalization* $\operatorname{trop}(f)$ of a polynomial $f \in K[x]$ to be the polynomial over the tropical semiring obtained by replacing each coefficient in f by its valuation. For instance, if f is the cubic in Example 7.3 then

$$\operatorname{trop}(f) \;=\; 1 \odot x^{\odot 3} \;\oplus\; 0 \odot x^{\odot 2} \;\oplus\; 1 \odot x \;\oplus\; 5. \tag{7.1}$$

To evaluate a tropical polynomial, one takes the minimum of its tropical monomials. And, tropical monomials are linear functions. In the example,

$$\operatorname{trop}(f)(u) \;=\; \min\bigl\{1+3u,\, 0+2u,\, 1+u,\, 5\,\bigr\} \quad \text{for all}\, u \in \mathbb{R}.$$

If $u = \operatorname{val}(c)$ for some $c \in K$ then we have $\operatorname{val}(f(c)) = (\operatorname{trop}(f))(u)$.

In the following lemma we are claiming that two functions $\mathbb{R} \to \mathbb{R}$ agree.

Lemma 7.4. *If $f, g \in K[x]$ then $\operatorname{trop}(fg) = \operatorname{trop}(f) \odot \operatorname{trop}(g)$.*

Proof. By enlarging the field K, we may assume that the value group equals $\mathbb{R}$. Let $a \in \mathbb{R}$ and choose any $c \in K$ with $\operatorname{val}(c) = a$. A computation shows

$$\operatorname{trop}(fg)(a) = \operatorname{val}((fg)(u)) = \operatorname{val}(f(u)) + \operatorname{val}(g(u)) = \operatorname{trop}(f)(a) \odot \operatorname{trop}(g)(a).$$

Hence the two functions agree on every argument $a \in \mathbb{R}$. □

Definition 7.5. Let $g(x)$ be a tropical polynomial, i.e. a function that is the minimum of finitely many linear functions. A number $u \in \mathbb{R}$ is called a *tropical root* of g if that minimum is attained at least twice.

The following facts relate classical root finding and tropical root finding.

Theorem 7.6. *Let $f \in K[x]$ and $g = \operatorname{trop}(f)$ its tropicalization. If $c \in K$ satisfies $f(c) = 0$ then $u = \operatorname{val}(c)$ is a tropical root of g. Conversely, if K is algebraically closed then every tropical root u of g arises from a zero c of f.*

Proof. Let $f(x) = \sum_{i=0}^{d} b_i x^i$ and suppose $f(c) = \sum_{i=0}^{d} b_i c^i$ is zero. Then $\infty = \operatorname{val}(f(c))$ but $\operatorname{val}(b_i c^i) < \infty$ for $i = 1, \ldots, d$. Using [**37**, Lemma 2.1.1], this implies $\operatorname{val}(b_i c^i) = \operatorname{val}(b_j c^j) \leq \operatorname{val}(b_k c^k)$ for some $i \neq j$ and all $k \neq i, j$. This means that $u = \operatorname{val}(c)$ is a tropical zero of the $g = \operatorname{trop}(f)$.

Our proof of the second statement follows that of [**37**, Proposition 3.1.5]. We assume that u is a tropical root. Since K is algebraically closed, we can factor $f(x) = \prod_{j=1}^{d}(a_j x - b_j)$. Since u is a tropical root of $\operatorname{trop}(f)$, it is also a tropical root of one of the tropicalized factors, by Lemma 7.4. Hence there is an index j such that $\operatorname{val}(a_j) \odot u = \operatorname{val}(b_j)$. This implies $u = \operatorname{val}(b_j) - \operatorname{val}(a_k)$. We now simply set $c = b_j / a_j$ in K. Then $f(c) = 0$ and $u = \operatorname{val}(c)$. □

Example 7.7 ($d = 3$)**.** Let f be as in Example 7.3 and $g = \operatorname{trop}(f)$ its tropicalization (7.1). The tropical roots of g are the rational numbers u such that of $1 + 3u,\, 0 + 2u,\, 1 + u$ and 5 is attained twice. The solutions are $u = -1$, $u = 1$, $u = 4$. These are the valuations of the three roots of f.

Fields with valuations provide a systematic way of speaking algebraically about logarithms. This explains the connection between classical arithmetic and tropical arithmetic. We shall return to this point in our exploration of varieties in Subsection 7.3. First, however, let us develop some more purely tropical machinery, in the familiar setting of matrices and linear algebra.

7.2. Linear Algebra

Vectors and matrices make sense over the tropical semiring. For instance, the tropical scalar product in $\mathbb{R}^3$ of a row vector with a column vector equals

$$\begin{aligned}(u_1, u_2, u_3) \odot (v_1, v_2, v_3)^{\mathrm{T}} \quad &= \quad u_1 \odot v_1 \;\oplus\; u_2 \odot v_2 \;\oplus\; u_3 \odot v_3 \\ &= \quad \min\bigl\{u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3\bigr\}.\end{aligned}$$

Here is the product of a column vector and a row vector of length three:

$$(7.2) \qquad \begin{aligned}&(u_1, u_2, u_3)^T \odot (v_1, v_2, v_3) \\ &= \begin{pmatrix} u_1 \odot v_1 & u_1 \odot v_2 & u_1 \odot v_3 \\ u_2 \odot v_1 & u_2 \odot v_2 & u_2 \odot v_3 \\ u_3 \odot v_1 & u_3 \odot v_2 & u_3 \odot v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 & u_1 + v_2 & u_1 + v_3 \\ u_2 + v_1 & u_2 + v_2 & u_2 + v_3 \\ u_3 + v_1 & u_3 + v_2 & u_3 + v_3 \end{pmatrix}.\end{aligned}$$

Any matrix which can be expressed as such a product has *tropical rank one.*

Fix a $d \times n$-matrix A. We may wish to find its image $\{A \odot \mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^n\}$ and to solve linear systems $A \odot \mathbf{x} = \mathbf{b}$ for various right hand sides $\mathbf{b}$. For an introduction to tropical linear systems see the books on *Max-linear Systems* by Butkovič [**6**] and *Essentials of Tropical Combinatorics* by Joswig [**29**].

For a first application of tropical linear algebra, consider the problem of finding shortest paths in a weighted directed graph G with n nodes. Every directed edge (i, j) in G has an associated length d_{ij} which is a non-negative real number. If (i, j) is not an edge of G then we set $d_{ij} = +\infty$. We represent G by its $n \times n$ *adjacency matrix* $D_G = \bigl(d_{ij}\bigr)$ with zeros on the diagonal. The off-diagonal entries are the edge lengths d_{ij}. The matrix D_G need not be symmetric; we allow $d_{ij} \neq d_{ji}$ for some i, j. If G is an undirected graph, then we represent it as a directed graph with two directed edges (i, j) and (j, i) for each undirected edge $\{i, j\}$. In that case, D_G is a symmetric matrix, where $d_{ij} = d_{ji}$ is the distance between node i and node j.

Consider the $n{\times}n$-matrix with entries in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ that results from tropically multiplying the given adjacency matrix D_G with itself $n-1$ times:

$$(7.3) \qquad D_G^{\odot(n-1)} \quad = \quad D_G \odot D_G \odot \cdots \odot D_G.$$

Proposition 7.8. *Let G be a weighted directed graph on n nodes with adjacency matrix D_G. The entry of the matrix $D_G^{\odot(n-1)}$ in row i and column j equals the length of a shortest path from node i to node j in the graph G.*

Proof. Let $d_{ij}^{(r)}$ denote the minimum length of any path from node i to node j using at most r edges in G. We have $d_{ij}^{(1)} = d_{ij}$ for any two nodes i and j. Since the edge weights d_{ij} were assumed to be non-negative, for each two nodes i, j there exists a shortest path from i to j that visits each node of G at most once. Hence the length of a shortest path from i to j equals $d_{ij}^{(n-1)}$.

For $r \geq 2$ we have a recursive formula for the length of a shortest path:

$$d_{ij}^{(r)} \quad = \quad \min\bigl\{d_{ik}^{(r-1)} + d_{kj} \,:\, k = 1, 2, \ldots, n\bigr\}. \tag{7.4}$$

Using tropical arithmetic, this formula can be rewritten as follows:

$$\begin{aligned} d_{ij}^{(r)} \quad &= \quad d_{i1}^{(r-1)} \odot d_{1j} \;\oplus\; d_{i2}^{(r-1)} \odot d_{2j} \;\oplus\; \cdots \;\oplus\; d_{in}^{(r-1)} \odot d_{nj}. \\ &= \quad (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, \ldots, d_{in}^{(r-1)}) \odot (d_{1j}, d_{2j}, \ldots, d_{nj})^T. \end{aligned}$$

From this it follows, by induction on r, that $d_{ij}^{(r)}$ equals the entry in row i and column j of the $n \times n$ matrix $D_G^{\odot r}$. Indeed, the right hand side of the recursive formula is the tropical product of row i of $D_G^{\odot(r-1)}$ and column j of D_G, which is the (i, j) entry of $D_G^{\odot r}$. In particular, $d_{ij}^{(n-1)}$ is the entry in row i and column j of $D_G^{\odot(n-1)}$. This proves the claim. □

The above algorithm belongs to what is known as *Dynamic Programming* in Computer Science. For us, it means performing the matrix multiplication

$$D_G^{\odot r} \quad = \quad D_G^{\odot(r-1)} \odot D_G \qquad \text{for } r = 2, \ldots, n-1.$$

We next consider the notion of the *tropical determinant.* Fix an $n \times n$ matrix $X = (x_{ij})$. As there is no negation in tropical arithmetic, we define this determinant as the tropical sum over the tropical diagonal products obtained by taking all $n!$ permutations π of $\{1, 2, \ldots, n\}$:

$$\operatorname{tropdet}(X) \quad := \quad \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)}. \tag{7.5}$$

Here S_n is the *symmetric group* of permutations of $\{1, 2, \ldots, n\}$. Evaluating the tropical determinant means solving the classical *assignment problem* of combinatorial optimization. Imagine a company that has n jobs and n workers, and each job needs to be assigned to exactly one of the workers. Let x_{ij} be the cost of assigning job i to worker j. The company wishes to find the cheapest assignment $\pi \in S_n$. The optimal total cost equals

$$\min\bigl\{x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{n\pi(n)} \;:\; \pi \in S_n\bigr\}. \tag{7.6}$$

That minimum is the tropical determinant (7.5) of the matrix $X = (x_{ij})$:

Proposition 7.9. *The tropical determinant solves the assignment problem.*

In the assignment problem we seek the minimum over $n!$ quantities. This appears to require exponentially many operations. However, there is a polynomial-time method, with run time $O(n^3)$, known as the *Hungarian Algorithm.* This algorithm maintains a price for each job and a partial assignment of workers and jobs. At each iteration, an unassigned worker is chosen and a shortest augmenting path from him to the set of jobs is chosen.

In classical arithmetic, the complexity of evaluating determinants and permanents differs greatly. The determinant of an $n \times n$ matrix can be computed in $O(n^3)$ steps, namely by *Gaussian elimination*, while computing the permanent of an $n \times n$ matrix is a hard problem. Leslie Valiant proved that computing permanents is *#P-complete*. In tropical arithmetic, computing the permanent is easier, thanks to the Hungarian Algorithm. We can think of this algorithm as a certain tropicalization of Gaussian Elimination.

Eigenvectors and eigenvalues of square matrices are central to linear algebra. The same is true in tropical linear algebra. We fix an $n \times n$-matrix $A = (a_{ij})$ over $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. An *eigenvalue* of A is a real number λ such that

$$A \odot \mathbf{v} \;=\; \lambda \odot \mathbf{v} \qquad \text{for some } \mathbf{v} \in \mathbb{R}^n. \tag{7.7}$$

We say that $\mathbf{v}$ is an *eigenvector* of the matrix A. The arithmetic operations in (7.7) are tropical. For instance, for $n = 2$, the left hand side of (7.7) is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} a_{11} \odot v_1 \oplus a_{12} \odot v_2 \\ a_{21} \odot v_1 \oplus a_{22} \odot v_2 \end{pmatrix} \;=\; \begin{pmatrix} \min\{a_{11} + v_1, a_{12} + v_2\} \\ \min\{a_{21} + v_1, a_{22} + v_2\} \end{pmatrix}.$$

The right hand side of (7.7) equals

$$\lambda \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} \lambda \odot v_1 \\ \lambda \odot v_2 \end{pmatrix} \;=\; \begin{pmatrix} \lambda + v_1 \\ \lambda + v_2 \end{pmatrix}.$$

Let $G(A)$ denote the directed graph with adjacency matrix A. Its nodes are labeled by $[n] = \{1, 2, \ldots, n\}$. There is an edge from node i to node j if and only if $a_{ij} < \infty$. The edge has length a_{ij}. In particular, $a_{ii} \neq \infty$ if and only if there is a loop at vertex i. The *normalized length* of a directed path $i_0, i_1, \ldots, i_k$ in $G(A)$ is $(a_{i_0 i_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k})/k$, computed in classical arithmetic. If $i_k = i_0$ then the path is a *directed cycle*, and this quantity is the normalized length of the cycle. Recall that a directed graph is *strongly connected* if there is a directed path from any node to any other node.

Theorem 7.10. *Let A be an $n \times n$-matrix such that $G(A)$ is strongly connected. Then A has precisely one eigenvalue $\lambda(A)$. It equals the minimum normalized length of a directed cycle.*

Proof. Let $\lambda = \lambda(A)$ be the minimum of the normalized lengths over all directed cycles in $G(A)$. We first prove that $\lambda(A)$ is the only possibility for an eigenvalue. Suppose that $\mathbf{z} \in \mathbb{R}^n$ is any eigenvector of A, and let γ be the corresponding eigenvalue. For any cycle $(i_1, i_2, \ldots, i_k, i_1)$ in $G(A)$ we have

$$a_{i_1 i_2} + z_{i_2} \geq \gamma + z_{i_1}, \;\; a_{i_2 i_3} + z_{i_3} \geq \gamma + z_{i_2},$$
$$a_{i_3 i_4} + z_{i_4} \geq \gamma + z_{i_3}, \; \ldots \;, \, a_{i_k i_1} + z_{i_1} \geq \gamma + z_{i_k}.$$

Adding the left-hand sides and the right-hand sides, we find that the normalized length of the cycle is greater than or equal to γ. In particular, we have

$\lambda(A) \geq \gamma$. For the reverse inequality, start with any index i_1. Since $\mathbf{z}$ is an eigenvector with eigenvalue γ, there exists i_2 such that $a_{i_1 i_2} + z_{i_2} = \gamma + z_{i_1}$. Likewise, there exists i_3 such that $a_{i_2 i_3} + z_{i_3} = \gamma + z_{i_2}$. We continue in this manner until we reach an index i_l which was already in the sequence, say, $i_k = i_l$ for $k < l$. By adding the equations along this cycle, we find that

$$\begin{aligned} &(a_{i_k i_{k+1}} + z_{i_{k+1}}) + (a_{i_{k+1} i_{k+2}} + z_{i_{k+2}}) + \cdots + (a_{i_{l-1} i_l} + z_{i_l}) \\ = \quad &(\gamma + z_{i_k}) \ + \ (\gamma + z_{i_{k+1}}) \ + \ \cdots \ + \ (\gamma + z_{i_{l-1}}). \end{aligned}$$

We conclude that the normalized length of the cycle $(i_k, i_{k+1}, \ldots, i_l = i_k)$ in $G(A)$ is equal to γ. In particular, $\gamma \geq \lambda(A)$. This proves that $\gamma = \lambda(A)$.

It remains to prove the existence of an eigenvector. Let B be the matrix obtained from A by (classically) subtracting $\lambda(A)$ from every entry in A. All cycles in $G(B)$ have non-negative length, and there exists a cycle of length zero. Using tropical matrix operations we define

$$B^{+} \ = \ B \oplus B^{\otimes 2} \oplus B^{\otimes 3} \oplus \cdots \oplus B^{\otimes n}.$$

This matrix is known as the *Kleene plus* of the matrix B. The entry B^{+}_{ij} in row i and column j of B^{+} is the length of a shortest path from node i to node j in the weighted directed graph $G(B)$. Here, we assume that a path contains some edges, thus the shortest path from i to i may be strictly positive. Since $G(B)$ is strongly connected, we have $B^{+}_{ij} < \infty$ for all i and j.

Fix any node j that lies on a zero length cycle of $G(B)$. Let $\mathbf{x} = B^{+}_{\cdot j}$ denote the jth column vector of the matrix B^{+}. We have $x_j = B^{+}_{jj} = 0$, as there is a path from j to itself of length zero, and there are no negative weight cycles. This implies $B^{+} \odot \mathbf{x} \leq B^{+}_{\cdot j} = \mathbf{x}$. Next note that $(B \odot \mathbf{x})_i = \min_l (B_{il} + x_l) = \min_l (B_{il} + B^{+}_{lj}) \geq B^{+}_{ij} = x_i$, since lengths of shortest paths obey the triangle inequality. In vector notation this states $B \odot \mathbf{x} \geq \mathbf{x}$. Since tropical linear maps preserve coordinatewise inequalities among vectors, we have $B^2 \odot \mathbf{x} \geq B \odot \mathbf{x}$, and $B^3 \odot \mathbf{x} \geq B^2 \odot \mathbf{x}$, etc. Therefore, $B^{+} \odot \mathbf{x} = B \odot \mathbf{x} \oplus B^2 \odot \mathbf{x} \oplus \cdots \oplus B^n \odot \mathbf{x} = B \odot \mathbf{x}$. This yields $\mathbf{x} \leq B \odot \mathbf{x} = B^{+} \odot \mathbf{x} \leq \mathbf{x}$. This means that $B \odot \mathbf{x} = \mathbf{x}$, so $\mathbf{x}$ is an eigenvector of B with eigenvalue 0. We conclude that $\mathbf{x}$ is an eigenvector with eigenvalue λ of our matrix A:

$$A \odot \mathbf{x} \ = \ (\lambda \odot B) \odot \mathbf{x} \ = \ \lambda \odot (B \odot \mathbf{x}) \ = \ \lambda \odot \mathbf{x}.$$

This completes the proof of Theorem 7.10. □

The eigenvalue λ of a tropical $n \times n$-matrix $A = (a_{ij})$ can be computed efficiently, using a *linear program* with $n+1$ decision variables $v_1, \ldots, v_n, \lambda$:

$$\text{(7.8)} \qquad \text{Maximize } \gamma \text{ subject to } \quad a_{ij} + v_j \geq \gamma + v_i \ \text{ for all } \ 1 \leq i, j \leq n.$$

Proposition 7.11. *The unique eigenvalue $\lambda(A)$ of the given $n \times n$-matrix $A = (a_{ij})$ coincides with the optimal value γ^* of the linear program* (7.8).

Proof. See [**37**, Proposition 5.1.2]. □

We next determine the *eigenspace* of the matrix A, which is the set

$$\mathrm{Eig}(A) \quad = \quad \big\{ \mathbf{x} \in \mathbb{R}^n \,:\, A \odot \mathbf{x} \,=\, \lambda(A) \odot \mathbf{x} \big\}.$$

The set $\mathrm{Eig}(A)$ is closed under tropical scalar multiplication: if $\mathbf{x} \in \mathrm{Eig}(A)$ and $c \in \mathbb{R}$ then $c \odot \mathbf{x}$ is also in $\mathrm{Eig}(A)$. We can thus identify $\mathrm{Eig}(A)$ with its image in the quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^{n-1}$. Here $\mathbf{1} = (1, 1, \ldots, 1)$. This space is called the *tropical projective torus*; cf. [**29**, Section 1.4]. We saw that every eigenvector of the matrix A is also an eigenvector of the matrix $B = (-\lambda(A)) \odot A$ and vice versa. Hence the eigenspace $\mathrm{Eig}(A)$ is equal to

$$\mathrm{Eig}(B) \quad = \quad \big\{ \mathbf{x} \in \mathbb{R}^n \,:\, B \odot \mathbf{x} \,=\, \mathbf{x} \big\}.$$

Theorem 7.12. *Let B_0^+ be the submatrix of the Kleene plus B^+ given by the columns whose diagonal entry B_{jj}^+ is zero. The image of this matrix, in tropical arithmetic, equals the eigenspace:* $\mathrm{Eig}(A) \,=\, \mathrm{Eig}(B) \,=\, \mathrm{Image}(B_0^+)$.

Proof. See [**37**, Theorem 5.1.3]. □

Example 7.13. We demonstrate the computation of eigenvectors for $n = 3$. In our first example, the minimal cycle lengths are attained by the loops:

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix} \quad \Rightarrow \quad \lambda(A) = 3 \quad \Rightarrow \quad B = B^+ = B_0^+ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenspace is the image of the column vectors of B. Its image in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ is the hexagon with vertices $(0,1,1)$, $(0,0,1)$, $(1,0,1)$, $(1,0,0)$, $(1,1,0)$ and $(0,1,0)$. In our second example, the winner is the cycle $1 \to 2 \to 1$:

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 4 & 3 \end{pmatrix} \Rightarrow \lambda(A) = 1 \Rightarrow B = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 3 & 2 \end{pmatrix} \Rightarrow B^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}.$$

The eigenspace of A is the tropical linear span of the first column of B_+:

$$\mathrm{Eig}(A) \,=\, \mathrm{Eig}(B) \,=\, \big\{ c \odot (0,0,3)^T \,:\, c \in \mathbb{R} \big\} \,=\, \big\{ (c, c, c+3)^T \,:\, c \in \mathbb{R} \big\}$$

So, here $\mathrm{Eig}(A)$ is just a single point in the tropical projective 2-torus $\mathbb{R}^3/\mathbb{R}\mathbf{1}$.

We computed the eigenspace of a square matrix as the image of another matrix. This motivates the study of images of tropical linear maps $\mathbb{R}^m \to \mathbb{R}^n$. Such images are not tropical linear spaces. They are known as *tropical polytopes*. Indeed, one defines *tropical convexity* in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ by taking tropical linear combinations. Tropical convexity is a rich and beautiful theory with many applications. For introductions see [**29**, Chapter 5] and [**37**, §5.2].

We give a brief illustration for m=n=3. The image of a 3×3-matrix A is the set of all tropical linear combinations of three vectors in $\mathbb{R}^3$. We represent this by its image in the plane $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. That image is a *tropical triangle*, because it is the tropical convex hull of three points in the plane. This triangle is degenerate if the three points are tropically collinear in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. This happens when the minimum in the tropical determinant (7.5) is attained twice. In that case, the matrix A is called *tropically singular*.

Example 7.14. Consider the tropical triangle in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ given by the matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad A' = \begin{pmatrix} -1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Each point in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ is represented uniquely by a column vector $(u, v, 0)$. This tropical triangle consists of the segment between $(-1, -1, 0)$ and $(0, 0, 0)$, the segment between $(0, 3, 0)$ and $(0, 1, 0)$, the segment between $(2, 1, 0)$ and $(1, 1, 0)$, and the classical triangle with vertices $(0, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$.

There are five combinatorial types of tropical triangles. Similarly, there are 35 types of tropical quadrilaterals. They are shown in [**37**, Figure 5.2.4].

7.3. Tropical Varieties

The previous section explored tropical counterparts of concepts from linear algebra. In what follows we move on to nonlinear algebra. Our aim is to introduce the tropical counterparts of algebraic varieties. Our point of departure is discussion of fields with valuation at the end of Section 7.1.

Example 7.15 (The p-adic valuation)**.** For every prime number p, the field $K = \mathbb{Q}$ of rational numbers has a valuation val_p with value group $\mathbb{Z}$. Indeed, every rational number c can be written uniquely as $c = p^u \cdot \frac{q}{r}$ where q and r are relatively prime integers not divisible by p and $u \in \mathbb{Z}$. We have $\mathrm{val}_p(c) = u$. The completion of $\mathbb{Q}$ with respect to the norm induced by val_p is the field of p-adic numbers. This field is important in number theory.

In what follows we assume that K is an algebraically closed field with a valuation whose value group is $\mathbb{Q}$. The Puiseux series field $K = \mathbb{C}\{\{t\}\}$ is the primary example. Consider any polynomial in n variables over K:

$$f \;=\; c_1\mathbf{x}^{\mathbf{a}_1} \,+\, c_2\mathbf{x}^{\mathbf{a}_2} \,+\, \cdots \,+\, c_s\mathbf{x}^{\mathbf{a}_s}. \tag{7.9}$$

The tropicalization of f is the following expression in tropical arithmetic:

$$\mathrm{trop}(f) \;=\; \mathrm{val}(c_1) \odot \mathbf{x}^{\odot\mathbf{a}_1} \,\oplus\, \mathrm{val}(c_2) \odot \mathbf{x}^{\odot\mathbf{a}_2} \,\oplus\, \cdots \oplus\, \mathrm{val}(c_s) \odot \mathbf{x}^{\odot\mathbf{a}_s}.$$

To evaluate the *tropical polynomial* $\mathrm{trop}(f)$ at a point $\mathbf{u} = (u_1, \ldots, u_n)$, we take the minimum of the s expressions

$$\mathrm{val}(c_i) \odot \mathbf{u}^{\odot\mathbf{a}_i} \;=\; \mathrm{val}(c_i) \odot u_1^{\odot a_{i1}} \odot \cdots \odot u_n^{\odot a_{in}} \;=\; \mathrm{val}(c_i) + a_{i1}u_1 + \cdots + a_{in}u_n,$$

where the index i runs over $\{1, \ldots, s\}$. If this minimum is attained at least twice then $\mathbf{u}$ is a *tropical zero* of $\mathrm{trop}(f)$. The special case $n = 1$ appeared in Section 7.1. The following result generalizes the first part of Theorem 7.6.

Proposition 7.16. *If* $\mathbf{z} = (z_1, \ldots, z_n) \in K^n$ *is a zero of a polynomial* f *in* $K[\mathbf{x}]$ *then its coordinatewise valuation* $\mathrm{val}(\mathbf{z}) = \big(\mathrm{val}(z_1), \ldots, \mathrm{val}(z_n)\big) \in \mathbb{Q}^n$ *is a tropical zero of* $\mathrm{trop}(f)$.

Proof. Note that the valuation of $c_i \mathbf{z}_i^{\mathbf{a}}$ equals $\mathrm{val}(c_i) \odot \mathbf{u}^{\odot \mathbf{a}_i}$. The sum of these r scalars is zero in K, so the terms of lowest valuation must cancel. This implies that the minimum valuation is attained by two or more of the expressions $\mathrm{val}(c_i) \odot \mathbf{u}^{\odot \mathbf{a}_i}$. By definition, this means that the vector $\mathbf{u} \in \mathbf{Q}^n$ is a tropical zero of $\mathrm{trop}(f)$. □

A celebrated result due to Kapranov states that the converse holds too. Namely, if $f \in K[\mathbf{x}]$ and $\mathbf{u} \in \mathbb{Q}^n$ is a tropical zero of $\mathrm{trop}(f)$ then there is a point $\mathbf{z} \in K^n$ such that $f(\mathbf{z}) = 0$ and $\mathrm{val}(\mathbf{z}) = \mathbf{u}$. We refer to [**37**, Theorem 3.1.3] for the proof and further details. For the $n = 1$ case see Theorem 7.6.

The element ∞ in the tropical semiring arises naturally from the arithmetic in the field K because $\mathrm{val}(0) = \infty$. Sometimes it is preferable to restrict tropical algebra to $\mathbb{R}$, or to $\mathbb{Q}$, thus excluding ∞. This is done by disallowing zero coordinates among the solutions of a polynomial system. For this, we set $K^* = K \backslash \{0\}$ and we introduce the *algebraic torus* $(K^*)^n$. The ring of polynomial functions on $(K^*)^n$ is the *Laurent polynomial ring*

$$K[\mathbf{x}^{\pm}] \;:=\; K[\, x_1^{\pm 1},\, x_2^{\pm 1},\, \ldots,\, x_n^{\pm 1}\,].$$

Its elements are polynomials as in (7.9) but we now allow negative integers among the coordinates of the exponent vectors $\mathbf{a}_i$.

In what follows we fix $K = \mathbb{C}\{\{t\}\}$, the field of Puiseux series. The extension to other fields is found in [**37**, §2.4]. Given any vector $\mathbf{u} \in \mathbb{R}^n$, the *initial form* $\mathrm{in}_{\mathbf{u}}(f)$ is the subsum of terms $\overline{c_i}\mathbf{x}^{\mathbf{a}_i}$ in (7.9) for which $\mathrm{val}(c_i) \odot \mathbf{u}^{\odot \mathbf{a}_i}$ is minimal. Here $\overline{c_i}$ is the term of lowest order in the Puiseux series c_i. For instance, if $c = 3t + 9t^3 - \frac{2}{3}t^4 + \ldots \in K$ is the second scalar displayed in Example 7.3 then $\overline{c} = 3t$. A *monomial* in the Laurent polynomial ring is a scalar times a product of variables with possibly negative coefficients.

Lemma 7.17. *For any Laurent polynomial* $f \in K[\mathbf{x}^{\pm}]$ *and any point* $\mathbf{u} \in \mathbb{R}^n$*, the following three conditions are equivalent:*

- *The initial form* $\mathrm{in}_{\mathbf{u}}(f)$ *is not a unit in* $K[\mathbf{x}^{\pm}]$.
- *The initial form* $\mathrm{in}_{\mathbf{u}}(f)$ *is not a monomial.*
- *The point* $\mathbf{u}$ *is a tropical zero of* $\mathrm{trop}(f)$.

Proof. Every monomial is invertible in the Laurent polynomial ring. To show the converse, we fix a lexicographic order on monomials. If $hg = 1$ then the product of the smallest monomials in the support of h and g must equal the product of the two largest. In particular, the smallest and the largest monomial appearing in h must be the same, i.e. h is a monomial.

The equivalence of the last two points in Lemma 7.17 follows from *Kapranov's Theorem*. By this we mean the converse to Proposition 7.16 what was mentioned above. For the precise statement see [**37**, Theorem 3.1.3]. □

Fix any ideal I in $K[\mathbf{x}^{\pm}]$ and let $\mathcal{V}(I)$ be its variety in the algebraic torus $(K^*)^n$. We define the *tropical variety* of I to be the following subset of $\mathbb{R}^n$:

$$\mathrm{trop}(\mathcal{V}(I)) \;=\; \big\{\, \mathbf{u} \in \mathbb{R}^n \;:\; \mathbf{u} \text{ is a tropical zero of } \mathrm{trop}(f) \text{ for all } f \in I \,\big\}.$$

We also refer to this set as the *tropicalization* of the variety $\mathcal{V}(I)$.

The study of tropical varieties is the subject of tropical algebraic geometry. Two important results are the *Fundamental Theorem* ([**37**, Theorem 3.2.3]) and the *Structure Theorem* ([**37**, Theorem 3.3.5]). The former extends Kapranov's Theorem. It states that the set of rational points in $\mathrm{trop}(\mathcal{V}(I))$ is the image of the classical variety $\mathcal{V}(I) \subset (K^*)^n$ under the coordinatewise valuation map. The latter states that the tropical variety $\mathrm{trop}(\mathcal{V}(I))$ is a balanced polyhedral complex. Furthermore, its dimension agrees with the dimension of $\mathcal{V}(I)$. Numerous concrete examples of such balanced polyhedral complexes are found in the textbooks [**29**] and [**37**].

Example 7.18. Fix $n = 9$ and let $\mathbf{x} = (x_{ij})$ be a 3×3-matrix whose entries are unknowns. Let I be the ideal in $K[\mathbf{x}^{\pm}]$ that is generated by the nine 2×2-minors of $\mathbf{x}$. Then $\mathcal{V}(I)$ is the 5-dimensional variety of 3×3-matrices of rank 1 in $(K^*)^{3\times 3}$. The tropical variety $\mathrm{trop}(\mathcal{V}(I))$ is the set of 3×3-matrices in (7.2), that is, matrices $\mathbf{u}$ of tropical rank one. This is the linear subspace of dimension 5 in $\mathbb{R}^{3\times 3}$ defined by the tropical 2×2-determinants $u_{ij} \odot u_{kl} \oplus u_{ik} \odot u_{kj}$. Of course, this minimum is attained twice if and only if $u_{ij} + u_{kl} - u_{ik} - u_{kj} = 0$. Every matrix $\mathbf{u} = (u_{ij})$ that satisfies these linear equations, and has its entries in $\mathbb{Q}$, arises as the valuation $\mathbf{u} = \mathrm{val}(\mathbf{z})$ of a rank one matrix $\mathbf{z} = (z_{ij})$ with entries in K^*. For instance, $\mathbf{z} = (t^{u_{ij}})$.

Consider the assignment problem in Proposition 7.9. The tropical variety $\mathrm{trop}(\mathcal{V}(I))$ represents scenarios where all six assignments for $n = 3$ have the same cost. The situation becomes more interesting when we pass from rank 1 to rank 2. Now only the two best assignments have the same cost.

To model this, let $J \subset K[\mathbf{x}^{\pm}]$ be the principal ideal generated by the determinant of $\mathbf{x}$. Then $\mathcal{V}(J)$ is a hypersurface of degree three in $(K^*)^{3\times 3}$.

The tropical hypersurface $\mathrm{trop}(\mathcal{V}(J))$ is defined by the tropical determinant

$$\begin{aligned}\mathrm{tropdet}(\mathbf{x}) \quad = \quad & x_{11} \odot x_{22} \odot x_{33} \;\oplus\; x_{11} \odot x_{23} \odot x_{32} \;\oplus\; x_{12} \odot x_{21} \odot x_{33} \\ & \oplus\, x_{12} \odot x_{23} \odot x_{31} \;\oplus\; x_{13} \odot x_{21} \odot x_{32} \;\oplus\; x_{13} \odot x_{22} \odot x_{31}.\end{aligned}$$

Thus $\mathrm{trop}(\mathcal{V}(J))$ is set of all 3×3-matrices $\mathbf{u} = (u_{ij})$ such that this minimum is attained twice. For such a matrix, there is more than one optimal assignment of the three workers to the three jobs in (7.5). The set $\mathrm{trop}(\mathcal{V}(J))$ is a polyhedral fan of dimension 8. It is a cone with apex $\mathrm{trop}(\mathcal{V}(I)) \simeq \mathbb{R}^5$ over the 2-dimensional polyhedral complex shown in Figure 1.

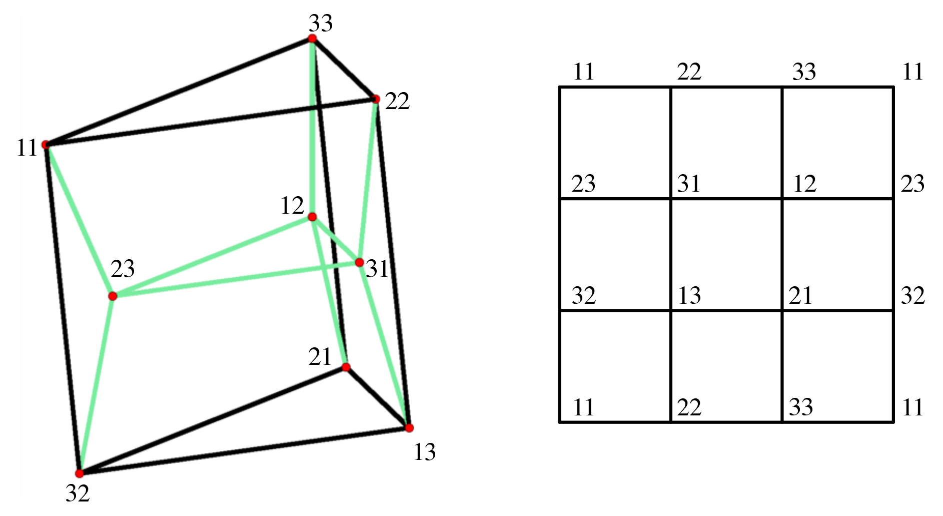


Figure 1. Combinatorial structure of the tropical hypersurface that is defined by the tropical 3×3-determinant.

The six triangles represent matrices $\mathbf{u}$ where the minimum of the six terms in $\mathrm{tropdet}(\mathbf{u})$ is attained by two permutations in S_3 with the same sign. The nine squares on the right in Figure 1 are glued to form a torus. These represent matrices $\mathbf{u}'$ where the minimum is attained by two permutations in S_3 with opposite signs. Concrete examples for the two cases are

$$\mathbf{u} \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}' \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here are rank 2 matrices over K that map to $\mathbf{u}$ and $\mathbf{u}'$ under tropicalization:

$$\mathbf{z} \;=\; \begin{pmatrix} t+1 & -1+t & 2t \\ t & 1 & 1+t \\ 1 & t & 1+t \end{pmatrix} \qquad \text{and} \qquad \mathbf{z}' \;=\; \begin{pmatrix} 1 & 2 & t \\ 2 & 4 & 5t \\ 3t & 6t & 7 \end{pmatrix}$$

The supports of the matrices $\mathbf{u} = \mathrm{trop}(\mathbf{z})$ and $\mathbf{u}' = \mathrm{trop}(\mathbf{z}')$ match the labels of the corresponding 2-cells in Figure 1. The matrix $\mathbf{u}$ has support $13, 21, 32$, which labels the bottom triangle on the left. The matrix $\mathbf{u}$ has support $13, 23, 31, 32$, which labels the middle left square on the right.

We close with a remark on lifting Proposition 7.8 from tropical algebra to algebra over the field $K = \mathbb{C}\{\{t\}\}$. Given a directed graph G with rational edge weights d_{ij}, we now define a new adjacency matrix A_G. The entry of A_G in row i and column j equals $t^{d_{ij}}$ if (i, j) is an edge of G, and 0 otherwise.

By construction, the valuation of the matrix A_G is the adjacency matrix D_G seen earlier in Section 7.1. Moreover, the tropical matrix power in (7.3) is the valuation of the corresponding power of the classical matrix A_G:

$$D_G^{\odot(n-1)} = (\mathrm{val}(A_G))^{\odot(n-1)} = \mathrm{val}\big(A_G^{n-1}\big). \tag{7.10}$$

Indeed, the (i, j) entry of A_G^{n-1} is the generating function for all paths. To be precise, this entry is the Puiseux polynomial $\sum_\ell c_\ell t^\ell$, where c_ℓ is the number of paths from node i to node j in the graph G that have length $\leq \ell$.

Exercises

(1) Let u, v, w be real numbers and let x, y, z be variables. What are the coefficients in the expansion of the expression $(u \odot x \oplus v \odot y \oplus w \odot z)^{\odot n}$ in tropical arithmetic?

(2) Prove that the tropical matrix multiplication is an associative operation.

(3) Draw the graph of the following function on the real plane: $\mathbb{R} \to \mathbb{R}$, $x \mapsto 1 \oplus 2 \odot x \oplus 3 \odot x^{\odot 2} \oplus 6 \odot x^{\odot 3} \oplus 10 \odot x^{\odot 4}$. What are the tropical zeros of this tropical polynomial?

(4) How would you define the tropical characteristic polynomial of a square matrix? Compute your polynomial for the 3×3-matrices in Example 7.13.

(5) Draw the graph of the function

$$\mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto 1 \oplus 2 \odot x \oplus 3 \odot y \oplus 6 \odot xy \oplus 10 \odot xy^{\odot 2}.$$

What are the tropical zeros of this tropical polynomial?

(6) Let G be the directed graph on n nodes with edge weights $d_{ij} = i \cdot j$ for $i, j \in \{1, 2, \ldots, n\}$. Compute the tropical powers $D_G^{\odot i}$ of the matrix D_G for $i = 1, 2, \ldots, n-1$. What are their tropical ranks? Interpret the entries of these matrices in terms of paths.

(7) Take the graph G from above with $n = 5$. Compute the powers A_G^i of the matrix A_G for $i < n$. What are their ranks? Interpret the entries in terms of paths. Verify equation (7.10).

(8) Take the graph G from above with $n = 3$. Find the eigenvalues and eigenspaces of the classical matrix A_G. Find the tropical eigenvalue and the tropical eigenspace of the matrix D_G. Do you see a relationship?

(9) Take the graph G from above with $n = 10$. Compute the determinant of A_G and the tropical determinant of D_G. Do you see a relationship?

(10) Take the graph G from above. The matrix D_G defines a tropical linear map from $\mathbb{R}^n$ to itself. Determine the image of this map for $n = 2, 3, 4$. Draw pictures in $\mathbb{R}^n/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^{n-1}$. These are tropical polytopes.

(11) Consider the quartic polynomial $f(x) = t + t^2x + t^3x^2 + t^6x^3 + t^{10}x^4$ in $K[x]$, where $K = \mathbb{C}\{\{t\}\}$. Identity its four roots. Write the first 10 terms of these Puiseux series. What are their valuations?

(12) Let J be the ideal generated by the determinant of a symmetric 3×3-matrix. This lives in a Laurent polynomial ring with six variables. Determine the tropical hypersurface $\text{trop}(\mathcal{V}(J))$. Write a discussion similar to Example 7.18. Draw the analog to Figure 1 for symmetric matrices.

(13) Analyze the complexity of the algorithm described in Proposition 7.8. Can you improve the computation of $D_G^{\odot(n-1)}$? What happens if some edge weights of G are negative? What happens if G contains cycles of negative total weight? How can you detect if such a cycle exists?

(14) The Wikipedia site for *Tropical Geometry* shows a tropical cubic curve. Find a tropical polynomial in two unknowns that defines this curve.

Chapter 8

Toric Varieties

Toric varieties are the simplest and most accessible varieties. They often appear in applications, both within mathematics and across the sciences. A toric variety is an irreducible variety that is parametrized by a vector of monomials. The relations among these monomials are binomials, i.e. polynomials with only two terms. Thus, an irreducible variety is toric if and only if its prime ideal is generated by binomials. Monomials and binomials correspond to points in an integer lattice, and we think of these as the vertices of a lattice polytope. Toric varieties appear prominently in optimization and statistics, thanks to the purely combinatorial description given above. This description also makes them a perfect "model organism" for algebraic geometers. They use toric varieties to test conjectures, teach geometric concepts, and compute invariants. For instance, the dimension and degree of a toric variety are the dimension and volume of the associated lattice polytope.

8.1. The Affine Story

The adjective *toric* derives from the noun *torus*. We begin by introducing tori from an algebraic perspective. We fix an algebraically closed field K and the Laurent polynomial ring $K[\mathbf{x}^{\pm}] = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The associated variety $(K^*)^n = \operatorname{Spec} K[\mathbf{x}^{\pm}]$ is the *algebraic torus* of dimension n over K.

The algebraic torus $(K^*)^n$ is a group under coordinatewise multiplication. The name torus comes from the special case when $n = 2$ and $K = \mathbb{C}$ is the complex numbers. Here, we have $(\mathbb{C}^*)^2 \simeq (\mathbb{R}_+ \times \mathbb{S}^1)^2$, where $\mathbb{S}^1$ is the unit circle. Thus usual topological torus $\mathbb{S}^1 \times \mathbb{S}^1$ is equal to the 2-dimensional algebraic torus after multiplication with the contractible factor $\mathbb{R}_+ \times \mathbb{R}_+$.

We recall from Section 7.3 that subvarieties of the algebraic torus $(K^*)^n$ are the objects one starts from when developing tropical algebraic geometry.

Definition 8.1 (Character of a torus)**.** A *character* of the algebraic torus $T = (K^*)^n$ is an algebraic map $\chi : T \to K^*$ that is also a group morphism.

In Exercise 1 we shall see that characters are given by Laurent monomials

$$\mathbf{x}^{\mathbf{b}} \;=\; x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}, \quad \text{where } \mathbf{b} \in \mathbb{Z}^n.$$

The characters of T are hence the elements of $\mathbb{Z}^n$. Under this correspondence, multiplication of characters becomes addition in the group $(\mathbb{Z}^n, +)$:

$$(\chi_1 + \chi_2)(\mathbf{x}) \;=\; \chi_1(\mathbf{x})\chi_2(\mathbf{x}).$$

A group isomorphic to $\mathbb{Z}^k$, for some k, is called *a lattice*. The lattice of characters of T will be denoted by M_T or simply M. As a subgroup of a free abelian group is free, any set of characters generates a sublattice $\tilde{M} \subset M$.

Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ be characters in $M_T \simeq \mathbb{Z}^n$. We write A for the $n \times p$ matrix whose columns are the vectors $\mathbf{a}_i$. The lattice $\tilde{M}$ generated by the characters $\mathbf{a}_i$ is the image of $\mathbb{Z}^p$ under the right multiplication by the matrix A.

Proposition 8.2. *The image of T in $(K^*)^p$ under the map $f : \mathbf{x} \to \mathbf{x}^A = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p})$ is also a torus $\tilde{T}$. The character lattice of $\tilde{T}$ is equal to $\tilde{M}$.*

Proof. The monomial map $f : T \mapsto (K^*)^p$ induces the ring homomorphism

$$f^* : K[y_1^{\pm 1}, \dots, y_p^{\pm 1}] \;\to\; K[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \quad y_i \mapsto \mathbf{x}^{\mathbf{a}_i}.$$

The spectrum of the image of the ring map f^* is the image $\tilde{T}$ we are interested in. Note that this image is the group algebra $K[\tilde{M}]$. By definition, this is the vector space over K with basis given by elements of $\tilde{M}$ and multiplication induced from addition in M_T. The lattice $\tilde{M}$ is isomorphic to the group $\mathbb{Z}^d$ for some integer $d \leq \min(n, p)$. We have $\tilde{T} = \operatorname{Spec} K[\tilde{M}] = (K^*)^d$. □

The d-dimensional torus $\tilde{T}$ lives in $(K^*)^p$. We are interested in its Zariski closure in the affine space K^p. Such an affine variety is a toric variety.

Definition 8.3. An *affine toric variety* is the closed image of a monomial map $(K^*)^n \to K^p$, $\mathbf{x} \mapsto \big(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p}\big)$, where $\mathbf{a}_i \in \mathbb{Z}^n$ and $K^* = K \backslash \{0\}$.

We specify a toric variety by an integer matrix $A \in \mathbb{Z}^{n \times p}$. The p columns of A represent characters of the torus $T = (K^*)^n$. Toric geometry relates the combinatorics of these lattice points with the geometry of the toric variety.

Example 8.4. Any affine space is a toric variety. The corresponding matrix A is the identity matrix. The cuspidal cubic curve $x^3 - y^2$ is a toric variety. It is the image of the map $z \mapsto (z^2, z^3)$ that is given by the matrix $A = \big(2 \;\; 3\big)$.

Proposition 8.5. *The dimension of the affine toric variety in Definition 8.3 is equal to the rank of the lattice $\tilde{M}$ that is spanned by $\mathbf{a}_1, \dots, \mathbf{a}_p$ in $\mathbb{Z}^n$.*

Proof. We saw in the proof of Proposition 8.2 that the torus $\tilde{T}$ has dimension $d = \text{rank}(\tilde{M})$. The toric variety is the Zariski closure of $\tilde{T}$. Its has dimension d, since passing to the Zariski closure preserves dimension. □

We defined toric varieties as closures in K^p of subtori of the torus $(K^*)^p$. In the notation of Proposition 8.2, the toric variety equals $\operatorname{Spec} K[S]$, where S is the monoid in M_T generated by the distinguished characters, i.e. the smallest set containing 0, the chosen characters and closed under addition.

Example 8.6. (1) The cuspidal curve defined by the equation $x^3 - y^2$ equals $\operatorname{Spec} K[z^2, z^3]$. The underlying monoid equals $\{0, 2, 3, 4, \dots\}$.

(2) The affine line is the closure of the image of the map

$$K^* \ni x \to x \in K.$$

Here the character lattice is $M = \mathbb{Z}$, the distinguished character corresponds to $1 \in M$ and the monoid equals $\{0, 1, 2, \dots\}$.

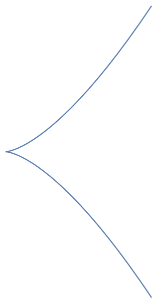


Figure 1. The cuspidal cubic curve

There is a fundamental difference between the cuspidal curve and affine line. The monoid for the cuspidal curve has a 'hole', namely the character 1.

Definition 8.7. A submonoid S in a lattice M is called *saturated* if and only if for any $x \in M$ and $k \in \mathbb{Z}_+$ the following implication holds:

$$kx \in S \Rightarrow x \in S.$$

An affine toric varieties $X = \operatorname{Spec} K[S]$ for which S is saturated (in the lattice $\tilde{M}$ it generates) is called *normal*. For the algebraic definition of normality see [**1**, Chapter 5]. Nonnormal varieties are always singular. For curves, the two notions coincide. Example 8.6 displays one normal (i.e. smooth) curve and one nonnormal (i.e. singular) curve, seen in Figure 1.

We next discuss the prime ideal of the toric variety X. This is computed from the characters that define X. In general, given a variety defined as a Zariski closure of the image of a map, finding the defining equations is a hard problem, known as *implicitization.* We discussed this in Chapter 4. The implicitization problem greatly simplifies when the variety is toric. The prime ideal I_X of the toric variety X lives in the polynomial ring $K[y_1, \dots, y_p]$. This *toric ideal* is the kernel of the restriction of f^* to this polynomial ring.

Lemma 8.8. *Let X be the toric variety defined by $A = (\mathbf{a}_1, \dots, \mathbf{a}_p)$. Then:*

(1) *any relation $\sum_i b_i \mathbf{a}_i = \sum_j c_j \mathbf{a}_j$, with positive integral coefficients $b_i, c_j \in \mathbb{Z}_+$ provides a binomial $\prod y_i^{b_i} - \prod y_j^{c_j}$ in the toric ideal I_X;*

(2) *every binomial in the ideal I_X is of the form described in point 1;*

(3) *the toric ideal I_X is generated by these binomials.*

Recall that a *binomial* is a polynomial with only two terms. Statement (2) is understood up to scaling: we can multiply the binomial by a constant.

Proof: Properties (1) and (2) follow from the fact that a polynomial vanishes on the toric variety X if and only if we obtain zero after substituting y_i by $\mathbf{x}^{\mathbf{a}_i}$. However, such a substitution turns monomials (in variables y) to monomials (in variables x). The fact that the monomials in $\mathbf{x}$ cancel is precisely encoded by the integral relations in point (1). Property (3) follows similarly, by induction on number of terms of a polynomial in the ideal of X. For a similar argument using a monomial ordering see [**52**, Lemma 4.1]. □

Example 8.9. Fix $n = 3, p = 7$. To specify a toric variety, we must choose characters $\mathbf{a}_1, \dots, \mathbf{a}_7 \in \mathbb{Z}^3$. Let us take the column vectors of the matrix

$$A \quad = \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

The associated toric variety X is a threefold in K^7. The toric ideal I_X equals

$$\begin{aligned} \langle\, y_1y_3 - y_2y_7\,,\; y_1y_4 - y_7^2\,,\; y_1y_5 - y_6y_7\,,\; y_2y_4 - y_3y_7\,,\; y_2y_5 - y_7^2\,, \\ y_2y_6 - y_1y_7\,,\; y_3y_5 - y_4y_7\,,\; y_3y_6 - y_7^2\,,\; y_4y_6 - y_5y_7\,\rangle. \end{aligned} \tag{8.1}$$

Each of these nine binomials vanishes under the substitution $y_i \mapsto \mathbf{x}^{\mathbf{a}_i}$. Using the methods in Section 4.2, we can check that I_X is the desired prime ideal. The toric variety X has dimension 3 and it lives in the affine space K^7.

The ideal I_X is homogeneous. Each of the nine binomials in (8.1) is homogeneous. This comes from the fact that the matrix A has column sums $(3, 3, 3, 3, 3, 3, 3)$. Geometrically speaking, the threefold X is a cone in K^7. We can therefore also regard X as a surface in the projective space $\mathbb{P}^6$. That surface is nonsingular and it has degree six. This passage from appropriate matrices A to projective toric varieties will be our theme in Section 8.2.

Theorem 8.10. *The toric ideals I_X are precisely the prime ideals generated by binomials $\mathbf{y}^{\mathbf{b}} - \mathbf{y}^{\mathbf{c}}$. Every such ideal defines a toric variety X as above.*

Proof. Let I be a prime ideal generated by a set of binomials $\mathbf{y}^{\mathbf{b}_i} - \mathbf{y}^{\mathbf{c}_i}$ in p variables $y_1, \ldots, y_p$. By Hilbert's Basis Theorem, there is a finite subset of minimal generators. For each such generator, the nonnegative integer vectors $\mathbf{b}_i$ and $\mathbf{c}_i$ have disjoint support, since I is prime. We write the difference vectors $\mathbf{b}_i - \mathbf{c}_i$ as the columns of a matrix B that has p rows.

Let A be any integer matrix of format $n \times p$ whose rows span the kernel of B under left multiplication. Here, the kernel is understood as a $\mathbb{Z}$-module (a.k.a. abelian group), so it is computed using integer linear algebra (e.g. the Hermite normal form algorithm). We claim that the columns of B span the kernel A under right multiplication. This is clear over $\mathbb{Q}$, but it also holds over $\mathbb{Z}$ by our hypothesis that I is a prime ideal. Otherwise, there would exist a vector $\mathbf{b} - \mathbf{c}$ that is not in the column space of B but some integer multiple $k\mathbf{b} - k\mathbf{c}$ is in that column span. Then the binomial $\mathbf{y}^{k\mathbf{b}} - \mathbf{y}^{k\mathbf{c}}$ is in the ideal I but none of its factors is, which is impossible for a prime ideal.

We now take X to be the toric variety X in K^p defined by the matrix A. The argument above shows that $I = I_X$, which gives the assertion in the theorem. For further details on this proof see [**12**, Proposition 1.1.11]. □

Definition 8.11. A *convex polyhedral cone* C in $\mathbb{R}^n$ is a subset of elements of the form $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$ where $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are fixed and $\lambda_1, \ldots, \lambda_k$ range over $\mathbb{R}_{\geq 0}$. We call C *rational* if the vectors $\mathbf{v}_i$ can be chosen in $\mathbb{Q}^n$. In what follows we refer to rational convex polyhedral cones simply as *cones*.

Definition 8.12. A *face* F of a cone $C \subset \mathbb{R}^n$ is a subset of the form

$$F \;=\; \bigl\{\mathbf{c} \in C : \ell(\mathbf{c}) = 0\bigr\},$$

where ℓ is a linear form that is nonnegative on C, i.e. $\ell(\mathbf{c}) \geq 0$ for all $\mathbf{c} \in C$. If $\dim C = n = \dim F + 1$, then ℓ is uniquely determined, up to scalar. In this case, F is called a *facet* and the hyperplane defined by ℓ is a *supporting hyperplane* of C. We point out that $\ell = 0$ gives $F = C$. Furthermore, any face of a cone is also a cone, and the relation "is a face of" is transitive.

Example 8.13. The orthant $C = \mathbb{R}^n_{\geq 0}$ is a cone. It has 2^n faces, ranging from the apex $\{0\}$ to the full cone C. There are $\binom{n}{i}$ faces of dimension i. Each of the n facets F arises by setting one coordinate to zero, so $F \simeq \mathbb{R}^{n-1}_{\geq 0}$.

By Proposition 8.2, the toric variety X is the closure in K^p of the torus $\tilde{T} \subset (K^*)^p$. The group $\tilde{T}$ acts both on itself and on K^n, and it hence also acts on its closure X. The *torus orbits* on X are the orbits of that action by $\tilde{T}$. We next provide a combinatorial and geometric description of the torus orbits. We assume that X is defined by an integer $n \times p$-matrix A as above. We write $C \subset \mathbb{R}^n$ for the cone that is generated by the p columns $\mathbf{a}_i$ of A.

Theorem 8.14. *The torus orbits in X are in bijection with the faces of the cone C. The orbit corresponding to a face F is $\{\mathbf{y} \in X : y_i{\neq}0 \iff \mathbf{a}_i{\in}F\}$. The closure of this orbit is the toric variety* $\operatorname{Spec} K[F{\cap}A]$ *whose parametrization is $(\mathbf{x}^{\mathbf{a}_i} \,:\, \mathbf{a}_i \in F)$. The dimension of this orbit equals* $\dim(F)$*. Moreover, the inclusion of orbit closures in X corresponds to inclusion of faces of C.*

Proof. For normal toric varieties, this appears in [**12**, Section 3.2]. However, normality is not needed. A direct argument shows that, if F is a face of C, then $\{\mathbf{y} \in X : y_i \neq 0 \iff \mathbf{a}_i \in F\}$ is a torus orbit. Furthermore, the binomials in I_X ensure that each point of X lies in one of these orbits. □

Example 8.15. Let X be the toric threefold in K^7 given in Example 8.9. The cone C is spanned in $\mathbb{R}^3$ by the columns of the 3×7 matrix A. It is the cone over a hexagon, so it has $14 = 1+6+6+1$ faces. The variety X is the disjoint union of 14 torus orbits, as follows. The face $F = \{0\}$ corresponds to the origin in K^7. The 1-dimensional face $F = \mathbb{R}_{\geq 0}\{\mathbf{a}_1\}$ corresponds to the curve $\{(t,0,0,0,0,0,0) \in X : t \in K^*\}$. The 2-dimensional face $F = \mathbb{R}_{\geq 0}\{\mathbf{a}_1, \mathbf{a}_2\}$ corresponds to the surface $\{(t,u,0,0,0,0,0) \in X : t,u \in K^*\}$. And, $F = C$ corresponds to the 3-dimensional torus $\tilde{T} = X \cap (K^*)^7$.

In conclusion, the geometry of X is read off from the cone C representing it.

8.2. Varieties from Polytopes

Projective toric varieties are obtained from affine toric varieties that are cones. They can be defined as follows. Let $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p)$ be an integer $n \times p$-matrix of rank n that has the vector $(1,1,\ldots,1)$ in its row span. Let $I_A \subset K[y_1, y_2, \ldots, y_p]$ be the prime ideal of polynomial relations among the Laurent monomials $\mathbf{x}^{\mathbf{a}}_1, \mathbf{x}^{\mathbf{a}}_2, \ldots, \mathbf{x}^{\mathbf{a}}_p$. This is a toric ideal. According to Lemma 8.8, I_A is generated by the homogeneous binomials $\mathbf{y}^{\mathbf{b}} - \mathbf{y}^{\mathbf{c}}$, where $\mathbf{b} - \mathbf{c}$ is in the kernel of A. We write $P = \operatorname{conv}(A)$ for the convex hull of the column vectors $\mathbf{a}_i$ in $\mathbb{R}^n$. By construction, P is a polytope of dimension $n-1$ with $\leq p$ vertices. For instance, in Example 8.9, the polytope P is a regular hexagon, and the surface X_A is the blow-up of $\mathbb{P}^2$ at three points.

Definition 8.16. A *projective toric variety* is any projective variety in $\mathbb{P}^{p-1}$ of the form $X_A = \mathcal{V}(I_A)$, where I_A is a homogeneous toric ideal as above.

Definition 8.17. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points. A polytope is a *lattice polytope* if it is the convex hull of points in $\mathbb{Z}^n$.

Given a projective toric variety X_A, we associate to it the polytope $P = \operatorname{conv}(A)$. Conversely, any lattice polytope can be coordinatized so that it spans the affine hyperplane $\{y_1 + y_2 + \cdots + y_n = k\}$ for some $k, n \in \mathbb{Z}_+$.

We then take $A = P \cap \mathbb{Z}^n$, and we associate the projective toric variety $X_P := X_A$ with the polytope P. This variety lives in $\mathbb{P}^{p-1}$ where $p = |A|$.

The class of varieties X_A is strictly larger than the class of varieties X_P. The reason is that A can be a proper subset of $\operatorname{conv}(A) \cap \mathbb{Z}^n$, However, the projective toric varieties of most interest to us are X_P for some polytope P.

Example 8.18. The *Veronese variety* and the *Segre variety* from classical algebraic geometry are two prominent examples of projective toric varieties.

We write Δ_{n-1} for the standard $(n-1)$-simplex, whose vertices are the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Fix $k \in \mathbb{Z}_{>0}$ and $P = k\Delta_{n-1}$. Then $A = P \cap \mathbb{Z}^n$ consists of the nonnegative integer vectors with coordinate sum k. Hence $p = |A| = \binom{n+k-1}{k}$. The toric variety X_P is the *k-th Veronese embedding* of $\mathbb{P}^{n-1}$. It has dimension $n-1$ and degree k^{n-1} in $\mathbb{P}^{p-1}$. Its toric ideal I_A consists of the polynomial relations among all monomials of degree k in n variables. For instance, if $n = k = 3$ then there are ten such monomials:

$$A \quad = \quad \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

The ideal defining this Veronese surface in $\mathbb{P}^9$ is generated by 27 quadrics:

$$I_A \;=\; \langle y_1y_4 - y_2^2, y_1y_5 - y_2y_3, y_1y_6 - y_3^2, y_1y_7 - y_2y_4, \ldots, y_7y_{10} - y_8y_9, y_8y_{10} - y_9^2 \rangle.$$

Next, fix $n_1, n_2 \in \mathbb{Z}_{>0}$ and set $n = n_1 + n_2$ and $p = n_1n_2$. Let $P = \Delta_{n_1} \times \Delta_{n_2}$ and write $\mathbf{e}_i$ and $\mathbf{e}'_j$ for the unit vectors in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Then $A = P \cap \mathbb{Z}^n = \{\mathbf{e}_i + \mathbf{e}'_j : 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq n_2\}$. The variety X_P is the *Segre embedding* of the product $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ into $\mathbb{P}^{p-1}$. The points on X_P are $n_1 \times n_2$ matrices of rank one, up to scaling. The associated toric ideal I_A is generated by the 2×2 minors of an $n_1 \times n_2$ matrix of unknowns. We here deviate slightly from our hypothesis, as the rank of A is $n-1$, not n.

In the algebraic geometry literature, it is often assumed that toric varieties are *normal*. This is motivated by the fact that normal toric varieties admit a nice intrinsic characterization, in terms of fans. In our setting, this hypothesis is generally not needed. Still, we present the relevant definition.

Definition 8.19. A lattice polytope P in $\mathbb{R}^n$ is called *normal* if, for any integer k and any point $\mathbf{u} \in kP \cap \mathbb{Z}^n$, there exist $\mathbf{u}_1, \ldots, \mathbf{u}_k \in P \cap \mathbb{Z}^n$ such that $\mathbf{u} = \sum_{i=1}^k \mathbf{u}_i$. In this case, the toric variety X_P is projectively normal.

The simplices and products of simplices in Example 8.18 are normal. Hence the Veronese variety and the Segre variety are projectively normal. Exercise 6 gives an example of a 3-dimensional lattice polytope that is not normal. All lattice polytopes of dimension 1 and 2 are normal. If P is normal then the polyhedral fan that characterizes its toric variety X_P intrinsically is the *normal fan* of P. We refer to [**12**, Section 3.1] for the basic theory.

We now return to the setting where I_A is any homogeneous toric ideal, $X_A \subset \mathbb{P}^{p-1}$ its toric variety, and $P = \operatorname{conv}(A)$ not necessarily normal. Let T denote the subset of X_A consisting of all points with non-zero coordinates. This is a torus of dimension $n-1$. That torus acts on X_A with finitely many orbits. Theorem 8.14 extends essentially verbatim to the projective case.

Corollary 8.20. *The torus orbits in X_A are in bijection with the faces of the polytope P. The orbit corresponding to a face F is $\{\mathbf{y} \in X_A : y_i \neq 0 \iff \mathbf{a}_i \in F\}$. The closure of this orbit is the projective toric variety with parametrization $(\mathbf{x}^{\mathbf{a}_i} : \mathbf{a}_i \in F)$. The dimension of this orbit equals $\dim(F)$. The inclusion of orbit closures in X_A corresponds to inclusion of faces of P.*

Proof. We apply Theorem 8.14 to the affine toric variety defined by I_A in K^p. This is the affine cone over $X_A \subset \mathbb{P}^{p-1}$. Its orbits correspond to the faces of the cone C over the polytope P. Note that $\dim(C) = n = \dim(P) + 1$. Each i-dimensional face F of P corresponds to an $(i+1)$-dimensional face of C, namely the cone over F. Likewise, each i-dimensional orbit in X_A corresponds to an $(i+1)$-dimensional orbit of the affine cone over X_A. These bijections, for $i = 0, 1, \ldots, n-1$, establish the desired bijection for P and X_A. The only face of C that is missing in P is the origin $\{0\}$. Likewise, the cone point of the affine cone over X_A disappears in X_A. □

Example 8.21. Consider the Segre threefold $X_A = \mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, given by $n_1 = 2$ and $n_2 = 3$ in Example 8.18. The toric ideal I_A is generated by the 2×2-minors of a 2×3-matrix of unknowns, and the polytope $P = \Delta_1 \times \Delta_2$ is a *triangular prism*. This 3-dimensional polytope has $21 = 6+9+5+1$ faces, one for each of the torus orbits on X_A. For instance, the five 2-dimensional orbits are given by setting one row or column of the 2×3-matrix to zero, and the 0-dimensional orbits in X_A are the matrices with one non-zero entry.

Corollary 8.20 establishes a combinatorial link between projective toric varieties and their lattice polytopes. In what follows we tighten this to a geometric link. We now fix $K = \mathbb{C}$, the complex numbers. We seek to argue that the geometry of the polytope coincides with the geometry of the toric variety. The key to this identification is the *moment map* from X_A onto P.

We work in the complex projective space $\mathbb{P}^{p-1}_{\mathbb{C}}$ with its homogeneous coordinates $\mathbf{y} = (y_1 : y_2 : \cdots : y_p)$. The following map onto $P = \operatorname{conv}(A)$ is defined via the usual Euclidean norm $|\cdot|$ on the complex plane $\mathbb{C} \simeq \mathbb{R}^2$:

$$\mathbb{P}^{p-1}_{\mathbb{C}} \to \mathbb{R}^n, \quad \mathbf{y} \mapsto \frac{1}{\sum_{i=1}^p |y_i|} \sum_{i=1}^p |y_i| \cdot \mathbf{a}_i. \tag{8.2}$$

This map is well-defined because the image is invariant under scaling the vector $\mathbf{y}$, and $\sum_{i=1}^{p} |y_i|$ is always positive. Its image is precisely the polytope P, since we are taking arbitrary convex combinations of the points $\mathbf{a}_i$ in $\mathbb{R}^n$.

Definition 8.22. The *algebraic moment map* $\mu_A : X_A \to \mathbb{R}^n$ is defined as the restriction of (8.2) from the ambient space $\mathbb{P}_{\mathbb{C}}^{p-1}$ to the toric variety X_A. We write $X_{A,\mathbb{R}}$ for the subset of real points in X_A. Its subset of nonnegative resp. positive points is denoted by $X_{A,\geq 0}$ resp. $X_{A,>0}$. These are semialgebraic sets in the real projective space $\mathbb{P}_{\mathbb{R}}^{p-1}$. To be precise, the *positive toric variety* $X_{A,>0}$ consists of all positive solutions, up to scale, of the binomial equations in I_A, and similarly for the *nonnegative toric variety* $X_{A,\geq 0}$.

The complex projective toric variety X_A maps naturally onto its nonnegative part $X_{A,\geq 0}$ under the coordinatewise absolute value map

$$(y_1 : y_2 : \cdots : y_p) \;\mapsto\; (|y_1| : |y_2| : \cdots : |y_p|). \tag{8.3}$$

The fibers of this map are real tori. Specifically, the fiber over each point in $X_{A,>0}$ is homeomorphic to the torus $(\mathbb{S}^1)^{n-1}$. This torus is a subgroup the complex torus $T \simeq (\mathbb{C}^*)^{n-1}$, and we can think of (8.3) as the quotient map

$$X_A \longrightarrow X_A/(\mathbb{S}^1)^{n-1} \;=\; X_{A,\geq 0}.$$

See [**12**, Proposition 12.2.3] for a formal statement for normal toric varieties. The algebraic moment map μ_A factors through the quotient map (8.3).

Theorem 8.23. *The restriction of the algebraic moment map μ_A to the nonnegative toric variety $X_{A,\geq 0}$ is a homeomorphism onto the polytope P.*

Proof. This is found in many sources. One of them is [**49**, Theorem 8.4]. □

Corollary 8.24. *If the linear system of equations $A\mathbf{y} = \mathbf{b}$ has a nonnegative solution $\mathbf{y} \in \mathbb{R}_{\geq 0}^{p}$ then it has unique solution $\hat{\mathbf{y}}$ in the toric variety X_A.*

Proof. Here we identify X_A with the affine cone over the projective toric variety defined by the $n \times p$ matrix A. The algebraic moment map μ_A lifts uniquely, by scaling, to a homeomorphism from this affine cone to the cone C over the polytope P. The system $A\mathbf{y} = \mathbf{b}$ has a nonnegative solution if and only if $\mathbf{b}$ lies in C. In this case, the point $\mathbf{b}$ has a unique preimage $\hat{\mathbf{y}} = \mu_A^{-1}(\mathbf{b})$ under the moment map. This preimage is the desired solution. □

Example 8.25. Let A be the $(n_1{+}n_2) \times (n_1 n_2)$ matrix for the polytope $P = \Delta_{n_1-1} \times \Delta_{n_2-1}$ as in Examples 8.18 and 8.21. This matrix A represents the linear map that takes an $n_1 \times n_2$ matrix $\mathbf{y}$ to its vector $\mathbf{b}$ of row and column sums. The polytopes $\{\mathbf{y} \in \mathbb{R}_{\geq 0}^{n_1 \times n_2} : A\mathbf{y} = \mathbf{b}\}$ are known as *transportation polytopes*. The points in the Segre variety X_A are the $n_1 \times n_2$ matrices $\mathbf{y}$ of rank one. In this case, Corollary 8.24 has the following interpretation: *Every transportation polytopes contains a unique rank one matrix* $\hat{\mathbf{y}}$.

Example 8.25 has important consequences in statistics. We saw this already for $n_1 = n_2 = m$ in Example 2.5. Suppose the nonnegative matrix $\mathbf{y}$ has entries that sum to 1. Then $\mathbf{y}$ is a joint distribution for two random variables that have n_1 and n_2 states respectively. The nonnegative variety $X_{A,\geq 0}$ is the independence model for these two random variables. The map A computes the sufficient statistics $\mathbf{b} = A\mathbf{y}$, i.e. the column vector of row sums and the row vector of column sums. The product of these vectors is the rank one matrix $\hat{\mathbf{y}} = \mu_A^{-1}(\mathbf{b})$. This is the *maximum likelihood estimate* for the empirical distribution $\mathbf{y}$ with respect to the independence model.

Example 8.26 ($n_1 = n_2 = 2$)**.** The independence model for two binary random variables is a quadratic surface in $\mathbb{P}^3_{\geq 0} = \Delta_3$. This is the nonnegative part $X_{A,\geq 0}$ of the Segre quadric $X_A = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. That surface meets the boundary of the ambient tetrahedron in four edges that form a 4-cycle. The moment map μ_A projects the tetrahedron onto a square. The 4-cycle is mapped onto the boundary of the square. The surface $X_{A,>0}$ is mapped bijectively onto the interior of the square. Figure 2 illustrates this scenario.

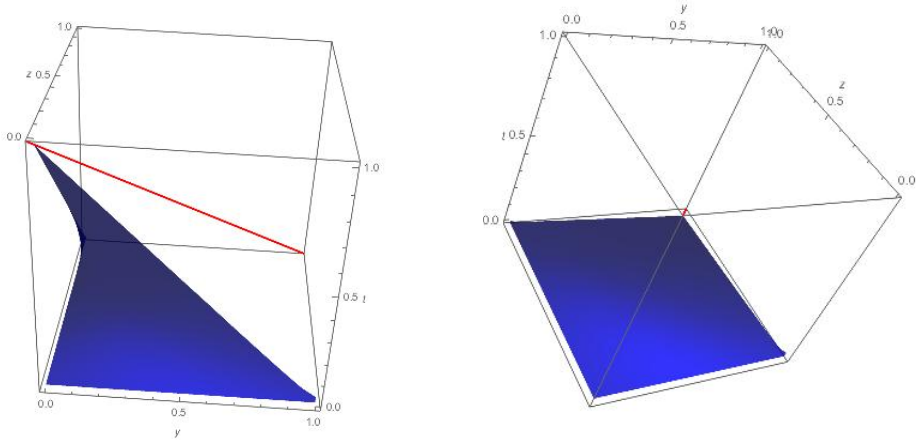


Figure 2. The moment map identifies the Segre quadric with a square.

Example 8.26 is an instance of a general construction in algebraic statistics. Projective toric varieties X_A correspond to a class of statistical models, referred to as *toric models* in [**42**] and as *log-linear models* in [**50**]. The inverse moment map μ_A^{-1} is the maximum likelihood estimator for the model $X_{A,\geq 0}$. Given a point $\mathbf{b}$ in the model polytope $P = \mathrm{conv}(A)$, the estimate $\mu_A^{-1}(\mathbf{b})$ is the *Birch point* in $X_{A,\geq 0}$. This distribution best explains the data with sufficient statistic $\mathbf{b}$. See [**42**, Proposition 1.9] and [**50**, Corollary 7.3.9].

Toric varieties are ubiquitous in applications. One explanation for this is the following observation which connects this chapter to the previous one.

Proposition 8.27. *Let X be an irreducible variety over a field K as in Section 7.3. If X is toric then its tropicalization* $\mathrm{Trop}(X)$ *is a linear space.*

Proof. If $X = \mathcal{V}(I_A)$ then every point in X has the form $(\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_p})$. The images of these points under coordinatewise valuation are $\mathbf{u}A$ where $\mathbf{u}=\mathrm{val}(\mathbf{x})$ runs over $\mathbb{Q}^n$. This implies that $\mathrm{Trop}(X)$ is the row space of A. □

In fact, the converse to this proposition also holds, with a slightly more inclusive definition of toric variety. Informally speaking, toric varieties are precisely those varieties that become linear spaces under taking logarithms.

8.3. The World is Toric

The occurrence of toric structures in an application can be either obvious or hidden. A typical example for the former is log-linear models in statistics. These are obviously toric, as seen around Example 8.26. In this section we discuss some scenarios where the toric structure is hidden, and it needs to be unearthed, often by a non-trivial chance of coordinates. Our style in this section is extremely informal. We briefly visit four fields where toric varieties arise. Under each header we focus on one concrete instance of a toric variety $X_A \subset \mathbb{P}^{p-1}$. The broader context is discussed alongside that example.

Chemical Reactions. Three chemical species $\sigma_1, \sigma_2, \sigma_3$ can form four chemical complexes $3\sigma_1, 3\sigma_2, 3\sigma_3, \sigma_1{+}\sigma_2{+}\sigma_3$. Each complex can react so as to transform into any other complex. We introduce unknowns c_1, c_2, c_3 for the species concentrations and K_1, K_2, K_3, K_4 to encode rate constants.

This chemical reaction system is modeled by the *toric balancing ideal*

$$I_A \;=\; \left(\left\langle 2 \times 2 \text{ minors of } \begin{pmatrix} K_1 & K_2 & K_3 & K_4 \\ c_1^3 & c_2^3 & c_3^3 & c_1c_2c_3 \end{pmatrix} \right\rangle \;:\; (c_1c_2c_3)^\infty \right).$$

This toric ideal has ten minimal generators, namely the six minors plus

$$c_1^2K_2K_3 - c_2c_3K_4^2\,,\; c_2^2K_1K_3 - c_1c_3K_4^2\,,\; c_3^2K_1K_2 - c_1c_2K_4^2\,,\; \underline{K_1K_2K_3 - K_4^3}.$$

The variety $X_A = \mathcal{V}(I_A)$ is a threefold of degree 13 in $\mathbb{P}^6$. The underlying 4×7 matrix A is found with the integer linear algebra method in the proof of Theorem 8.10. The polytope $P = \mathrm{conv}(A)$ is a triangular prism. One triangle face is Δ_2 with vertices labeled by c_1, c_2, c_3. The other triangle face is $3\Delta_2$ with vertices K_1, K_2, K_3 and centroid K_4. The underlined cubic generates the *moduli ideal*, which identifies the *toric dynamical systems*.

The mathematical theory of chemical reaction network systems with mass action kinetics is an important domain of application for nonlinear algebra. For an introduction we refer to the text book by Dickenstein and Feliu [**17**]. The term “toric dynamical systems” was coined in the article

[**13**]. In the chemical literature, these are known as complex balancing mass action systems. The toric ideals above where introduced in [**13**, Section 2].

Gaussian MLE. Let $n = 5, p = 10$ and consider the integer matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}. \tag{8.4}$$

The columns of A are labeled $y_{01}, y_{02}, \ldots, y_{34}$. These are our coordinates for $\mathbb{P}^9$. The polytope $P = \mathrm{conv}(A)$ is the *second hypersimplex* of dimension 4. It has f-vector $(10, 30, 30, 10)$ but it is not self-dual. Its toric ideal I_A has 10 quadratic generators $y_{ij}y_{kl} - y_{ik}y_{jl}$, and X_A is a fourfold of degree 11 in $\mathbb{P}^9$.

This toric variety of the second hypersimplex of dimension m arises when studying Gaussian distributions on $\mathbb{R}^m$ with structured covariance matrix Σ. Consider the model given by prescribing all off-diagonal entries to be equal. Thus, for $m = 4$, we are interested in the linear space of symmetric matrices

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_0 & \sigma_0 & \sigma_0 \\ \sigma_0 & \sigma_2 & \sigma_0 & \sigma_0 \\ \sigma_0 & \sigma_0 & \sigma_3 & \sigma_0 \\ \sigma_0 & \sigma_0 & \sigma_0 & \sigma_4 \end{bmatrix}. \tag{8.5}$$

Given a sample covariance matrix S, one seeks to maximize the log-likelihood

$$\ell(\Sigma) \; = \; -\log\,\det(\Sigma) - \mathrm{trace}(S\Sigma^{-1}) \; = \; \log\,\det(K) - \mathrm{trace}(SK).$$

This function is convex in the *concentration matrix* $K = \Sigma^{-1}$. For that reason, we study the variety of matrices K whose inverse has structure (8.5). This is a non-linear projective variety of dimension m in $\mathbb{P}^{\binom{m}{2}}$. Defining polynomials are obtained by equating off-diagonal entries in the adjoint of K. These are complicated expressions with many terms of degree $m - 1$.

We find that this is a toric variety, after the linear change of coordinates

$$K = \begin{bmatrix} y_{01}{+}y_{12}{+}y_{13}{+}y_{14} & -y_{12} & -y_{13} & -y_{14} \\ -y_{12} & y_{02}{+}y_{12}{+}y_{23}{+}y_{24} & -y_{23} & -y_{24} \\ -y_{13} & -y_{23} & y_{03}{+}y_{13}{+}y_{23}{+}y_{34} & -y_{34} \\ -y_{14} & -y_{24} & -y_{34} & y_{04}{+}y_{14}{+}y_{24}{+}y_{34} \end{bmatrix}.$$

For any m, this is the reduced Laplacian matrix of the complete graph on $m+1$ nodes. It follows from [**56**, Theorem 1.2] that the variety of matrices K whose inverse is constant outside the diagonal equals the toric variety of the second hypersimplex. Using the coordinates y_{ij}, the inverse of the matrix Σ in (8.5) satisfies the ten quadratic binomials in I_A. The article [**56**] establishes this toric structure for a larger class of Gaussian models, one for each rooted tree, thus contributing to likelihood inference for such models.

Phylogenetics. Group-based models in phylogenetics are varieties that become toric after a linear change of coordinates. The nonlinear algebra of this transformation was pioneered in [**54**]. For the relevant background from molecular biology we refer to [**42**, Chapter 4]. The following case study is taken from [**54**, Example 3]. The *Jukes-Cantor model* for the claw tree $K_{1,3}$ is a model for three binary random variables. Its eight joint probabilities are

$$\begin{aligned} p_{000} &= \pi_0\alpha_0\beta_0\gamma_0 + \pi_1\alpha_1\beta_1\gamma_1, & p_{001} &= \pi_0\alpha_0\beta_0\gamma_1 + \pi_1\alpha_1\beta_1\gamma_0\,, \\ p_{010} &= \pi_0\alpha_0\beta_1\gamma_0 + \pi_1\alpha_1\beta_0\gamma_1, & p_{011} &= \pi_0\alpha_0\beta_1\gamma_1 + \pi_1\alpha_1\beta_0\gamma_0\,, \\ p_{100} &= \pi_0\alpha_1\beta_0\gamma_0 + \pi_1\alpha_0\beta_1\gamma_1, & p_{101} &= \pi_0\alpha_1\beta_0\gamma_1 + \pi_1\alpha_0\beta_1\gamma_0\,, \\ p_{110} &= \pi_0\alpha_1\beta_1\gamma_0 + \pi_1\alpha_0\beta_0\gamma_1, & p_{111} &= \pi_0\alpha_1\beta_1\gamma_1 + \pi_1\alpha_0\beta_0\gamma_0. \end{aligned}$$

Here π_0 and $\pi_1 = 1-\pi_0$ give the root distribution. The other model parameters $\alpha_0 = 1-\alpha_1$, $\beta_0 = 1-\beta_1$ and $\gamma_0 = 1-\gamma_1$ are the transition probabilities from the root to the three leaves. Since all parameters are nonnegative, the model is a 4-dimensional semialgebraic subset of the probability simplex Δ_7. The *Fourier transform* gives a change of coordinates in the parameter space,

$$\begin{gathered} \pi_0 = \tfrac{1}{2}(r_0+r_1),\ \pi_1 = \tfrac{1}{2}(r_0-r_1),\ \alpha_0 = \tfrac{1}{2}(a_0+a_1),\ \alpha_1 = \tfrac{1}{2}(a_0-a_1), \\ \beta_0 = \tfrac{1}{2}(b_0+b_1),\ \beta_1 = \tfrac{1}{2}(b_0-b_1),\ \gamma_0 = \tfrac{1}{2}(c_0+c_1),\ \gamma_1 = \tfrac{1}{2}(c_0-c_1), \end{gathered}$$

and it also gives a linear change of coordinates in the probability space:

$$p_{ijk} \quad = \quad \sum_{r=0}^{1}\sum_{s=0}^{1}\sum_{t=0}^{1}(-1)^{ir+js+kt}\cdot y_{rst}.$$

After these coordinate changes, the parametrization is now toric:

$$\begin{aligned} &y_{000} = r_0a_0b_0c_0, \quad y_{001} = r_1a_0b_0c_1, \quad y_{010} = r_1a_0b_1c_0, \quad y_{011} = r_0a_0b_1c_1, \\ &y_{100} = r_1a_1b_0c_0, \quad y_{101} = r_0a_1b_0c_1, \quad y_{110} = r_0a_1b_1c_0, \quad y_{111} = r_1a_1b_1c_1. \end{aligned}$$

This corresponds to a matrix $A \in \{0,1\}^{8\times 8}$ of rank 5. The toric ideal equals

$$I_A \;=\; \langle y_{001}y_{110} - y_{000}y_{111},\, y_{010}y_{101} - y_{000}y_{111},\, y_{100}y_{011} - y_{000}y_{111}\rangle.$$

Hence X_A is a complete intersection of codimension 3 and degree 8 in $\mathbb{P}^7$. The study of such phylogenetics models is an active area of research.

Paths and Signatures. Let $n = 6$, $p = 19$ and consider the monomial map

$$\begin{aligned} y_{ijk} &= a_ia_ja_k \quad &&\text{for} \quad 1 \le i \le j \le k \le 3, \\ z_{k;ij} &= a_kb_{ij} \quad &&\text{for} \quad k = 1,2,3 \text{ and } 1 \le i < j \le 3. \end{aligned}$$

This defines a toric variety X_A of dimension 5 and degree 24 in $\mathbb{P}^{18}$. The matrix $A \in \{0,1\}^{6\times 19}$ has rows indexed by $a_1, a_2, a_3, b_{12}, b_{13}, b_{23}$ and $19 = 10+9$ columns indexed by $y_{111}, y_{112}, \ldots, y_{333}$ and $z_{1;12}, z_{1;13}, \ldots, z_{3;23}$. The toric ideal I_A is generated by 81 binomial quadrics, namely the 2×2-minors of

$$\begin{pmatrix} y_{111} & y_{112} & y_{113} & y_{122} & y_{123} & y_{133} & z_{1;12} & z_{1;13} & z_{1;23} \\ y_{112} & y_{122} & y_{123} & y_{222} & y_{223} & y_{233} & z_{2;12} & z_{2;13} & z_{2;23} \\ y_{113} & y_{123} & y_{133} & y_{223} & y_{233} & y_{333} & z_{3;12} & z_{3;13} & z_{3;23} \end{pmatrix}. \tag{8.6}$$

Let $\tilde{X}_A$ be the join of X_A with $\mathbb{P}^7$. This is a 13-dimensional toric variety of degree 24 in $\mathbb{P}^{26}$. It is defined by the same ideal I_A but now in 27 variables. We replace these by the entries of a $3 \times 3 \times 3$ tensor $\sigma = (\sigma_{ijk})$ as follows:

$$\begin{aligned} y_{ijk} &= \sigma_{kij} + \sigma_{ikj} + \sigma_{ijk} + \sigma_{kji} + \sigma_{jki} + \sigma_{jik}, \\ z_{k;ij} &= \tfrac{1}{2}(\sigma_{kij} + \sigma_{ikj} + \sigma_{ijk}) - \tfrac{1}{2}(\sigma_{kji} + \sigma_{jki} + \sigma_{jik}). \end{aligned} \tag{8.7}$$

The resulting variety $\mathcal{U}_{3,3}$ is the universal variety of third-order signature tensors of arbitrary paths in $\mathbb{R}^3$. Such tensors play an important role in *stochastic analysis*, especially in the Hairer-Lyons theory of rough paths.

A natural generalization of the universal variety is the *rough Veronese variety* which was studied and shown to be toric by Colmenarejo et al. in [**9**]. This variety is a variant of the classical Veronese variety, but adapted to the study of rough paths. For an introduction to this theory we see [**9**, Section 1] and the references therein. Returning to the example above, in the notation of [**9**, Section 2], we have $\mathcal{U}_{3,3} = \mathcal{R}_{3,3,3} = \tilde{X}_A \subset \mathbb{P}^{26}$ and $\mathcal{R}_{3,3,2} = X_A \subset \mathbb{P}^{18}$. The equations defining these varieties are obtained by substituting (8.7) into (8.6). This 3×10 matrix has rank ≤ 1 for the signatures of all paths in $\mathbb{R}^3$.

Exercises

(1) Prove that every character χ of the torus $T = (K^*)^n$ is given by a Laurent monomial $\mathbf{x}^\mathbf{b} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, for some integer vector $\mathbf{b}$ in $\mathbb{Z}^n$.

(2) Show that every polynomial in the ideal I_A of an affine toric variety is a K-linear combination of binomials. This gives item (3) in Lemma 8.8.

(3) Describe the ideal of the Segre variety $\mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{a_s-1}$ inside $\mathbb{P}^{a_1 \cdots a_s - 1}$. What is the degree of this toric variety? Describe its lattice polytope P.

(4) There is a natural bijection between (convex, rational, polyhedral) cones in $\mathbb{R}^d$ and finitely generated saturated monoids in $\mathbb{Z}^d$. Prove this fact.

(5) Determine the toric ideal I_A and the toric variety X_A for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

(6) Show that the 3-dimensional lattice polytope $P = \mathrm{conv}(A)$ is not normal. Draw this polytope and determine its f-vector.

(7) Prove that every 2-dimensional lattice polytope is normal.

(8) Prove the following fact which generalizes the previous exercise: for any k-dimensional lattice polytope P, the scaled polytope $(k-1)P$ is normal.

(9) Determine the number of lattice points in kP where P is the polytope in Exercises (5)-(6). Show that this number is a cubic polynomial in k.

(10) Prove the following theorem due to Mumford in the case of toric varieties; Let X be a projective toric variety. For r large enough the r-th Veronese reembeding $v_r(X)$ of X is defined by quadratic equations.

(11) Compute an explicit Gröbner basis for the toric ideal I_A, where A is the matrix in Example 8.4. Is your initial monomial ideal $\mathrm{in}(I_A)$ radical?

(12) Compute the inverse of the matrix Σ in 8.5, and verify that its entries satisfy the ten quadratic binomials given by the second hypersimplex.

(13) Let A be the 3×7-matrix in Example 8.9. Can you give a formula for the inverse moment map μ_A^{-1}? Is there an expression in radicals?

(14) (a) Compute the number of points of a projective toric variety X_P over a finite field, in terms of the f-vector of the associated polytope P.

(b) * Assuming that X_P is smooth and $K = \mathbb{C}$, use the Weil conjectures (which are theorems, thanks to Grothendieck and Deligne), to give a formula for Betti numbers of X_P, again in terms of the f-vector.

(15) Give an example of a toric threefold X_A in $\mathbb{P}^6$ that has degree 11. Draw the polytope $P = \mathrm{conv}(A)$. Can you arrange for X_A to be smooth?

Chapter 9

Tensors

Tensors are ubiquitous in many different branches of modern mathematics. They are higher dimensional analogs of matrices. Just as matrices are basic objects in *linear algebra*, tensors are fundamental for *nonlinear algebra*. One reason they appear so late in this book is that we already saw them in disguise: homogeneous polynomials are symmetric tensors. In this chapter we show that basic attributes of matrices, like eigenvectors and rank, can be defined also for tensors. HowAnnever, their behavior is far more interesting now. We also discuss applications of tensors, with focus on a a central open algorithmic problem: how fast can one multiply two matrices? As always, linear algebra is our door to nonlinear algebra. Further, the new nonlinear tools will be applied to revisit fundamental questions in linear algebra.

9.1. Eigenvectors

In this section we extend the familiar concepts of eigenvectors, rank and singular values from matrices to the setting of tensors. We start by reviewing some basics of linear algebra, beginning with the study of symmetric matrices. Recall that symmetric matrices uniquely represent quadratic forms.

For instance, consider the following quadratic form in three variables:

$$Q \;=\; 2x^2 + 7y^2 + 23z^2 + 6xy + 10xz + 22yz. \tag{9.1}$$

This quadratic form is represented by a symmetric 3 × 3-matrix as follows:

$$Q \;=\; \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{9.2}$$

The gradient of the quadratic form Q is the vector of its partial derivatives. Thus, the gradient is a vector of linear forms. It defines a linear map from K^3 to itself. Up to multiplication by 2, this is the linear map one usually associates with a square matrix. For the quadratic form in (9.1) we have

$$\nabla Q \;=\; \begin{pmatrix} \partial Q/\partial x \\ \partial Q/\partial y \\ \partial Q/\partial z \end{pmatrix} \;=\; 2\cdot \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In this section, the field K is usually $\mathbb{R}$ or $\mathbb{C}$. We call $\mathbf{v} \in \mathbb{R}^n$ an *eigenvector* of Q if $\mathbf{v}$ is mapped to a scalar multiple of $\mathbf{v}$ by the gradient map:

$$(\nabla Q)(\mathbf{v}) \;=\; \lambda \cdot \mathbf{v} \quad \text{for some } \lambda \in K.$$

Just like in the earlier chapters, it is convenient to replace the affine space K^n with the projective space $\mathbb{P}^{n-1}$. Thus, two nonzero vectors are identified if they are parallel. From Q we obtain a rational self-map of projective space:

$$\nabla Q \,:\, \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}. \tag{9.3}$$

The dashed arrow means that this is a rational map. If Q is rank-deficient then the linear map has a kernel. These are the points where the gradient ∇Q vanishes. They are the *base points* of the map (9.3). If Q has full rank then ∇Q is a regular map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$, i.e. it is defined on all of $\mathbb{P}^{n-1}$. We conclude our discussion with the following remark on the gradient map:

Remark 9.1. The eigenvectors of Q are the fixed points ($\lambda \neq 0$) and base points ($\lambda = 0$) of the gradient map ∇Q in (9.3). These points $\mathbf{v}$ live in $\mathbb{P}^{n-1}$.

Symmetric $n \times n$ matrices often appear in statistics. Consider n real-valued random variables $X_1, \ldots, X_n$. Their *covariance matrix* is the matrix Σ whose (i,j) entry is $\mathrm{cov}[X_i, X_j] = E[(X_i - E[X_i])(X_j - E[X_j])]$. We note that Σ is positive semidefinite, i.e. all its eigenvalues are nonnegative.

A $n \times n$-matrix usually has n linearly independent *eigenvectors*, provided the underlying field K is algebraically closed. If the matrix is real and symmetric, then its eigenvectors have real coordinates and are *orthogonal.* For a rectangular matrix, one considers pairs of *singular vectors*, defined below; one on the left and one on the right. The number of these singular vector pairs is equal to the smaller of the two matrix dimensions.

Eigenvectors and singular vectors are familiar from linear algebra, where they are taught in concert with *eigenvalues* and *singular values.* Numerical linear algebra is the foundation of applied mathematics and scientific computing. Specifically, the concept of eigenvectors, and numerical algorithms for computing them, became a key technology during the 20th century.

Singular vectors are associated to rectangular matrices. We review their definition through the lens of Remark 9.1. We begin with the observation that each rectangular matrix represents a bilinear form, e.g.

$$B \;=\; 2ux{+}3uy{+}5uz{+}3vx{+}7vy{+}11vz \;=\; \begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{9.4}$$

The gradient of the bilinear form defines an endomorphism on the direct sum of the row space and the column space. This fuses left multiplication and right multiplication by our matrix into a single linear map.

For the example (9.4), the gradient is the following vector of linear forms

$$\nabla B = \Big(\,(\frac{\partial B}{\partial u}, \frac{\partial B}{\partial v})\,,\,(\frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial B}{\partial z})\Big). \tag{9.5}$$

The associated endomorphism has the form $\nabla B \,:\, K^3 \oplus K^2 \to K^2 \oplus K^3$. This gradient map takes the pair $\big((x,y,z),(u,v)\big)$ to the pair in (9.5), i.e. to

$$((2x+3y+5z, 3x+7y+11z), (2u+3v, 3u+7v, 5u+11v))\,.$$

More generally, let B be an $m\times n$-matrix over K. Consider the equations

$$B\mathbf{x} = \lambda\mathbf{y} \quad \text{und} \quad B^t\mathbf{y} = \lambda\mathbf{x}, \tag{9.6}$$

where λ is a scalar, $\mathbf{x}$ is a vector in $\mathbb{R}^n$, and $\mathbf{y}$ is a vector in $\mathbb{R}^m$. These are our unknowns. Given a solution to (9.6), we see that $\mathbf{x}$ is an eigenvector of B^tB, $\mathbf{y}$ is an eigenvector of BB^t, and λ^2 is a common eigenvalue of these two symmetric matrices. Assuming $K = \mathbb{R}$, its nonnegative square root $\lambda \geq 0$ is a *singular value* of B. Associated to λ are the *right singular vector* $\mathbf{x}$ and the *left singular vector* $\mathbf{y}$. In analogy to Remark 9.1, the process of solving (9.6) has the following dynamical interpretation:

Remark 9.2. The singular vector pairs $(\mathbf{x},\mathbf{y})$ of a rectangular matrix B of size $m \times n$ are the fixed points of the gradient map ∇B of the associated bilinear form. This is now a self-map on a product of projective spaces:

$$\nabla B \,:\, \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \;\longrightarrow\; \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$$
$$(\mathbf{x},\mathbf{y}) \;\mapsto\; \Big(\,(\frac{\partial B}{\partial x_1}, \ldots, \frac{\partial B}{\partial x_n})\,,\,(\frac{\partial B}{\partial y_1}, \ldots \frac{\partial B}{\partial y_m})\Big).$$

We summarize our brief review of linear algebra in the following points:

- Symmetric matrices Q represent quadratic forms.
- Rectangular matrices B represent bilinear forms.
- Their gradients ∇Q and ∇B specify the linear maps one usually identifies with the matrices Q and B.
- Fixed points of these maps are *eigenvectors* and *singular vectors.*

- These fixed points are computed via orthogonal decompositions:
$$Q \;=\; O\cdot \mathrm{diag}\cdot O^t \quad \text{and} \quad B \;=\; O_1\cdot \mathrm{diag}\cdot O_2.$$
Here O, O_1 and O_2 are orthogonal matrices. The formulas above are known as the *spectral decomposition* and the *singular value decomposition*. These objects are usually defined for $K=\mathbb{R}$.

In the age of Big Data, the role of matrices is increasingly played by *tensors*, that is, multidimensional arrays of numbers. Principal component analysis tells us that the eigenvectors of a covariance matrix $Q = BB^t$ give directions in which the data B is most spread. One hopes to identify similar features for tensor data. This has encouraged engineers and scientists to spice up their linear algebra tool box with a pinch of algebraic geometry.

The spectral theory of tensors is the theme of the following discussion. This theory was pioneered around 2005 by Lek-Heng Lim and Liqun Qi. We refer to the textbook [**44**] for background and context. Our aim is to generalize familiar notions, such as rank, eigenvectors and singular vectors, from matrices to tensors. Specifically, we address the following two questions. The answers to these two questions are provided in Examples 9.7 and 9.13.

Question 9.3. *How many eigenvectors does a* $3\times 3\times 3$*-tensor have?*

Question 9.4. *How many singular vector triples does a* $3{\times}3{\times}3$*-tensor have?*

A *tensor* is a d-dimensional array $T = (t_{i_1 i_2 \cdots i_d})$. Here the entries $t_{i_1 i_2\cdots i_d}$ are elements in the ground field K. The set of all tensors of format $n_1{\times}n_2{\times}\cdots{\times}n_d$ form a vector space of dimension $n_1 n_2\cdots n_d$ over K. For $d=1,2$ we get vectors and matrices. A tensor has *rank 1* if it is the outer product of d vectors, written $T = \mathbf{u}\otimes\mathbf{v}\otimes\cdots\otimes\mathbf{w}$, or, in coordinates,
$$t_{i_1 i_2\cdots i_d} \;=\; u_{i_1} v_{i_2}\cdots w_{i_d}.$$
The problem of *tensor decomposition* is the following. We wish to express a given tensor T as a sum of rank 1 tensors, using as few summands as possible. That minimal number of rank 1 summands needed to represent T is the *rank* of T. We will discuss this topic in detail in the next section.

An $n{\times}n{\times}\cdots{\times}n$-tensor $T = (t_{i_1 i_2\cdots i_d})$ is called *symmetric* if it is unchanged under permuting the indices. The space $\mathrm{Sym}_d(\mathbb{R}^n)$ of such symmetric tensors has dimension $\binom{n+d-1}{d}$. It is identified with the space of homogeneous polynomials of degree d in n variables, written as

$$T \;=\; \sum_{i_1,\ldots,i_d=1}^{n} t_{i_1 i_2\cdots i_d}\cdot x_{i_1} x_{i_2}\cdots x_{i_d}. \tag{9.7}$$

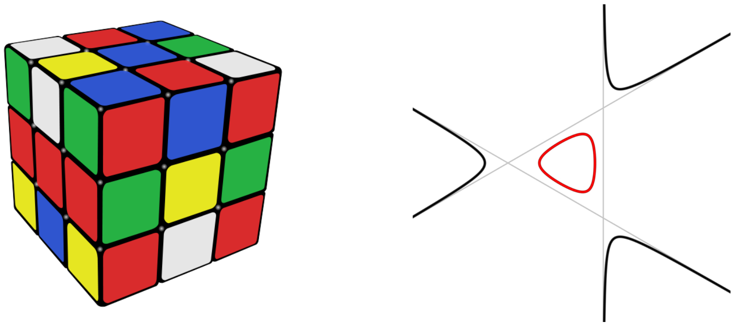


Figure 1. A symmetric 3×3×3 tensor represents a cubic curve in $\mathbb{P}^2$.

Example 9.5. A tensor T of format $3{\times}3{\times}3$ has 27 entries. If T is a symmetric tensor then it has at most ten distinct entries, one for each coefficient of the associated cubic polynomial in three variables. This polynomial defines a cubic curve in the projective plane $\mathbb{P}^2$, as indicated in Figure 1.

Symmetric tensor decomposition writes T as sum of powers of linear forms:

$$T \;=\; \sum_{j=1}^{r} \lambda_j \mathbf{v}_j^{\otimes d} \;=\; \sum_{j=1}^{r} \lambda_j (v_{1j}x_1 + v_{2j}x_2 + \cdots + v_{nj}x_n)^d. \tag{9.8}$$

As before, the gradient of T defines a linear map $\nabla T : K^n \to K^n$. A vector $\mathbf{v} \in K^n$ is an *eigenvector* of T if $(\nabla T)(\mathbf{v}) = \lambda \cdot \mathbf{v}$ for some $\lambda \in K$.

Eigenvectors of tensors arise naturally in optimization. Consider the problem of maximizing a real homogeneous polynomial T over the unit sphere in $\mathbb{R}^n$. If λ denotes a Lagrange multiplier, then one sees that the eigenvectors of T are the critical points of this optimization problem. One can check the values of T at these points to find global maxima and minima.

We find it convenient to replace K^n by the projective space $\mathbb{P}^{n-1}$. The gradient map is then a rational map from this projective space to itself:

$$\nabla T \;:\; \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}.$$

The eigenvectors of T are *fixed points* ($\lambda \neq 0$) and *base points* ($\lambda = 0$) of ∇T. Thus the spectral theory of tensors is closely related to the study of dynamical systems on $\mathbb{P}^{n-1}$. The matrix case ($d = 2$) appeared in (9.3). By the Spectral Theorem, a real quadratic form T has a real decomposition (9.8) with $d = 2$. Here r is the rank, the λ_j are the eigenvalues of T, and the eigenvectors $\mathbf{v}_j = (v_{1j}, v_{2j}, \ldots, v_{nj})$ are orthonormal. We can compute this by *power iteration*, namely, by applying ∇T until a fixed point is reached.

For $d \geq 3$, one can still use the power iteration to compute eigenvectors of T. However, the eigenvectors are usually not the vectors $\mathbf{v}_i$ in the low rank decomposition (9.8). One exception arises when the symmetric tensor is *odeco*, or orthogonally decomposable. This means that T has the form (9.8), where $r = n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis of $\mathbb{R}^n$. These basis vectors are the attractors of the dynamical system ∇T, provided $\lambda_j > 0$.

The following gives a count of the eigenvectors of a symmetric tensor.

Theorem 9.6 (Cartwright-Sturmfels [**8**])**.** *If K is algebraically closed, then the number of eigenvectors of a general tensor $T \in \mathrm{Sym}_d(\mathbb{K}^n)$ equals*

$$\frac{(d-1)^n - 1}{d-2} \quad = \quad \sum_{i=0}^{n-1} (d-1)^i.$$

Example 9.7 ($n = d = 3$)**.** The Fermat cubic $T = x^3 + y^3 + z^3$ is an odeco tensor. Its gradient map is the regular map that squares each coordinate: $\nabla T : \mathbb{P}^2 \to \mathbb{P}^2$, $(x : y : z) \mapsto (x^2 : y^2 : z^2)$. This dynamical system has $7 = 1 + 2 + 2^2$ fixed points, of which only the first three are attractors:

$$(1:0:0), (0:1:0), (0:0:1),\ (1:1:0), (1:0:1), (0:1:1),\ (1:1:1).$$

We conclude that T has seven eigenvectors, as predicted by Theorem 9.6.

It is known that all eigenvectors can be real for suitable tensors. This was proved by Khozhasov [**30**] using the theory of *harmonic polynomials*. For $n = 3$, this can be seen as follows. Let T be a product of linear forms in x, y, z, defining d lines in $\mathbb{P}^2_{\mathbb{R}}$. The $\binom{d}{2}$ vertices of the line arrangement are base points of ∇T, and each of the $\binom{d}{2} + 1$ regions contain one fixed point. This accounts for all $1 + (d{-}1) + (d{-}1)^2$ eigenvectors, which are hence real.

Example 9.8. Let $d = 4$ and fix the quartic $T = xyz(x + y + z)$, which is a symmetric $3{\times}3{\times}3{\times}3$ tensor. Its curve in $\mathbb{P}^2$ is an arrangement of four lines. All $13 = 6 + 7$ eigenvectors of T are real. The 6 vertices of the arrangement are the base points of ∇T. Each of the 7 regions contains one fixed point.

For special tensors T, two of the eigenvectors in Theorem 9.6 may coincide. This corresponds to vanishing of the *eigendiscriminant*, a big polynomial in the $t_{i_1 i_2 \cdots i_d}$. In the matrix case ($d = 2$), this is the discriminant of the characteristic polynomial of an $n{\times}n$-matrix [**53**, §7.5]. For $3{\times}3{\times}3$ tensors, the eigendiscriminant is a polynomial of degree 24 in 27 unknowns.

Theorem 9.9 (Abo-Seigal-Sturmfels [**2**])**.** *The eigendiscriminant is an irreducible homogeneous polynomial of degree $n(n-1)(d-1)^{n-1}$ in the $t_{i_1 i_2 \cdots i_d}$.*

Example 9.10 ($n = 2$)**.** The eigendiscriminant of a binary form $T(x, y)$ of degree d is the discriminant of $x\frac{\partial T}{\partial y} - y\frac{\partial T}{\partial x}$, so it has degree $2d - 2$ in T.

Singular value decomposition is a central notion in linear algebra and its applications. Remark 9.2 casts the singular vector pairs of a matrix as fixed points of a self-map of a product of two projective spaces. Consider now a

d-dimensional tensor T in $\mathbb{R}^{n_1\times\cdots\times n_d}$. It corresponds to a multilinear form. The *singular vector tuples* of T are the fixed points of the gradient map

$$\nabla T \,:\, \mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_d-1} \dashrightarrow \mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_d-1}.$$

Example 9.11. The trilinear form $T = x_1y_1z_1 + x_2y_2z_2$ is interpreted as a $2\times2\times2$ tensor. The gradient ∇T of this trilinear form is the rational map

$$\begin{array}{ccc} \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 & \dashrightarrow & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1, \\ \big((x_1:x_2),(y_1:y_2),(z_1:z_2)\big) & \mapsto & \big((y_1z_1:y_2z_2),(x_1z_1:x_2z_2),(x_1y_1:x_2y_2)\big). \end{array}$$

This map has six fixed points, namely $\big((1{:}0),(1{:}0),(1{:}0)\big)$, $\big((0{:}1),(0{:}1),(0{:}1)\big)$, $\big((1{:}1),(1{:}1),(1{:}1)\big)$, $\big((1{:}1),(1{:}{-1}),(1{:}{-1})\big)$, $\big((1{:}{-1}),(1{:}1),(1{:}{-1})\big)$, and $\big((1{:}{-1}),(1{:}{-1}),(1{:}1)\big)$. These are the singular vector triples of the tensor T.

Here is a formula for the expected number of singular vector tuples.

Theorem 9.12 (Friedland and Ottaviani [**21**])**.** *For a general $n_1\times\cdots\times n_d$-tensor T over an algebraically closed field K, the number of singular vector tuples is the coefficient of the monomial $z_1^{n_1-1}\cdots z_d^{n_d-1}$ in the polynomial*

$$\prod_{i=1}^{d}\frac{(\widehat{z_i})^{n_i}-z_i^{n_i}}{\widehat{z_i}-z_i} \quad \textit{where} \quad \widehat{z_i} = z_1+\cdots+z_{i-1}+z_{i+1}+\cdots+z_d.$$

We finish our study of spectral theory of tensors by answering Question 2.

Example 9.13. Let $d{=}n_1{=}n_2{=}n_3{=}3$. The polynomial in Theorem 9.12 is

$$(\widehat{z_1}^2{+}\widehat{z_1}z_1{+}z_1^2)(\widehat{z_2}^2{+}\widehat{z_2}z_2{+}z_2^2)(\widehat{z_3}^2{+}\widehat{z_3}z_3{+}z_3^2) \;=\; \cdots + \mathbf{37}z_1^2z_2^2z_3^2 + \cdots$$

Therefore, a general $3\times3\times3$-tensor has exactly 37 triples of singular vectors. Likewise, a general $3\times3\times3\times3$-tensor has 997 quadruples of singular vectors.

9.2. Tensor Rank

There are many ways to define the rank of an $a\times b$ matrix M over a field K:

(1) the smallest integer r such that all $(r+1)\times(r+1)$ minors vanish,

(2) the dimension of the image of the induced linear map $K^a\to K^b$,

(3) the dimension of the image of the induced linear map $K^b\to K^a$,

(4) the smallest r such that $M = UW$ where $U\in K^{a\times r}$ and $W\in K^{r\times b}$.

The first point implies that matrices of rank at most r form a variety. The last point implies that a matrix of rank r is a sum of r matrices of rank one. This is also true for symmetric matrices: a symmetric matrix of rank r is a sum of r symmetric rank one matrices. Another fact is that a real matrix of rank r has also rank r when viewed over $\mathbb{C}$. This seems obvious, but a priori, it is not clear why there is no shorter decomposition into rank one matrices with entries in $\mathbb{C}$. Our aim is to study these issues for arbitrary tensors.

In this section we work in a tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_d$ of finite-dimensional vector spaces V_i over a field K. We sometimes identify $V_i \simeq K^{n_i}$, and thus $V_1 \otimes V_2 \otimes \cdots \otimes V_d \simeq K^{n_1 \times n_2 \times \cdots \times n_d}$. A tensor T in this space has rank one if it is the outer product of d vectors, i.e. $T = \mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{w}$. In coordinates, this means that the entries of T factor as

$$t_{i_1 i_2 \cdots i_d} \;=\; u_{i_1} v_{i_2} \cdots w_{i_d}.$$

Tensors of rank at most one form an affine variety. It is the affine cone over the Segre variety $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1}$ in $\mathbb{P}^{n_1 \cdots n_d - 1}$. In fact, from Chapter 2 and 8 we know the equations of this variety. They are binomial quadrics, namely the 2×2 minors of all flattenings of T.

The flattenings have the following invariant description. Let I be any subset of $[d] = \{1, 2, \ldots, d\}$. The corresponding flattening is the linear map

$$K^{\prod_{i \in I} n_i} \;=\; \bigotimes_{i \in I} V_i^* \;\longrightarrow\; \bigotimes_{i \in [d] \setminus I} V_i \;=\; K^{\prod_{i \in [d] \setminus I} n_i} \tag{9.9}$$

that is defined by T. Thus a tensor T has rank one if and only if all $2^d - 2$ flattenings of T are matrices of rank one. There is a similar result for tensors of rank two, due to Landsberg and Manivel, but for all higher ranks only one direction is true: the rank of T is bounded below by that of any flattening.

Example 9.14. A tensor $T = (t_{ijk}) \in V_1 \otimes V_2 \otimes V_3$ induces the linear map

$$V_1^* \to V_2 \otimes V_3\,, \quad \mathbf{e}_i^* \mapsto (t_{ijk})_{j,k} \;=\; \sum_{j,k} t_{ijk} \cdot \mathbf{f}_j \otimes \mathbf{g}_k, \tag{9.10}$$

where $(\mathbf{e}_i), (\mathbf{f}_j), (\mathbf{g}_k)$ are respectively bases of V_1, V_2, V_3. This is the case $I = \{1\}$ in (9.9). We think of (9.10) as an $n_1 \times n_2 n_3$ matrix. The transpose of this matrix is the $n_2 n_3 \times n_1$ matrix that corresponds to $I = \{2, 3\}$ in (9.9). Thus, a three-way tensor has three distinct flattenings, up to transposing.

We conclude that rank one tensors behave in a very nice way. However, arbitrary tensors exhibit rather strange properties. Recall that the rank of a tensor T is the minimal r such that T is the sum of r rank one tensors. For instance, the three-way tensors of rank ≤ 2 are the tensors of the form

$$T \quad = \quad \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} + \mathbf{d} \otimes \mathbf{e} \otimes \mathbf{f}. \tag{9.11}$$

We shall see that the set of these tensors is not Zariski closed in $K^{n_1 \times n_2 \times n_3}$.

Example 9.15. Let $d = 3$ and $V_1 = V_2 = V_3 = \mathbb{C}^2$ with basis $\{\mathbf{e}_0, \mathbf{e}_1\}$. The following $2 \times 2 \times 2$ tensor is known in quantum physics as the *W-state*:

$$W \;=\; \mathbf{e}_0 \otimes \mathbf{e}_0 \otimes \mathbf{e}_1 \;+\; \mathbf{e}_0 \otimes \mathbf{e}_1 \otimes \mathbf{e}_0 \;+\; \mathbf{e}_1 \otimes \mathbf{e}_0 \otimes \mathbf{e}_0. \tag{9.12}$$

This representation shows that W has rank at most three. In fact, $\operatorname{rk} W = 3$, as the reader is asked to prove in Exercise 9. To do so, equate W with T

in (9.11). This gives an inconsistent system of 8 cubic equations in the 12 unknown coordinates $a_0, a_1, b_0, \ldots, f_1$ of the vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ in (9.11).

However, there exist rank two tensors arbitrarily close to W. We have

$$\frac{1}{\epsilon}((\mathbf{e}_0 + \epsilon\mathbf{e}_1) \otimes (\mathbf{e}_0 + \epsilon\mathbf{e}_1) \otimes (\mathbf{e}_0 + \epsilon\mathbf{e}_1) - \mathbf{e}_0 \otimes \mathbf{e}_0 \otimes \mathbf{e}_0) \quad =$$

$$W \,+\, \epsilon(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_0 + \mathbf{e}_1 \otimes \mathbf{e}_0 \otimes \mathbf{e}_1 + \mathbf{e}_0 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1) \,+\, \epsilon^2\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1.$$

This is an identity for all $\epsilon \neq 0$. In particular, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big((\mathbf{e}_0 + \epsilon\mathbf{e}_1) \otimes (\mathbf{e}_0 + \epsilon\mathbf{e}_1) \otimes (\mathbf{e}_0 + \epsilon\mathbf{e}_1) - \mathbf{e}_0 \otimes \mathbf{e}_0 \otimes \mathbf{e}_0\big) \;=\; W.$$

We conclude that the W-state is a tensor of rank three, but it can be approximated with arbitrary precision by a sequence of tensors of rank two.

Definition 9.16. The *border rank* $\mathrm{brk}(T)$ of a complex tensor T is the smallest r such that T lies in the closure of the set of tensors of rank r.

The notion of border rank requires a topology on the space of tensors. The geometric locus of tensors of border rank $\leq r$ is the closure of the locus of tensors of rank $\leq r$. Over the complex numbers, by Chevalley's Theorem 4.18, it does not matter if we take Euclidean or Zariski topology: the closures coincide. However, the situation is different over the real numbers. To prove this, we shall use the hyperdeterminant, denoted Det, from Example 4.10.

Lemma 9.17. *The hyperdeterminant of the rank two tensor in (9.11) equals*

$$\mathrm{Det}(T) \quad = \quad (a_0d_1 - a_1d_0)^2(b_0e_1 - b_1e_0)^2(c_0f_1 - c_1f_0)^2. \tag{9.13}$$

Proof. We have an explicit formula for $\mathrm{Det}(T)$ is a homogeneneous polynomial of degree 4 in the eight tensor entries t_{ijk}. If we substitute $t_{ijk} = a_ib_jc_k + d_ie_jf_k$ and factor, then we obtain the above product of degree 12. □

Corollary 9.18. *Let* $T \in \mathbb{R}^{2\times 2\times 2}$. *If* T *has real rank* ≤ 2 *then* $\mathrm{Det}(T) \geq 0$.

Proof. Our hypothesis says that T has a representation (9.11) over $\mathbb{R}$. The expression (9.13) for $\mathrm{Det}(T)$ is a square in $\mathbb{R}$. It is hence nonnegative. □

Example 9.19. Let $i = \sqrt{-1}$. The following tensor has rank two in $\mathbb{C}^{2\times 2\times 2}$:

$$T \;=\; \frac{1}{2}\big((\mathbf{e}_1 + i\mathbf{e}_0)^{\otimes 3} \,+\, (\mathbf{e}_1 - i\mathbf{e}_0)^{\otimes 3}\big) \;=\; \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \,-\, W.$$

Note that this tensor is real. By substituting its coefficients into the formula for the hyperdeterminant in Example 4.10, we find that $\mathrm{Det}(T) \;=\; -4 \;<\; 0$. Corollary 9.18 implies that the real rank of T is ≥ 3. Thus the tensor T has the property that its complex rank is strictly smaller than its real rank.

In Exercise 10, the reader is asked to prove that the set of rank two tensors is Zariski dense in the space of $2 \times 2 \times 2$ tensors. This holds for any infinite field K. If $K = \mathbb{C}$ then they are also dense in the Euclidean topology. In fact, a tensor T has complex rank ≤ 2 if and only if $\mathrm{Det}(T) \neq 0$. Note that the W-state has rank 3 and it satisfies $\mathrm{Det}(W) = 0$. If $K = \mathbb{R}$ then we must distinguish the two cases $\mathrm{Det}(T) > 0$ and $\mathrm{Det}(T) < 0$. In the former case, T has real rank two. In the latter case, T has real rank three.

Our discussion has the following interpretation in projective geometry. Tensors of rank one form the Segre threefold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^7$. Exercise 10 says that the secant variety of X fills $\mathbb{P}^7$. However, the tangential variety of X, which the union of all tangent spaces to X, has dimension six. It is the hypersurface $\{\mathrm{Det}(T) = 0\}$ in $\mathbb{P}^7$. The W-state is a point on that hypersurface. The line $\{\lambda \mathbf{e}_0 \otimes \mathbf{e}_0 \otimes \mathbf{e}_0 + W \,:\, \lambda \in K\}$ crosses this hypersurface transversally. If $K = \mathbb{R}$ then the real rank depends only on the sign of λ.

To conclude, unlike in the case of matrices or rank one tensors, we have:

- Tensors of rank at most r may not form a closed set.
- A real tensor can have smaller rank when seen as a complex tensor.
- Real tensors of bounded real border rank form semialgebraic sets.

We have described rank one tensors as the Segre product of projective spaces. It is natural to ask for a geometric description of tensors of rank at most r.

Definition 9.20 (Secant Variety)**.** Let X be any projective variety in $\mathbb{P}^n$. The *k-th secant variety* of X is the closure of the set of k-secant planes to X:

$$\sigma_k(X) \quad := \quad \overline{\bigcup_{p_1,\ldots,p_k \in X} \langle p_1, \ldots, p_k \rangle}. \tag{9.14}$$

Note that $X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_{\dim\langle X \rangle}(X) = \langle X \rangle$. These containments are strict until $\sigma_r(X)$ equals the linear span $\langle X \rangle$ of the variety X.

If X is the Segre variety, then the union in (9.14) is the set of tensors of rank $\leq r$. Its closure $\sigma_r(X)$ is the set of tensors of border rank $\leq r$. It is a major open problem to determine the ideal of $\sigma_r(X)$. This would provide a test for a tensor to have border rank r. The simplest equations for $\sigma_r(X)$ are the $(r+1) \times (r+1)$ minors of the various flattenings, as in (9.9). These have degree $r+1$. No polynomials of degree $\leq r$ vanish on $\sigma_r(X)$.

If X is the Veronese variety then we obtain the notion of *symmetric rank.* A symmetric tensor T is in X if the following equivalent conditions hold:

(1) The rank of T as a tensor is equal to one.

(2) $T = \mathbf{v} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{v}$ for some vector $\mathbf{v}$.

(3) T, represented as a polynomial, is a power of a linear form.

The symmetric rank of T is the smallest r such that $T \in \sigma_r(X)$. The rank of T is a lower bound for the symmetric rank of T, and ditto for border rank.

It was a longstanding question, known as Comon's Conjecture, whether the rank of a symmetric tensor is always equal to its symmetric rank. It turns out that the answer is no. A counterexample was constructed by Shitov in [**47**]. The border rank analogue of Comon's Conjecture remains open.

It is easy to prove that general tensors have high rank and border rank. But it is extremely hard to find explicit examples. In particular, it is not known how to provide examples of complex $n \times n \times n$ tensors T with either

- rank greater than $3n$, or
- border rank greater than $2n$.

By Exercise 12, a general tensor in $K^{n\times n\times n}$ has border rank quadratic in n. We offer a case study on tensor ranks in the case visualized in Figure 1.

Example 9.21 ($3 \times 3 \times 3$ tensors)**.** Fix an algebraically closed field K. Tensors of format $3 \times 3 \times 3$ are points in the projective space $\mathbb{P}^{26}$. The 6-dimensional Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ consists of all tensors of rank 1. Tensors of rank 2 form the variety $\sigma_2(X)$, which has dimension 13 and degree 783. Its ideal is generated by the 3×3-minors of the three flattenings. These flattenings are 3×9-matrices, like $\big[A\,|\,B\,|\,C\,\big]$ where $A = (t_{ij1})$, $B = (t_{ij2})$, $C = (t_{ij3})$ are the 3×3 matrices that are obtained as the slices in our Rubik's cube. These 3×3-minors span a space of cubics that has dimension 222.

The variety $\sigma_3(X)$ of rank 3 tensors has dimension 20. Its ideal is generated by a collection of quartic polynomials, namely the entries of the 3×3-matrices $A\cdot \mathrm{adj}(B)\cdot C - C\cdot \mathrm{adj}(B)\cdot A$, where we allow all possible ways of slicing the tensor. Finally, there is the variety $\sigma_4(X)$ of rank 4 tensors. This is a hypersurface of degree 9 in $\mathbb{P}^{26}$. Its defining polynomial is known as the *Strassen invariant.* The Strassen invariant can be computed as

$$\det(B)^2 \cdot \det\big(A \cdot B^{-1}C \,-\, C \cdot B^{-1}A\big).$$

The expression has 9216 terms and it is independent of the choice of slicing. The fifth secant variety $\sigma_5(X)$ is equal to $\mathbb{P}^{26}$. In other words, the set of tensors of rank ≤ 5 is dense in the space of all $3 \times 3 \times 3$ tensors.

We now restrict the rank stratification to the space of symmetric tensors.

Example 9.22 (Ternary Cubics)**.** Symmetric $3\times 3\times 3$ tensors T are ternary cubics, that is, homogeneous polynomials of degree three in three variables. We regard them as points in $\mathbb{P}^9 = \mathbb{P}(\mathrm{Sym}_3(K^3))$. Their ranks coincide with their symmetric ranks, i.e. Comon's Conjecture is true in this tiny case.

The three flattenings $\big[A\,|\,B\,|\,C\,\big]$ in Example 9.21 are now all equal. After removing redundant columns, this becomes a 3×6 matrix, known as *Hankel* *matrix* or *catelecticant*. The ideal of 2×2-minors of the Hankel matrix is generated by the 27 binomial quadrics seen for $A = 3\Delta_2$ in Example 8.18. Its variety is the Veronese surface $X \simeq \mathbb{P}^2$ whose points in $\mathbb{P}^9$ are the cubics of rank 1. The secant variety $\sigma_2(X)$ has dimension 5, its points are cubics of rank ≤ 2, and it is defined by the 3×3 minors of the Hankel matrix. Finally, the variety $\sigma_3(X)$ of rank 3 cubics is a quartic hypersurface in $\mathbb{P}^9$. Its defining polynomial is the classsical *Aronhold invariant*. This has 25 terms and it can be obtained by specializing any of the entries of $A \cdot \mathrm{adj}(B) \cdot C - C \cdot \mathrm{adj}(B) \cdot A$.

We have already discussed the distinction between complex rank and real rank. A further refinement of the latter is the notion of *nonnegative rank*. This very important in applications, e.g. in statistics, where one deals with probabilities. A tensor $T = (t_{i_1 i_2 \cdots i_d})$ is called *nonnegative* if its entries $t_{i_1 i_2 \cdots i_d}$ are all nonnegative. A *nonnegative rank* of a nonnegative tensor T is the minimal number r of nonnegative rank one tensors that sum up to T. In generally, nonnegative rank is larger than real rank, even for matrices.

9.3. Matrix Multiplication

The multiplication of two matrices is a bilinear operation. In this section we identify this operation with a very special tensor. We will use this to explain how tensors may be regarded as computational problems, tensor decompositions as algorithms, and tensor rank as a complexity measure.

Determining the rank of a tensor is an important computational problem in non-linear algebra. In general one cannot hope for an efficient solution, as it is NP-hard [**27**]. However, special cases are of particular interest. The most well-known and important one is the matrix multiplication tensor.

Let $\mathrm{Mat}_{a,b} \simeq K^{a \times b}$ be the space of $a \times b$ matrices over a field K. The operation of matrix multiplication is a bilinear map $\mathrm{Mat}_{a,b} \times \mathrm{Mat}_{b,c} \to \mathrm{Mat}_{a,c}$. Hence, matrix multiplication is an element of the vector space

$$\mathrm{Hom}(\mathrm{Mat}_{a,b} \times \mathrm{Mat}_{b,c}, \mathrm{Mat}_{a,c}) \;=\; \mathrm{Mat}^*_{a,b} \otimes \mathrm{Mat}^*_{b,c} \otimes \mathrm{Mat}_{a,c}.$$

This is a canonical isomorphism. We write $M_{a,b,c}$ for the element in the tensor space on the right hand side. This special three-way tensor is the *matrix multiplication tensor*. To simplify notation we write $M_n := M_{n,n,n}$ for the tensor that represents the multiplication of two square matrices.

Let $\{\mathbf{e}_{ij}\}$, $\{\mathbf{f}_{jk}\}$ and $\{\mathbf{g}_{ik}\}$ be the standard bases of the spaces $\mathrm{Mat}^*_{a,b}$, $\mathrm{Mat}^*_{b,c}$ and $\mathrm{Mat}_{a,c}$. Thus $\mathbf{g}_{ik}$ is the $a \times c$ matrix whose entries are zero except for a one in row i and column k. The other two bases are dual to such matrix units. The matrix multiplication tensor has the following representation:

$$M_{a,b,c} \quad = \sum_{i=1,j=1,k=1}^{a,b,c} \mathbf{e}_{ij} \otimes \mathbf{f}_{jk} \otimes \mathbf{g}_{ik}. \tag{9.15}$$

Another presentation is suggested in Exercise 13.

Example 9.23. Consider the tensor M_2 that represents multiplication of 2×2 matrices. Fixing the ordered basis $(\mathbf{e}_{00}, \mathbf{e}_{01}, \mathbf{e_{10}}, \mathbf{e_{11}})$ for $\mathrm{Mat}_{2,2} \simeq K^4$, we can write M_2 explicitly as a $4 \times 4 \times 4$ tensor with entries in $\{0, 1\}$. Among the 64 entries in this tensor, there are precisely eight ones and 56 zeros.

The rank one decomposition of the tensor $M_{a,b,c}$ given in (9.15) can be interpreted as an *algorithm* for computing the product of the two matrices:

- To carry out matrix multiplication, one needs to add abc partial results labelled by (i, j, k) in $\{1, \ldots, a\} \times \{1, \ldots, b\} \times \{1, \ldots, c\}$.
- In step (i, j, k) one performs one multiplication. Namely, one multiplies the (i, j)-th entry of the first matrix with the (j, k)-th entry of the second. The result is stored in the (k, i)-th entry of the third.

This is the familiar classical algorithm for multiplying two matrices. It performs $abc - 1$ additions and abc multiplications. For $a = b = c = n$, its running time is $O(n^3)$. We note that the number of multiplications is *exactly* equal to the number of rank one tensors appearing in the decomposition.

What if we present $M_{a,b,c}$ in a different way? Could it be that the number of multiplications we need is smaller than abc. Equivalently, is the rank of $M_{a,b,c}$ smaller than abc? Half a century ago, Volker Strassen set out on the quest to prove that this is not possible. He quickly realized that the case of arbitrary a, b, c is extremely hard and focused on the first nontrivial case $a = b = c = 2$. For that tensor, he discovered a most surprising formula:

$$
\begin{aligned}
M_2 \quad = \quad & (\mathbf{e}_{11} + \mathbf{e}_{22}) \otimes (\mathbf{f}_{11} + \mathbf{f}_{22}) \otimes (\mathbf{g}_{11} + \mathbf{g}_{22}) \\
& + (\mathbf{e}_{21} + \mathbf{e}_{22}) \otimes \mathbf{f}_{11} \otimes (\mathbf{g}_{21} - \mathbf{g}_{22}) \\
& + \mathbf{e}_{11} \otimes (\mathbf{f}_{12} - \mathbf{f}_{22}) \otimes (\mathbf{g}_{12} + \mathbf{g}_{22}) \\
& + \mathbf{e}_{22} \otimes (\mathbf{f}_{21} - \mathbf{f}_{11}) \otimes (\mathbf{g}_{11} + \mathbf{g}_{21}) \\
& + (\mathbf{e}_{11} + \mathbf{e}_{12}) \otimes \mathbf{f}_{22} \otimes (\mathbf{g}_{12} - \mathbf{g}_{11}) \\
& + (\mathbf{e}_{21} - \mathbf{e}_{11}) \otimes (\mathbf{f}_{11} + \mathbf{f}_{12}) \otimes \mathbf{g}_{22} \\
& + (\mathbf{e}_{12} - \mathbf{e}_{22}) \otimes (\mathbf{f}_{21} + \mathbf{f}_{22}) \otimes \mathbf{g}_{11}.
\end{aligned} \tag{9.16}
$$

Thus the rank of the matrix multiplication tensor M_2 is less than 8. In fact, it is known that the rank and border rank of M_2 are both exactly 7. The latter is a highly nontrivial statement. We are not aware of any easy proof.

Why could such a decomposition be interesting? It furnishes an algorithm to multiply 2×2 matrices that adds *seven* partial results. We describe only the first two, as we are sure the reader can reconstruct the other five:

(1) Add the $(1, 1)$ entry to the $(2, 2)$ entry of the first matrix and multiply by the sum of the $(1, 1)$ and $(2, 2)$ entries of the second matrix. Retain this in the $(1, 1)$ and $(2, 2)$ entries of the first partial result.

(2) Add $the(2,1)$ entry of the first matrix to the $(2,2)$ entry and multiply by the $(1,1)$ entry of the second matrix. Put the result in the $(2,1)$ entry and negated $(2,2)$ entry of the second partial result.

Computing each partial result requires only *one multiplication.* Although we improved on the number of multiplications, we increased the number of additions (and subtractions) to 21. Why should this be exciting? The reason is that multiplication of 2×2 matrices is not our final aim.

We would like to multiply very large matrices. Consider two 512×512 matrices. How to multiply them? We may regard our matrices as 2×2 matrices with entries *that are* 256×256 *matrices* and apply Strassen's Algorithm! We will have to add a lot of 256×256 matrices, but we only need to perform *seven* mutliplications of such matrices. Further, these multiplications may be done *recursively* applying the same algorithm, reducing to multiplication of 128×128 matrices, etc. Anyone who tried multiplying or adding very large matrices knows that it is beneficial to trade mutliplication even for many additions. This is in fact a theorem: the complexity of the (optimal) algorithm to multiply matrices is governed by the rank of M_n.

The asymptotics of these quantities is measured by the constant

$$\begin{aligned}\omega &= \inf\{\tau : \text{ the complexity of multiplying two } n \times n \text{ matrices is } O(n^\tau)\} \\ &= \inf\{\tau : \text{rank of } M_n = O(n^\tau)\}.\end{aligned}$$

This quantity is known as the *exponent of matrix multiplication.* The naive algorithm shows that $\omega \leq 3$. However Strassen's Algorithm, as described above, gives $\omega \leq \log_2 7$. As matrices are of size n^2, we also know that $\omega \geq 2$.

The central conjecture in this field says that the lower bound is attained:

Conjecture 9.24. *The constant ω is equal to two.*

The conjecture would imply that it is not much harder to multiply very large matrices then to add them (or even output the result)! At this point we note that our story is really relevant for scientific computing. Strassen's Algorithm is implemented and used in practice to multiply large matrices.

A careful reader might now have an idea how to proceed with a proof of Conjecture 9.24. As Strassen looked at 2×2 matrices, we should focus on larger matrices, say 3×3. The disappointing fact is that despite many attempts, no one knows either the rank or the border rank of the $9 \times 9 \times 9$ tensor M_3. For the current best estimates we refer to [**33, 34, 35, 48**].

For each fixed n, deciding if rank (resp. border rank) of M_n is $\leq r$ means deciding if M_n belongs to the image (resp. closed image) of a particular polynomial map. Thus, the methods of Chapter 4 apply. However, as tensor spaces are high-dimensional, such computations are impossible to carry out in practice, even for $n = 3$. What one can use instead is representation theory, as described in Chapter 10. The optimal estimates for ω are beyond the scope of this book. Currently we know $2 \leq \omega < 2.38$. It is fascinating that the upper bounds are based on border rank and nonconstructive methods: one proves the existence of an algorithm without explicitly providing it.

In general, we lack methods to show that a tensor has high rank or high border rank. To prove that $\omega > 2$ we would need to show that the rank of the tensor $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$ grows superlinearly with the dimension n^2 of the space of matrices. However, we currently cannot even prove that any (explicit) given tensor has rank greater than $3n^2$. Some methods to obtain bounds for the rank of the tensor will be discussed in Section 10.3.

Exercises

(1) Fix the quadratic form Q in (9.1). Compute all the maxima and minima of Q on the unit 2-sphere. Find all fixed points of the gradient map $\nabla Q : \mathbb{P}^2 \to \mathbb{P}^2$. How are these two questions related?

(2) Compute all fixed points of the map $\nabla B : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1$ that is given by the bilinear form B in (9.4). What are the singular vectors?

(3) Consider the $3 \times 3 \times 2 \times 2$ tensor defined by the multilinear form $T = x_1y_1z_1w_1 + x_2y_2z_2w_2$. Determine all quadruples of singular vectors of T.

(4) For $d = 2, 3, 4$, pick random symmetric tensors of formats $d{\times}d{\times}d$ and $d{\times}d{\times}d{\times}d$ with entries in $\mathbb{R}$. Compute all eigenvectors of your tensors.

(5) Prove Theorem 9.6.

(6) Find an explicit real $3{\times}3{\times}3{\times}3$ tensor with precisely 13 real eigenvectors.

(7) Find the number of singular vector tuples for your tensors in Problem 4.

(8) Compute the eigendiscriminants for symmetric tensors of format 2×2 and $2 \times 2 \times 2$ and $2 \times 2 \times 2 \times 2$. Write them explicitly as homogeneous polynomials in these entries of an unknown tensor.

(9) Prove that the rank of the W-state equals three. Hint: Show that the polynomial system $W = T$ described in Example 9.15 has no solution.

(10) Show that the Zariski closure of the set of tensors of rank two in $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ is the whole space. Hint: Use the Jacobian of the parametrization.

(11) Find the equation of the tangential variety to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^{11}$.

(12) Prove that in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$:
 (a) there exists a tensor of border rank at least $\frac{1}{3}n^2$,
 (b) every tensor has rank at most n^2.

(13) Linear maps from V_1 to V_2 are identified with tensors in $V_1^* \otimes V_2$. The composition of linear maps in $V_1 \to V_2 \to V_3$ may be regarded as a map

$$(V_1^* \otimes V_2) \times (V_2^* \otimes V_3) \to (V_1^* \otimes V_3).$$

Hence, the matrix multiplication tensor $M_{\dim V_1, \dim V_2, \dim V_3}$ belongs to

$$(V_1^* \otimes V_2)^* \otimes (V_2^* \otimes V_3)^* \otimes (V_1^* \otimes V_3) = (V_1 \otimes V_1^*) \otimes (V_2 \otimes V_2^*) \otimes (V_3 \otimes V_3^*).$$

(a) How to interpret $M_{\dim V_1, \dim V_2, \dim V_3}$ as an element of the last space? Do not refer to the basis, but only linear maps $V_i \to V_i$. Hint: The identity map is a distinguished element in $V_i^* \otimes V_i$.

(b) Provide a natural isomorphism $\mathrm{Mat}_{a,b}^* \simeq \mathrm{Mat}_{b,a}$.

(c) The tensor $M_{a,b,c}$ can also be identified with a trilinear map

$$\mathrm{Mat}_{a,b} \times \mathrm{Mat}_{b,c} \times \mathrm{Mat}_{c,a} \to K.$$

Describe this trilinear map without referring to coordinates.

(14) The matrix multiplication tensor $M_{2,2,3}$ has format $4 \times 6 \times 6$. Write this tensor explicitly in coordinates. What do you know about its rank?

(15) Expand the Aronhold invariant and the Strassen invariant in monomials.

(16) Compute the ideal of the secant variety $\sigma_2(X)$ where $X = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the Segre variety in $\mathbb{P}^{17}$. How about the same question for $\sigma_3(X)$?

(17) How can you test whether a complex $4 \times 4 \times 4$ tensor has rank ≤ 4?

Chapter 10

Representation Theory

Symmetry is the key to many applications and computations. While this is true across the mathematical sciences, it is especially pertinent in nonlinear algebra. In its most basic form, symmetry is expressed via the action of a group acting linearly on a vector space. The study of such actions is the subject of *representation theory.* For instance, the symmetric group on $n+1$ letters acts on n-dimensional space by the rotations and reflections that fix a regular n-simplex. The map that takes each group element to its associated $n \times n$ matrix is the representation of the group. The matrix representations of the groups we study here can be simultaneously block-diagonalized. The blocks are irreducible representations. Identifying these blocks is tantamount for exploiting symmetry in explicit computations. Our objective in this chapter is to give a first introduction to representation theory.

10.1. Irreducible Representations

The most important groups we study in this chapter are:

- $\mathrm{GL}(V) = \mathrm{GL}(\dim V)$ — the group of linear isomorphisms of a finite-dimensional vector space V. This group has the structure of an algebraic variety, given by Exercise 8 in Chapter 2.
- $\mathrm{SL}(V) = \mathrm{SL}(\dim V)$ — the group of volume– and orientation–preserving linear automorphisms of V, with the structure of an algebraic variety given by the equation $\det A = 1$;
- S_n — the group of permutations of a set with n elements; this is an algebraic variety consisting of $n!$ distinct points in $\mathrm{GL}(n)$, namely the $n \times n$ permutation matrices.

The groups that we consider have two structures: of an abstract group and of an algebraic variety. We note that basic group operations, like inverse or group action, are in fact morphisms of algebraic varieties. We call such groups *algebraic*. In what follows, we restrict our attention to algebraic groups and morphisms between them that are both group morphisms and morphisms of algebraic varieties. We work over an algebraically closed field K of characteristic zero.

In general, the following strategy to study an object can be very powerful:

- consider all maps from (resp. to) this object into (resp. from) another basic object.

This general approach could be seen as motivation to study homotopy, homology or the theory of embeddings. For groups, we obtain the following central definition.

Definition 10.1. *A representation* of a group G is a morphism $G \to \mathrm{GL}(V)$.

Given a representation $\rho : G \to \mathrm{GL}(V)$, every element of g induces a linear map $\rho(g) : V \to V$. It is useful to think about a representation as a map $G \times V \to V$ with the notation

$$gv \ := \ \rho(g)(v) \in V.$$

Here, we have the natural compatibilities

$$(g_1g_2)v = g_1(g_2v) \quad \text{and} \quad g(\lambda v_1 + v_2) = \lambda g v_1 + g v_2,$$

where $\lambda \in K$, $v, v_1, v_2 \in V$ and $g, g_1, g_2 \in G$. We say that the group G *acts on* the vector space V. If the action follows from the context then we call V a *representation* of G.

Example 10.2. The groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ act (by linear change of coordinates) on the space $V = K[x_1, \ldots, x_n]_k \simeq K^{\binom{n+k-1}{k}}$ of homogeneous polynomials of degree k in n variables. Using the monomial basis on V, the representation ρ maps a small matrix, of size $n \times n$, to a large matrix, with rows and columns indexed by monomials of degree k. The entries in that large matrix are homogeneous polynomials of degree k in the entries of the small matrix. We recommend working this out for $n = k = 2$. This representation ρ of $\mathrm{GL}(n)$ plays an important role in classical *Invariant Theory*, the topic to be studied in the next Chapter 11.

A *morphism* f between representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ is a linear map $f : V_1 \to V_2$ that is compatible with the group action:

$$f(\rho_1(g)(v)) = \rho_2(g)(f(v)) \quad \text{for all } g \in G \text{ and } v \in V_1.$$

This can also be written as $f(gv) = gf(v)$. The kernel and cokernel of f is also a representation of G—cf. Exercise 3.

Our first aim is to describe the basic building blocks of representations.

Definition 10.3. A *subrepresentation* of a representation V of a group G is a linear subspace $W \subset V$ such that the action of G restricts to W, i.e.

$$gw \in W \text{ for all } w \in W \text{ and } g \in G.$$

For any representation V, the subspaces 0 and V are always subrepresentations.

Definition 10.4. A representation V is called *irreducible* if and only if 0 and V are its only subrepresentations.

We next show that there are no nonzero morphisms between nonisomorphic irreducible representations.

Lemma 10.5 (Schur's Lemma)**.** *Let V_1 and V_2 be irreducible representations of a group G. If $f : V_1 \to V_2$ is a morphism of representations then either f is an isomorphism or $f = 0$. Further, any two isomorphisms between V_1 and V_2 differ by a scalar multiple.*

Proof. Both the kernel $\ker f$ and the image $\operatorname{im} f$ are representations. As V_1 is irreducible, either $\ker f = V_1$ or f is injective. In the latter case, $\operatorname{im} f \simeq V_1$ is a nontrivial subrepresentation of V_2, hence f is also surjective, i.e. it is a linear isomorphism. The inverse of f, as a linear map, is also the inverse as morphism of representations.

For the last part, consider two isomorphisms f_1 and f_2. We may assume that f_1 is the identity on V_1. If v is the eigenvector of f_2 with eigenvalue $\lambda \in K$ then

$$f_2(v) \;=\; \lambda v \;=\; \lambda f_1(v).$$

Consider the morphism of representations $f := f_2 - \lambda f_1$. Clearly, $v \in \ker f$. Hence, by the first part, $f_2 - \lambda f_1$ is the zero map, and hence $f_2 = \lambda f_1$. □

Theorem 10.6 (Maschke's theorem)**.** *Let V be a finite-dimensional representation of a finite group G. There exists a direct sum decomposition*

$$V \;=\; \bigoplus V_i,$$

where each V_i is an irreducible representation of G.

Remark 10.7. We recall that we work under the assumption that the field is algebraically closed and of characteristic zero, which makes representation theory much better behaved. Representation theory in finite characteristic is considerably more complicated.

Proof of Theorem 10.6. By induction on the dimension, it is enough to prove the following statement: if W is a subrepresentation of V, then there exists a subrepresentation W' such that $V = W \oplus W'$.

Let $\pi : V \to W$ be any linear (surjective) projection. Let $\tilde{\pi} : V \to W$ be defined by:

$$\tilde{\pi}(v) \;=\; \frac{1}{|G|}\sum_{g\in G} \rho(g)_{|W} \circ \pi \circ \rho(g)^{-1}.$$

We note that $\tilde{\pi}$ is a morphism of representations and $V = W \oplus \ker \tilde{\pi}$. □

Remark 10.8. The existence of decomposition into irreducible components holds not only for finite groups. It also holds for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. One possible proof is similar to the one above and is known as the *unitarian trick*. It was introduced by Hurwitz and generalized by Weyl.

A representation of an arbitrary group that allows such a decomposition is called *semi-simple* or *completely reducible*. If all representations of G have this property then the group G is called *reductive*.

Example 10.9. The group $G = (\mathbb{C}, +)$ is not reductive. Indeed, let us consider the following representation:

$$G \ni a \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2).$$

Clearly, the one dimensional subspace of $\mathbb{C}^2$ spanned by the second basis vector is invariant under the group action. However, it does not allow a complement - cf. Exercise 13.

The decomposition into irreducible representations in Maschke's Theorem is not unique. The following example makes this clear.

Example 10.10. Any group G acts on any vector space V trivially by $gv = v$. Any subspace of V is a subrepresentation. The irreducible subrepresentations are the 1-dimensional subspaces of V. Hence, any decomposition into 1-dimensional subspaces $V = \bigoplus_{i=1}^{\dim V} K^1$ is a decomposition into irreducible representations, but there is no distinguished one.

As we will see, the reason for nonuniqueness, is the fact that distinct V_i's appearing in the decomposition may be isomorphic. Let us group the isomorphic V_i's together obtaining:

$$V \;=\; \bigoplus V_j^{\times a_j}, \tag{10.1}$$

where $V_{j_0} \simeq V_{j_1}$ if and only if $j_0 = j_1$. The subrepresentations $V_j^{\times a_j}$ are called *isotypic components*. The number a_j is the *multiplicity* of the irreducible representation V_j in V.

Corollary 10.11 (to Schur's Lemma)**.** *The isotypic components and multiplicities of a semi-simple representation V are well defined, i.e. do not depend on the choice of the decomposition into irreducible representations.*

Proof. Consider two decompositions:

$$V = \bigoplus_j V_j^{\times a_j} = \bigoplus_k V_k^{\times b_k}.$$

Allowing a_j, b_k to be equal to zero, we may assume that all irreducible representations occur and that the indexing in both sums $\bigoplus$ is the same. First we prove that for a given irreducible representation V_i we have $a_i = b_i$. The restriction of identity gives us an injective map:

$$m : V_i^{\times a_i} \to \bigoplus_k V_k^{\times b_k}.$$

By Schur's Lemma, the composition of m with the projection

$$\pi_s : \bigoplus_k V_k^{\times b_k} \to V_s^{\times b_s}$$

equals zero, unless $s = i$. Hence, $\operatorname{im} m \subset V_i^{\times b_i}$. In particular, by dimension count, $a_i \leq b_i$. Analogously $b_i \leq a_i$, i.e. the multiplicities do not depend on the decomposition. Further, the composition $\pi_s \circ m$ is an isomorphism if $s = i$ and is zero if $s \neq i$. It follows that $\operatorname{im} m = V_i^{\times b_i}$. Thus, the identity maps isotypic components to (the same) isotypic components. □

Our next aim is to understand the irreducible representations of a given group G. The following definition provides us with the most important tool.

Definition 10.12 (Character)**.** Let $\rho : G \to GL(V)$ be a representation of G. The *character* $\chi_\rho = \chi_V$ of ρ is the function $G \to K$ obtained by composing ρ with the trace function Tr:

$$\chi_\rho(g) \;=\; \operatorname{Tr}(\rho(g)).$$

The analogy to characters studied in Chapter 8 will be presented in Remark 10.14.

Properties of the trace of a square matrix imply the following about characters:

- If $V = \bigoplus V_i$ then $\chi_V = \sum \chi_{V_i}$.
- If g_1 and g_2 are conjugate elements of G, then $\chi(g_1) = \chi(g_2)$ for any character χ.
- If V_1, V_2 are representations with characters χ_1, χ_2 then their tensor product $V_1 \otimes V_2$ is also a representation, and its character is the product $\chi_1\chi_2$.

- We have $\chi_V(e) = \dim V$, where $e \in G$ is the neutral element.

For a finite group G, we fix the following scalar product on the complex vector space $\mathbb{C}^G$ of all functions from G to $\mathbb{C}$:

$$\langle \chi_1, \chi_2 \rangle \;:=\; \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}. \tag{10.2}$$

It turns out that characters of all irreducible representations of G are orthonormal with respect to this scalar product. For details we refer to Serre's book [**45**, Chapter 2]. In particular, the characters of irreducible representations are linearly independent elements in $\mathbb{C}^G$. Hence, we can find the multiplicities a_j in the isotypic decomposition $V = \bigoplus_j V_j^{a_j}$ by decomposing the character:

$$\chi_V \;=\; \sum_j a_j \chi_j.$$

For any finite group G there are finitely many irreducible representations - the sum of squares of their dimensions equals the order of the group [**45**, Chapter 2.5, Corollary 2]. A *class function* is a function $G \to K$ that is constant on conjugacy classes. Characters in fact form a basis of the space of class functions. Often (all) characters are represented in a table, which makes the decomposition very easy, if we know the character of a representation.

Example 10.13. Consider the group S_3 of permutations of three elements. There are three conjugacy classes: the class of identity (with one element), the class of a 3-cycle (with two elements) and the class of any transposition (with three elements). Hence, there are three irreducible representations. The first is the trivial representation $gv = v$, the second is the sign representation $gv = (\operatorname{sgn} g)v$, and the third is the two-dimensional representation, given by the symmetries of a regular triangle. Each column in the table below represents a function $S_3 \to \mathbb{C}$ that is constant on conjugacy classes. We present the character table for the symmetric group S_3:

	Trivial representation	Sign repr.	2-dimensional repr.
identity	1	1	2
cycles (ijk)	1	1	-1
transpositions (ij)	1	-1	0

.

The reader should make sure to check these functions are orthonormal with respect to the inner product (10.2). In fact, one builds the character table of a finite group by exploiting the orthonormality of the columns. In this manner, one obtains the 5×5 character table for S_4 and the 7×7 character table for S_5.

These ideas generalize to $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. In this cases, we cannot represent the characters by tables as there are infinitely many conjugacy classes. However, we can represent each character χ by its values on the Zariski dense subset of diagonalizable matrices. Hence, we fix a torus $T = (K^*)^n \subset \mathrm{GL}(n)$ and restrict the character to T. As χ is constant on any conjugacy class and any diagonalizable matrix is conjugate to an element of T, the function $\chi_{|T}$ characterizes χ. Therefore, given any representation W of $\mathrm{GL}(n)$, we restrict the group and regard W as a representation of T. By Exercise 1 and Corollary 10.11 we know that, as a representation of T, the space W decomposes:

$$W \;=\; \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} W_{\mathbf{b}}^{a_{\mathbf{b}}}, \tag{10.3}$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ takes w to $\mathbf{t}^{\mathbf{b}}w$ for $w \in W_{\mathbf{b}}$. The isotypic components $W_{\mathbf{b}}^{a_{\mathbf{b}}}$ for the T-action are called *weight spaces*. The characters $\mathbf{b}$ of T for which $a_{\mathbf{b}} \neq 0$ are called *weights*.

Remark 10.14. Let T be the torus of diagonal matrices $\mathbf{t} = \mathrm{diag}(t_1, \ldots, t_n)$ in $\mathrm{GL}(n)$. If χ is a character of $\mathrm{GL}(n)$ then its restriction to T is the function $\chi|_T : T \to K$, $\mathbf{t} \mapsto \mathrm{Tr}(\rho(\mathbf{t}))$. Here Tr denotes the trace of a (large) square matrix. The restricted character $\chi|_T$ equals

$$\chi|_T(\mathbf{t}) \;=\; \sum_{\mathbf{b}\in\mathbb{Z}^n} a_{\mathbf{b}}\mathbf{t}^{\mathbf{b}}.$$

This Laurent polynomial in $t_1, \ldots, t_n$ is invariant under permuting its n unknowns.

Example 10.15. Following Example 10.2, we consider the action of $\mathrm{GL}(n)$ on homogeneous polynomials of degree k. Let χ be its character. Then $\chi|_T$ is the *complete symmetric polynomial* of degree k, i.e., $\chi|_T(\mathbf{t})$ is the sum of all monomials $\mathbf{t}^{\mathbf{a}}$ where $\mathbf{a} \in \mathbb{N}^n$ and $|\mathbf{a}| = k$.

Example 10.16. The group $\mathrm{GL}(n)$ acts naturally on the kth exterior power $V = \bigwedge^k K^n$. Write ρ for this representation and χ for its character. We identify V with $K^{\binom{n}{k}}$ by fixing the standard basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$. The image $\rho(g)$ of an $n \times n$-matrix $g = (g_{ij})$ is the *kth compound matrix* or *kth exterior power*, whose entries are the (suitably signed) $k \times k$ minors of g. We note that the determinant of $\rho(g)$ equals $\det(g)^{\binom{n-1}{k-1}}$. The restricted character $\chi|_T(\mathbf{t})$ is the kth elementary symmetric polynomial in $t_1, \ldots, t_n$.

For a concrete example, let $k = 2$. Then $\rho(g)$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix whose rows and columns are labeled by ordered pairs from $\{1, 2, \ldots, n\}$, and whose entry in row $(i < j)$ and column $(k < l)$ equals $g_{ik}g_{jl} - g_{il}g_{jk}$. We have $\det(\rho(g)) = \det(g)^{n-1}$ and $\chi|_T(\mathbf{t}) = \sum_{i<j} t_it_j$. For $k = 1$ we have

$\rho(g) = g$, so $\chi|_T(\mathbf{t}) = t_1 + t_2 + \cdots + t_n$. Finally, for $k = n$, we get the one dimensional representation where $\rho(g)$ is the 1×1-matrix with entry $\det(g)$, so we have $\chi|_T(\mathbf{t}) = t_1 t_2 \cdots t_n$. The latter gives the trivial representation when restricted to $\mathrm{SL}(n)$.

Let ρ be any representation of $\mathrm{GL}(n)$. We fix the lexicographic order on the set of weights $\mathbf{b}$ that occur in ρ. Of particular importance is *the highest weight*. The corresponding eigenvectors $w \in W_{\mathbf{b}}$ in (10.3) are called *highest weight vectors*. They span the *highest weight space*. In Example 10.15, the highest weight is $(d, 0, \ldots, 0) \in \mathbb{Z}^n$, and a highest weight vector is the monomial x_1^d. In Example 10.16, the highest weight is $(1, \ldots, 1, 0, \ldots, 0)$, and a highest weight vector is $e_1 \wedge \cdots \wedge e_k$. In both cases, the highest weight space is 1-dimensional. We note that the highest weight vector does not depend on n, provided it exists (e.g. $n \geq k$ in the exterior power case).

Example 10.17 (Adjoint representation)**.** The space $V = K^{n \times n}$ of $n \times n$ matrices M forms a representation of $GL(n)$ under the action by conjugation, where $\rho(g)(M) := gMg^{-1}$. This is the *adjoint representation.* The weights, known as *roots* in this case, are t_i/t_j with the highest weight $(1, 0, \ldots, 0, -1)$. If we restrict it to $\mathrm{SL}(V)$ we have $t_n^{-1} = \prod_{i=1}^{n-1} t_i$ and the highest weight becomes $(2, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$. Again, the highest weight space is 1-dimensional.

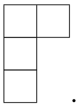
The following result provides a characterization of irreducible representations.

Proposition 10.18. *Every irreducible representation of* $\mathrm{SL}(V)$ *is determined (up to isomorphism) by its highest weight, and the highest weight space is* 1*-dimensional. A weight* $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$ *is the highest weight for some irreducible representation if and only if* $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$.

Proof. For the proof we refer to [**22**, Chapter 15]. □

Here is a combinatorial tool for building irreducible representations from highest weights:

Definition 10.19. A *Young diagram* with k-rows is a nonincreasing sequence of k positive integers. It is usually presented in the following graphical form, e.g. for a sequence $(2, 1, 1)$:



This particular Young diagram encodes the weight $(2, 1, 1)$. For $\mathrm{SL}(4)$ it represents the adjoint representation. We note that for $\mathrm{SL}(5)$ the same

Young diagram would *not* represent the adjoint representation, however the highest weight vector would be the same.

Proposition 10.18 tells us that the irreducible representations of $\mathrm{SL}(n)$ are in bijection with the Young diagrams with at most $n-1$ rows. Representations of $\mathrm{GL}(n)$ are not very different: first, every irreducible representation V of $\mathrm{GL}(n)$ is also an irreducible representation of $\mathrm{SL}(n)$, so it has a corresponding Young diagram λ. However, different representations of $\mathrm{GL}(n)$ give the same representation of $\mathrm{SL}(n)$ if they differ by a power of the determinant. Precisely, consider a representation $\rho : \mathrm{SL}(n) \to GL(V)$ with associated Young diagram λ. We have the following representations of $\mathrm{GL}(n)$ for any $a \in \mathbb{Z}$:

$$\rho_a(g) \ := \ (\det g)^a \cdot \sqrt[n]{\det g} \cdot \rho\big(\frac{1}{\sqrt[n]{\det g}} \cdot g\big).$$

Here, the argument of ρ is in $\mathrm{SL}(n)$. The irreducible representations of $\mathrm{GL}(n)$ are in bijection with pairs of: a Young diagram with at most n rows and an integer $a \in \mathbb{Z}$.

The 1-dimensional representation $g \mapsto \det(g)$ of $\mathrm{GL}(n)$ corresponds to a Young diagram with one column and n rows. Thus for $a \geq 0$ the representation ρ_a may be represented by a Young diagram λ extended by a columns of height n. The representation of $\mathrm{GL}(U)$ corresponding to a Young diagram λ is denoted by $S^\lambda(U)$.

Given a Young diagram λ, we write χ_λ for character of the irreducible representation $S^\lambda(U)$. This is a symmetric polynomial in $\mathbf{t} = (t_1, \ldots, t_n)$, known as the *Schur polynomial* of λ. Schur polynomials include the complete symmetric polynomials in Example 10.15, for $\lambda = (n)$, and the elementary symmetric polynomials in Example 10.16, for $\lambda = (1, 1, \ldots, 1)$.

Here is an explicit formula for Schur polynomials.

Proposition 10.20. *The Schur polynomial for λ is the following ratio of $n \times n$ determinants:*

$$\chi_\lambda(\mathbf{t}) \quad = \quad \frac{\det\big(\, t_i^{\lambda_j+n-j} \,\big)_{1\leq i,j\leq n}}{\det\big(\, t_i^{n-j} \,\big)_{1\leq i,j\leq n}}.$$

If λ has less than n rows, we extend it by zeros.

We can find the decomposition (10.1) of a representation V into irreducibles by writing the character χ_V as linear combination of Schur functions χ_λ with nonnegative integer coefficients a_j. These coefficients are the multiplicities. This expression is unique because the Schur polynomials form a $\mathbb{Z}$-linear basis for the ring of symmetric polynomials in n variables.

Example 10.21. Let $n = 3$. The Schur polynomial for $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the ternary form

$$\chi_\lambda(\mathbf{t}) \quad = \quad \frac{1}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} \cdot \det \begin{pmatrix} t_1^{\lambda_1+2} & t_1^{\lambda_2+1} & t_1^{\lambda_3} \\ t_2^{\lambda_1+2} & t_2^{\lambda_2+1} & t_2^{\lambda_3} \\ t_3^{\lambda_1+2} & t_3^{\lambda_2+1} & t_3^{\lambda_3} \end{pmatrix}.$$

From this, we compute the three Schur polynomials of degree $|\lambda| = 3$ as follows:

$$\begin{aligned} \chi_{(3,0,0)} &= t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3 + t_1^2 t_3 + t_1 t_2 t_3 + t_2^2 t_3 + t_1 t_3^2 + t_2 t_3^2 + t_3^3 \\ \chi_{(2,1,0)} &= (t_1 + t_2)(t_1 + t_3)(t_2 + t_3) \\ \chi_{(1,1,1)} &= t_1 t_2 t_3 \end{aligned}$$

The action of GL(3) on $U = K^3$ induces an action on the 27-dimensional space $U^{\otimes 3}$ of $3 \times 3 \times 3$-tensors. As characters are multiplicative under tensor product, its character equals

$$\chi_{U^{\otimes 3}} \;=\; (t_1 + t_2 + t_3)^3 \;=\; \chi_{(3,0,0)} \;+\; 2 \cdot \chi_{(2,1,0)} \;+\; \chi_{(1,1,1)}.$$

From this decomposition into Schur polynomials, we conclude the irreducible decomposition

$$U^{\otimes 3} \;=\; S^{(3)}(U) \;\oplus\; \bigl(S^{(21)}(U) \oplus S^{(21)}(U)\bigr) \;\oplus\; S^{(111)}(U). \tag{10.4}$$

The first summand is the symmetric tensors, the last summand is the antisymmetric tensors, and the middle summand consists of two copies of the adjoint representation (Example 10.17).

The irreducible representations $S^\lambda(U)$ of $\mathrm{SL}(U)$ come together with nice algebraic varieties. The group $\mathrm{SL}(U)$ acts also on the projective space $\mathbb{P}(S^\lambda(U))$. The letter action has unique closed orbit, namely the orbit of the highest weight vector. Particular examples are:

(1) The orbit of $[e_1 \cdots e_1] \in \mathbb{P}(S^k(U))$. This is the k-th Veronese embedding of $\mathbb{P}(U)$.

(2) The orbit of $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\wedge^k(U))$ is the Grassmannian $G(k, U)$ in its Plücker embedding. Here $\lambda = (1, \ldots, 1)$ as in Example 10.16.

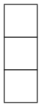
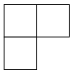
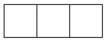
This result provides us with a unified approach to homogeneous varieties. It could be also used to build some of the representations. Fix a Young diagram λ and let $k\lambda$ be a Young diagram where each row is scaled by k. Given the homogeneous variety X in $\mathbb{P}(S^\lambda(U))$ we can take the k-th Veronese map v_k of this projective space and the linear span of $v_k(X)$ is $S^{k\lambda}(V)$. A special case of this construction is point (1) above where $X = \mathbb{P}(U)$.

10.2. Schur-Weyl Duality

In this section we present a beautiful connection between finite groups - S_n and Lie groups - $SL(n)$ or $GL(n)$. This is the *Schur-Weyl* duality. We refer readers interested in the topic to [**22**, Chapter 4].

Before stating it let us go back to irreducible representations of S_n. Their characters form a basis of class functions. Hence the number of irreducible representations equals the number of conjugacy classes. Each conjugacy class can be encoded by lengths of cycles in a decomposition of a permutation into cycles. These can be further represented by a Young diagram with n boxes: the first row represents the length of the longest cycle, the last of the shortest. Thus, the number of irreducible representations of S_n equals the number of Young diagrams with n boxes.

Example 10.22. For S_3 we have three conjugacy classes:

- Identity (1)(2)(3) with the Young diagram ;
- Transpositions, e.g. (12)(3) with the Young diagram  ;
- 3-cycles, e.g. (123) with the Young diagram .

We shall exhibit a natural bijection between Young diagrams with n boxes and irreducible representations of S_n. Before we see how to construct it, let us assume that to each such Young diagram λ we can associate a representation S_λ of S_n.

Fix a vector space U and consider the n-fold tensor product $U^{\otimes n}$. There are two groups acting on it: $\mathrm{GL}(U)$ - on each factor - and S_n - by permuting factors. Schur-Weyl duality provides a simultaneous decomposition of the space of tensors with respect to both groups.

Theorem 10.23 (Schur-Weyl duality)**.** *Let U be a vector space of dimension at least n. Then*

$$U^{\otimes n} = \sum_{|\lambda|=n} S_\lambda \otimes S^\lambda(U), \tag{10.5}$$

where the sum is over all Young diagrams with precisely n boxes.

When $n = 2$ and $\dim U \geq 2$ we obtain $U^{\otimes 2} = S^2(U) \oplus \bigwedge^2 U$, as there are only two irreducible representations of S_2, both 1-dimensional. This recovers the fact every $n \times n$ matrix is uniquely the sum of a symmetric matrix and a skew-symmetric matrix. The S_2 action on the matrix space

$U^{\otimes 2}$ is transposition, which acts trivially on $S^2(U)$ and changes the sign on $\bigwedge^2 U$.

The case $n = 3$ is the first interesting one. The three irreducible representations S_λ of S_3 in Example 10.13 correspond to the three outer summands in (10.4). Note that $\dim(S_\lambda) = 2$ for $\lambda = (2,1)$. The middle summand in (10.4) is the 16-dimensional space $S_{(21)} \otimes S^{(21)}(U)$.

By Schur-Weyl duality, the multiplicity of $S^\lambda(U)$ in $U^{\otimes n}$ equals the dimension of S_λ. This provides us with a method for defining S_λ. Consider the decomposition of $U^{\otimes n}$ as a $\mathrm{GL}(U)$ representation, into isotypic components. Here the a_λ can be found using Schur functions:

$$U^{\otimes n} \;=\; \oplus_\lambda (S^\lambda(U))^{a_\lambda}.$$

For each isotypic component $(S^\lambda(U))^{a_\lambda}$ consider the highest weight space, i.e. eigenvectors of the torus action with weight λ. The permutation group S_n acts on the highest weight space. This representation of S_n is irreducible, and we find that it is precisely S_λ.

Coming back to the example of matrices ($n = 2$), the highest weight vectors are as follows:

- The highest weight vector $e_1e_1 = e_1 \otimes e_1$ of $S^2(U)$ is invariant with respect to transposition, i.e. it provides the trivial representation of the two-element group S_2.
- The highest weight vector $e_1 \wedge e_2 = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$ of $\wedge^2(U)$ changes sign when transposed, i.e. it provides the sign representation of the two-element group S_2.

Example 10.24 ($n = 3$)**.** Let $\lambda = (2,1)$. The isotypic component $(S^{(21)}(U))^2$ in the middle of (10.4) has a 2-dimensional subspace S_λ of highest weight vectors. One possible basis of this space consists of the tensors $e_{112}+e_{211}-2e_{121}$ and $e_{121} + e_{211} - 2e_{112}$, where $e_{ijk} := e_i \otimes e_j \otimes e_k$.

10.3. Exploiting Symmetry

In this section we will show how representation theory can guide us to obtain *lower bounds* for complexity problems—precisely matrix multiplication discussed in Chapter 9.3.

Let $W_1 = A^* \otimes B$, $W_2 = B^* \otimes C$, $W_3 = A^* \otimes C$ be respectively the space of linear maps from A to B, from B to C and from A to C. For simplicity, we assume $\dim A = \dim B = \dim C = n$. Explicitly, we describe ideas how to bound the border rank of the matrix multiplication tensor $M_n \in W_1^* \otimes W_2^* \otimes W_3$, following [**36**]. We note that M_n is an invariant tensor with respect to the $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ action. It may be thought of as a morphism of representations:

$$M_n : B^* \otimes C \to A^* \otimes C \otimes A \otimes B^*.$$

The reader is encouraged to check that in the above map the C factor just goes by the identity map. This was part of Exercise 13 in Chapter 9. Thus it is essential to consider:

$$M_n' : B^* \to A^* \otimes W_1^*$$

and $M_n = M_n' \otimes Id_C$. In terms of linear maps $\operatorname{rk} M_n = n \operatorname{rk} M_n'$. Further, the rank of M_n as a tensor is bounded below by the rank of M_n as a linear map. This gives the trivial bound $\operatorname{rk} M_n \geq n^2$. The problem with this approach is that M_n' is represented as a nonsquare $n \times n^2$ matrix, hence obtaining good lower rank bounds seems impossible. This is where representations come into play. First we turn M_n' into much larger matrix by tensoring with $\bigwedge^k W_1^*$:

$$M_n \otimes id_{\bigwedge^k W_1^*} : B^* \otimes \bigwedge^k W_1^* \to A^* \otimes W_1^* \otimes \bigwedge^k W_1^*.$$

The new map is represented by an $n\binom{n^2}{k} \times n^2\binom{n^2}{k}$ matrix. This is still far from a square matrix. The next idea is to consider an equivariant projection: $W_1^* \otimes \bigwedge^k W_1^* \to \bigwedge^{k+1} W_1^*$. Landsberg and Ottaviani find a subspace of W_1^* on which M_n' composed with the above projection becomes injective. This is also achieved via representation theory—for a more combinatorial proof we refer to [**34**].

First, one identifies $A \simeq S^{n-1}\mathbb{C}^2$ and $B^* \simeq S^{n-1}\mathbb{C}^2$ with the space of homogeneous polynomials of degree $n-1$ in two variables. We have a natural multiplication map: $S^{n-1}\mathbb{C}^2 \otimes S^{n-1}\mathbb{C}^2 \to S^{2n-2}\mathbb{C}^2$. By Schur's Lemma, or simply by dualizing the map above, we obtain a $2n-1$ dimensional subspace $S^{2n-2}\mathbb{C}^2$ inside W_1^*. A direct computation reveals that indeed the induced map:

$$B^* \otimes \bigwedge^{n-1} S^{2n-2}\mathbb{C}^2 \to A^* \otimes \bigwedge^{n} S^{2n-2}\mathbb{C}^2$$

is injective, hence an isomorphism. In particular, the map:

$$\bigwedge^{n-1} S^{2n-2}\mathbb{C}^2 \otimes B^* \otimes C \to A^* \otimes C \otimes \left(\bigwedge^{n} S^{2n-2}\mathbb{C}^2\right)$$

has full rank equal to $n^2\binom{2n-1}{n-1}$. To sum up the final map was obtained in the following steps:

(1) Consider M_n as a tensor in the space $W_1^* \otimes W_2^* \otimes W_3$.

(2) Restrict it to a special subspace $\mathbb{C}^{2n-1} \otimes W_2^* \otimes W_3$.

(3) Represent it as a linear map $W_2 \to W_3 \otimes \mathbb{C}^{2n-1}$.

(4) Tensor the map by the identity map on $\bigwedge^{n-1} \mathbb{C}^{2n-1}$.

(5) Contract the codomain through $\mathbb{C}^{2n-1} \otimes \bigwedge^{n-1} \mathbb{C}^{2n-1} \to \bigwedge^{n} \mathbb{C}^{2n-1}$.

(6) The result is the final map:

$$\bigwedge^{n-1} \mathbb{C}^{2n-1} \otimes W_2 \to W_3 \otimes \bigwedge^{n} \mathbb{C}^{2n-1}.$$

The above procedure may be applied to any tensor, not only M_n. Irrespective of the choice of $\mathbb{C}^{2n-1} \subset W_1^*$ the reader should check that a rank one tensor gives rise to the final map of rank at most $\binom{2n-2}{n-1}$. Hence, we obtain the following proposition.

Proposition 10.25 ([**36**]). *The border rank of the tensor M_n is at least:*

$$n^2\binom{2n-1}{n-1}/\binom{2n-2}{n-1} = 2n^2 - n$$

We stress the fact that there are many further applications of representation theory. In fact, the book of Serre [**45**] to which we referred several times grew out of lectures for quantum chemists and physicists. For applications in probability theory and statistics we refer to [**16**].

Exercises

(1) (a) Prove that, over an algebraically closed field, every irreducible representation of an abelian group is 1-dimensional.

(b) Explain the correspondence between the characters of a torus $T = (\mathbb{C}^*)^n$, as defined in Chapter 8, and the irreducible representations of T.

(2) Derive the character table of the symmetric group S_4. Hint:

$$1^2+1^2+2^2+3^2+3^2 = 24.$$

What is the geometric meaning of the 3-dimensional irreducible representations?

(3) Let $f : V_1 \to V_2$ be a morphism between two representations of a group G.

- Prove that the kernel, image and cokernel of f are also representations.
- Prove that morphisms of two representations are closed under taking scalar multiples and sums, i.e. they form a vector space.

(4) Derive the character table of the symmetric group S_5. Hint:

$$1^2+1^2+4^2+4^2+5^2+5^2+6^2 = 120.$$

Can you write matrices $\rho(g)$ for the 6-dimensional irreducible representation?

(5) Let V_1 and V_2 be two representations of a group G.
   (a) Prove that linear morphisms $\mathrm{Hom}(V_1, V_2)$ have also a structure of a representation. How can you characterize morphisms of representations inside $\mathrm{Hom}(V_1, V_2)$?
   (b) In terms of multiplicities of isotypic components of V_1 and V_2, what is the dimension of the space of morphisms among these two representations?
   (c) Conclude that the multiplicity of an irreducible representation W in V_1 equals the dimension of morphisms of representations $W \to V_1$ (or equivalently of $V_1 \to W$).

(6) Let V be a representation of $\mathrm{GL}(n)$. Its character χ_V is a Laurent polynomial in $t_1, \ldots, t_n$. Show that the vector spaces $S^2(V)$ and $\bigwedge^2 V$ are also representations of $\mathrm{GL}(V)$, and compute the characters $\chi_{S^2(V)}$ and $\chi_{\bigwedge^2 V}$ in terms of χ_V.

(7) Describe the 2-dimensional irreducible representation from Example 10.13 explicitly, by assigning a 2×2 matrix to each of the six permutations of $\{1, 2, 3\}$.

(8) Consider the representation ρ of $\mathrm{GL}(3)$ action on $\bigwedge^3 K^6$? What is the highest weight? What is the associated Young diagram? Find the entries of the 20×20-matrix $\rho(g)$.

(9) Is every $2 \times 2 \times 2$ tensor the sum of a symmetric and a skew-symmetric tensor?

(10) If $U = K^n$, what is the dimension of $S^{\tiny\yng(2,1)}(U)$? Give a formula in terms of n.

(11) What is the dimensions of the vector space $S^3(S^3(K^3))$? Find a weight basis. Write down the character of this representation of $\mathrm{GL}(3)$. Can you decompose it into Schur polynomials?

(12) What are the orbits of points in the adjoint representation? Are they closed? What is the dimension of a general orbit? What is the vanishing ideal of such an orbit, e.g. for $n = 3$?

(13) Show that the representation $\mathbb{C}^2$ in Example 10.9 is not a sum of irreducible representations.

Chapter 11

Invariant Theory

What is geometry? An answer to this question was proposed by Felix Klein's *Erlanger Programm.* According to Klein, a quantity is geometric if it is invariant under the action of an underlying group of transformations. Thus, in short, geometry is invariant theory. For example, Euclidean geometry is the study of quantities, expressed in the coordinates of points, that are invariant under the Euclidean group. From the modern point of view, invariant theory can be seen as a branch of representation theory. However, that view does not do justice to the tremendous utility of invariant theory for dealing with geometric objects. In particular, in algebraic geometry, invariants are used to construct quotients of algebraic varieties modulo groups that act on them. This results in a concise description of orbit spaces. The study of such spaces is called *Geometric Invariant Theory.* Our aim in this chapter is to give a first introduction to this theory, starting with actions by finite groups.

11.1. Finite Groups

We fix the polynomial ring $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ over a field K of characteristic zero. The group $\mathrm{GL}(n, K)$ of invertible $n \times n$ matrices acts on K^n. This induces an action by G on the ring of polynomial functions on K^n. Namely, if $\sigma = (\sigma_{ij})$ is a matrix in $\mathrm{GL}(n, K)$ and f is a polynomial in $K[\mathbf{x}]$ then σf is the polynomial that is obtained from f by replacing the variable x_i by the linear form $\sum_{j=1}^{n} \sigma_{ij} x_j$ for $i = 1, \ldots, n$.

Let G be a subgroup of $\mathrm{GL}(n, K)$. A polynomial $f \in K[\mathbf{x}]$ is an *invariant* of the group G if $\sigma f = f$ for all $\sigma \in G$. We write $K[\mathbf{x}]^G$ for the set of all such invariants. This set is a subring because the sum of two invariants is again an invariant, and same for the product.

In this chapter we discuss two scenarios. In this section we consider finite groups G, and in the next one we consider representations of nice, i.e. reductive, infinite groups like $\mathrm{SL}(d, K)$ and $\mathrm{SO}(d, K)$. A celebrated theorem of Hilbert shows that the invariant ring is finitely generated in this case. After two initial examples, we begin by proving this for finite groups G.

Example 11.1. Let G be the group of $n \times n$ permutation matrices. The invariant ring $K[\mathbf{x}]^G$ consists of all polynomials f that are invariant under permuting the coordinates, i.e.

$$f(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n}) \,=\, f(x_1, x_2, \ldots, x_n) \quad \text{for all permutations } \pi \text{ of } \{1, 2, \ldots, n\}.$$

Such polynomials are called *symmetric*. The invariant ring $K[\mathbf{x}]^G$ is generated by the n elementary symmetric polynomials $E_1, \ldots, E_n$. These are the coefficients of the following auxiliary polynomial in one variable z:

$$(z + x_1)(z + x_2) \cdots (z + x_n) \quad = \quad z^n + \sum_{i=1}^{n} E_i(\mathbf{x}) z^{n-i}. \tag{11.1}$$

We also set $E_0 = 1$. Alternatively, $K[\mathbf{x}]^G$ can also be generated by the power sums

$$P_j(\mathbf{x}) \,=\, x_1^j + x_2^j + \cdots + x_n^j \qquad \text{for } j = 1, 2, \ldots, n.$$

The formulas that connect the E_i and the P_j are known as *Newton's Identities*:

(11.2)

$$\begin{array}{rrcl} & kE_k & = & \sum_{i=1}^{k} (-1)^{i-1} E_{k-i} P_i \\ \text{and} & P_k & = & (-1)^{k-1} k E_k + \sum_{i=1}^{k-1} (-1)^{k-1-i} E_{k-i} P_i \quad \text{for } 1 \leq k \leq n. \end{array}$$

Next, we provide geometric motivation to study the ring of invariants. Suppose a group G acts on an n dimensional vector space K^n. Our aim is to describe the space of orbits Q, i.e. a geometric object, which points correspond to orbits. We are not claiming that such a space has always a structure of a variety, but let us assume this for a moment. Following the approach presented in Chapters 1 and 2 we try to describe Q through polynomial functions on it. By assigning to a point an orbit to which it belongs we expect a map $K^n \to Q$. This provides us with a map from the ring of polynomial functions $K[Q]$ on Q to $K[\mathbf{x}]$. Note that a function on Q gives rise to a polynomial that is constant on the orbits of G. As invariants are exactly the polynomial functions that are constant along G-orbits, we see that $K[Q]$ maps to $K[\mathbf{x}]^G$. Thus invariants offer an algebraic view on the space of orbits. We may define Q as the spectrum of $K[\mathbf{x}]^G$. In the quotient space $Q = K^n /\!/ G$, the (closed) points should correspond to the orbits. This interpretation is only informal, as the details are very subtle. Note in particular, that we have not proved that indeed there is a bijection between (closed) points of Q and orbits. Making it all precise is the aim of *Geometric Invariant Theory.*

Example 11.2. For $n = 2$, consider the following representation of the *cyclic group of order* 4:

$$G \quad = \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{11.3}$$

These are the rotational symmetries of the square. Its invariant ring is generated by

$$I_1 = x_1^2 + x_2^2\,, \quad I_2 = x_1^2 x_2^2\,, \quad I_3 = x_1^3 x_2 - x_1 x_2^3.$$

These three invariants are algebraically dependent. Using their relation we can write

$$K[x_1, x_2]^G \;=\; K[I_1, I_2, I_3] \;\simeq\; K[y_1, y_2, y_3]/\langle y_1^2 y_2 - 4y_2^2 - y_3^2 \rangle. \tag{11.4}$$

The spectrum of the ring (11.4) corresponds to the cubic surface in K^3 defined by the equation $y_1^2 y_2 = 4y_2^2 + y_3^2$. The points on this surface are in one-to-one correspondence with the G-orbits on K^2.

In what follows, let G be a finite subgroup of $\mathrm{GL}(n, K)$. One can create invariants by averaging polynomials. The *Reynolds operator*, denoted by a star, is

$$* \,:\, K[\mathbf{x}] \;\to\; K[\mathbf{x}]^G\,, \quad f \;\mapsto\; f^* \;:=\; \frac{1}{|G|} \sum_{\sigma \in G} \sigma f. \tag{11.5}$$

Each of the following properties of the Reynolds operator is easily verified:

Lemma 11.3. *The Reynolds operators $*$ has the following three properties:*

(a) *The map $*$ is K-linear, i.e. $(\lambda f + \nu g)^* = \lambda f^* + \nu g^*$ for all $f, g \in K[\mathbf{x}]$ and $\lambda, \nu \in K$.*

(b) *The map $*$ restricts to the identity on $K[\mathbf{x}]^G$, i.e. $I^* = I$ for all invariant polynomials I.*

(c) *The map $*$ is a $K[\mathbf{x}]^G$-module homomorphism, i.e. $(fI)^* = f^* I$ for all $f \in K[\mathbf{x}]$ and $I \in K[\mathbf{x}]^G$.*

The following result from 1890 marks the beginning of Commutative Algebra.

Theorem 11.4 (Hilbert's Finiteness Theorem)**.** *The invariant ring $K[\mathbf{x}]^G$ of any finite matrix group $G \subset \mathrm{GL}(n, K)$ is finitely generated as a K-algebra.*

We present the proof under the hypothesis that K has characteristic zero. However, the result holds for every field K. For a proof see [**15**]. This is known as *modular invariant theory.*

Proof. Let $\mathcal{I}_G = \langle K[\mathbf{x}]^G_+ \rangle$ be the ideal in $K[\mathbf{x}]$ that is generated by all homogeneous invariants of positive degree. By Lemma 11.3 (a), every invariant is a K-linear combination of symmetrized monomials $(\mathbf{x}^\mathbf{a})^*$. These homogeneous invariants are the images of monomials under the Reynolds operator. Thus $\mathcal{I}_G$ is generated by the set $\{\, (\mathbf{x}^\mathbf{a})^* \,:\, \mathbf{a} \in \mathbb{N}^n \backslash \{0\}\}$. By Hilbert's Basis Theorem 1.14, the ideal $\mathcal{I}_G$ is finitely generated, so that a finite subset of $\mathbf{a}$ in $\mathbb{N}^n$ suffices. In conclusion, there exist invariants $I_1, I_2, \ldots, I_m$ such that $\mathcal{I}_G = \langle I_1, I_2, \ldots, I_m \rangle$.

We claim that these m invariants generate the invariant ring $K[\mathbf{x}]^G$ as a K-algebra. Suppose the contrary, and let I be a homogeneous element of minimal degree in $K[\mathbf{x}]^G \backslash K[I_1, I_2, \ldots, I_m]$. Since $I \in \mathcal{I}_G$, we have $I = \sum_{j=1}^m f_j I_j$ for some homogeneous polynomials $f_j \in K[\mathbf{x}]$ whose degrees are all strictly less than $\deg(I)$.

Applying the Reynolds operator on both sides of the equation $I = \sum_{j=1}^m f_j I_j$, we obtain

$$I \,=\, I^* \,=\, \bigl(\sum_{j=1}^m f_j I_j\,\bigr)^* \,=\, \sum_{j=1}^m f_j^* I_j.$$

Here we are using the properties (b) and (c) in Lemma 11.3. The new coefficients f_j^* are homogeneous invariants whose degrees are less than $\deg(I)$. From the minimality assumption on the degree of I, we get $f_j^* \in K[I_1, \ldots, I_m]$ for $j = 1, \ldots, m$. This implies $I \in K[I_1, \ldots, I_m]$, which is a contradiction to our assumption. This completes the proof of Theorem 11.4. $\square$

Theorem 11.5 (Noether's Degree Bound). *If G is finite and* $\mathrm{char}(K) = 0$ *then the invariant ring $K[\mathbf{x}]^G$ is generated by homogeneous invariants of degree $\leq |G|$.*

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be new variables. For any $d \in \mathbb{N}$, we consider the expression

$$\begin{array}{rcl} S_d(\mathbf{u}, \mathbf{x}) & = & \bigl[\,(u_1x_1 + \cdots + u_nx_n)^d\,\bigr]^* \\ & = & \frac{1}{|G|}\sum_{\sigma\in G}\bigl[u_1(\sigma x_1) + \cdots + u_n(\sigma x_n)\bigr]^d. \end{array}$$

This is a polynomial in $\mathbf{u}$ whose coefficients are polynomials in $\mathbf{x}$. Up to a multiplicative constant, they are the invariants $(\mathbf{x}^\mathbf{a})^*$ where $|\mathbf{a}| = d$. All polynomials in $\mathbf{u}$ are fixed under $*$.

Consider the $|G|$ expressions $u_1(\sigma x_1)+\cdots+u_n(\sigma x_n)$, one for each group element $\sigma \in G$. The polynomial $S_d(\mathbf{u}, \mathbf{x})$, up to a multiplicative constant, is the dth power sum of these expressions. The power sums for $d > |G|$ are polynomials in the first $|G|$ power sums. Such a representation is derived from Newton's Identities (11.2). It implies that all $\mathbf{u}$-coefficients of $S_d(\mathbf{u}, \mathbf{x})$ for $d > |G|$ are polynomial functions in the $\mathbf{u}$-coefficients of $S_d(\mathbf{u}, \mathbf{x})$ for $d \leq |G|$. Hence all invariants $(\mathbf{x}^{\mathbf{a}})^*$ with $|\mathbf{a}| > |G|$ are polynomial functions (over K) in the invariants $(\mathbf{x}^{\mathbf{b}})^*$ with $|\mathbf{b}| \leq |G|$. This proves the claim. □

We note that Example 11.2 attains Noether's degree bound. The cyclic group in that example has order 4, and the invariant ring requires a generator of degree 4.

Our next theorem is a useful tool for constructing the invariant ring. It says that we can count invariants by averaging the reciprocal characteristic polynomials of the group elements.

Theorem 11.6 (Molien)**.** *The Hilbert series of the invariant ring $K[\mathbf{x}]^G$ equals*

$$\sum_{d=0}^{\infty} \dim_K\big(K[\mathbf{x}]_d^G\big) z^d \quad = \quad \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det(\mathrm{id} - z\sigma)}. \tag{11.6}$$

The coefficient of z^d in this formal generating function is the number of linearly independent invariants of degree d.

Proof. See [**51**, Theorem 2.2.1]. □

Example 11.7. Consider the cyclic group $G = \mathbb{Z}_4$ in Example 11.2. For the four matrices σ in Example 11.2, the quadratic polynomials $\det(\mathrm{id} - z\sigma)$ are $(1-z)^2$, $(1+z)^2$ and twice $1+z^2$. Adding up their reciprocals and dividing by $|G| = 4$, we see that the Hilbert series of $K[\mathbf{x}]^G$ is

$$\frac{1 + z^4}{(1 - z^2)(1 - z^4)} \quad = \quad 1 + z^2 + 3z^4 + 3z^6 + 5z^8 + \cdots . \tag{11.7}$$

This agrees with the Hilbert series of the ring on the right in (11.4), where $\deg(y_1) = 2$ and $\deg(y_2) = \deg(y_3) = 4$. We see that every element of the ring can be uniquely represented as a polynomial in y_1 and y_2 or y_3 times such a polynomial. One says that the ring is a free module with basis $\{1, y_3\}$ over $K[y_1, y_2]$. This explains the numerator and denominator on the left of (11.7), and it proves that I_1, I_2, I_3 do indeed generate $K[\mathbf{x}]^G$.

11.2. Classical Invariant Theory

Hilbert's Finiteness Theorem also holds for an infinite group $G \subset \mathrm{GL}(n, K)$ that has a Reynolds operator $*$ satisfying the properties (a), (b) and (c) in Lemma 11.3. This is the case when G is *reductive.* Indeed, homogeneous polynomials of degree d in n variables form a representation $K[\mathbf{x}]_d$ and

$K[\mathbf{x}]_d^G$ is a subrepresentation. If we know there exists a complementary subrepresentation H such that $H \oplus K[\mathbf{x}]_d^G = K[\mathbf{x}]_d$ we may define $*$ as a projection with kernel H. Summing over all possible d and extending $*$ linearly we obtain the Reynolds operator.

Corollary 11.8. *Fix a reductive group G of $n\times n$-matrices. If $\{g_1, g_2, \ldots, g_m\}$ is a set of homogeneous polynomials that generates the ideal $\mathcal{I}_G$ then its image $\{g_1^*, g_2^*, \ldots, g_m^*\}$ under the Reynolds operator generates the invariant ring $K[\mathbf{x}]^G$ as a K-algebra.*

Proof. Let $M = \langle x_1, \ldots, x_n\rangle$ be the homogeneous maximal ideal in $K[\mathbf{x}]$, and consider the finite-dimensional vector space $\mathcal{I}_G/M\mathcal{I}_G$. It has a basis of invariants since $\mathcal{I}_G$ is generated by invariants. This means that the Reynolds operator acts as the identity on $\mathcal{I}_G/M\mathcal{I}_G$. The images of $g_1, g_2, \ldots, g_m$ also span $\mathcal{I}_G/M\mathcal{I}_G$ as a vector space, and hence so do the invariants $g_1^*, g_2^*, \ldots, g_m^*$. By Nakayama's Lemma, we find that $g_1^*, g_2^*, \ldots, g_m^*$ generate the ideal $\mathcal{I}_G$. As in the proof of Theorem 11.4, we conclude that $g_1^*, g_2^*, \ldots, g_m^*$ generate the K-algebra $K[\mathbf{x}]^G$. □

Classical invariant theory was primarily concerned with the case when G is a representation of the group $\mathrm{SL}(d, K)$ of $d \times d$-matrices with determinant 1. Here d is an integer that is usually much smaller than n and K is a field of characteristic zero. This means that G is the image of a group homomorphism $\mathrm{SL}(d, K) \to \mathrm{GL}(n, K)$. It is known that $\mathrm{SL}(d, K)$ is a reductive group, i.e. there also exists an averaging operator $*: K[\mathbf{x}] \to K[\mathbf{x}]^G$ which has the same formal properties as the averaging operator of a finite group, stated in Lemma 11.3.

That Reynolds operator $*$ can be realized either by integration or by differentiating. In the first realization, one replaces the sum in (11.5) by an integral. Namely, one takes $K = \mathbb{C}$ and one integrates over the compact subgroup $\mathrm{SU}(d, \mathbb{C})$ with respect to Haar measure. The same kind of integral also works in Theorem 11.6. If $G = \mathrm{SL}(d, \mathbb{C})$ then one can compute the Hilbert series of the invariant ring by averaging reciprocal characteristic polynomials.

An alternative to integrating with respect to Haar measure on $\mathrm{SU}(d, \mathbb{C})$ is a certain differential operator known as *Cayley's Ω-process.* This process, which is explained in [**51**, Section 4.3], can also be used to transform arbitrary polynomials into invariants.

A third method for computing invariants is plain old linear algebra. Indeed, suppose we fix an integer $d \in \mathbb{N}$ and we seek a basis for the space $K[\mathbf{x}]_d^G$ of homogeneous invariants of degree d. We then pick a general polynomial f of degree d with unknown coefficients, and we examine the equations

$\sigma f = f$ for $\sigma \in G$. Each of these translates into a linear system of equations in the unknown coefficients of f. By taking enough matrices σ, we obtain a linear system of equations whose solutions are precisely the invariants of degree d. In the case when G is a connected Lie group, like $\mathrm{SL}(d, \mathbb{C})$, one can replace the condition $\sigma f = f$ by requiring that f is annihilated by the associated *Lie algebra*. Setting up these linear equations and solving them is usually quite efficient on small examples. See [**51**, Section 4.5].

In what follows we take the matrix group to be an n-dimensional polynomial representation of $G = \mathrm{SL}(d, K)$ for some $d, n \in \mathbb{N}$. Each of these is a direct sum of irreducible representations, one for each integer partition, as seen in Chapter 10.

Example 11.9. Let $U = (K^d)^m$ be the space of $d \times m$-matrices. Thus U is the direct sum of m copies of the defining representation of G. The group G acts on U by matrix multiplication on the left. This induces an action on the ring $K[U]$ of polynomials in the entries of a $d \times m$ matrix of variables. If $m < d$ then this action has no non-constant invariants. If $m \geq d$ then the $\binom{m}{d}$ maximal minors of the $d \times m$ matrix are invariants. This invariance holds because the determinant of the product of two $d \times d$-matrices is the product of the determinants. It is known that the invariant ring $K[U]^G$ is generated by these $\binom{m}{d}$ determinants. This result is the First Fundamental Theorem of Invariant Theory; cf [**51**, Section 3.2].

Note that we already encountered the ring $K[U]^G$ in Chapter 5. It is the coordinate ring of the Grassmannian of d-dimensional subspaces in K^m. Thus, $K[U]^G$ is isomorphic to a polynomial ring in $\binom{m}{d}$ variables, modulo the ideal of quadratic Plücker relations.

Arguably, the most important irreducible representations of the group $G = \mathrm{SL}(d, K)$ are the p-th symmetric powers of the defining representation K^d, where $p \in \mathbb{N}$. We denote such a symmetric power by $V = K[u_1, \ldots, u_d]_p = \mathrm{Sym}_p(K^d)$. Its elements are homogeneous polynomials of degree p in d variables. The G-module V has dimension $n = \binom{p+d-1}{p}$. The monomials form a basis. The action of G on V is simply by linear change of coordinates.

Example 11.10 (d=2, p=3)**.** Fix the space $V = \mathrm{Sym}_3(K^2)$ of binary cubics

$$f(u_1, u_2) \quad = \quad x_1 u_1^3 \,+\, x_2 u_1^2 u_2 \,+\, x_3 u_1 u_2^2 \,+\, x_4 u_2^3. \tag{11.8}$$

The coefficients x_i are the coordinates on $V \simeq K^4$. The way we set things up, the group $\mathrm{SL}(2, K)$ acts on this space by left multiplication, in its guise

as the group G of 4×4-matrices of the form
(11.9)

$$\phi(\sigma) \;=\; \begin{pmatrix} \sigma_{11}^3 & \sigma_{11}^2\sigma_{12} & \sigma_{11}\sigma_{12}^2 & \sigma_{12}^3 \\ 3\sigma_{11}^2\sigma_{21} & \sigma_{11}^2\sigma_{22} + 2\sigma_{11}\sigma_{12}\sigma_{21} & \sigma_{12}^2\sigma_{21} + 2\sigma_{11}\sigma_{12}\sigma_{22} & 3\sigma_{12}^2\sigma_{22} \\ 3\sigma_{11}\sigma_{21}^2 & \sigma_{12}\sigma_{21}^2 + 2\sigma_{11}\sigma_{21}\sigma_{22} & \sigma_{11}\sigma_{22}^2 + 2\sigma_{12}\sigma_{21}\sigma_{22} & 3\sigma_{12}\sigma_{22}^2 \\ \sigma_{21}^3 & \sigma_{21}^2\sigma_{22} & \sigma_{21}\sigma_{22}^2 & \sigma_{22}^3. \end{pmatrix}.$$

For $\sigma \in G = \mathrm{SL}(2, K)$, the determinant of this 4×4-matrix equals $(\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21})^6 = 1$. The G-action on V is given by $x \mapsto \phi(\sigma)x$ where x is the column vector $(x_1, x_2, x_3, x_4)^T$. One invariant under this action is the discriminant of the binary cubic $f(u_1, u_2)$, which is

$$(11.10) \qquad \Delta \;=\; 27x_1^2x_4^2 \;-\; 18x_1x_2x_3x_4 \;+\; 4x_1x_3^3 \;+\; 4x_2^3x_4 \;-\; x_2^2x_3^2.$$

It turns out that the discriminant generates the invariant ring, i.e. $K[\mathbf{x}]^G = K[\Delta]$.

Invariants of binary forms ($d = 2$) are a well-studied subject in invariant theory. Complete lists of generators for the invariant ring are known up to degree $p = 10$. For $p = 2$, there is also only the discriminant $\Delta = x_2^2 - 4x_1x_3$. For $p = 4$, we have two generating invariants of degree 2 and 3 respectively. For $p = 10$, the invariant ring has 106 minimal generators.

11.3. Geometric Invariant Theory

According to Felix Klein, invariant theory plays a fundamental role for geometry. Namely, a polynomial in the coordinates of a space is invariant under the group of interest if and only if that polynomial expresses a geometric property. For instance, consider the space V of binary cubics f in Example 11.10. The hypersurface defined by f in $\mathbb{P}^1$ consists of three points. The vanishing of the invariant Δ means that these three points are not all distinct.

In geometric invariant theory, one considers the variety $\mathcal{V}(\mathcal{I}_G)$ defined by all homogeneous invariants of positive degree. This variety is known as the *nullcone*. Its points are known as *unstable points*. For a finite group G, the nullcone consists just of the origin, $V(\mathcal{I}_G) = \{0\}$. For $G = \mathrm{SL}(d, K)$ the situation is more interesting, and the geometry of the nullcone is very important for understanding the invariant ring $K[\mathbf{x}]^G$. Corollary 11.8 says, more or less, that computing $K[\mathbf{x}]^G$ is equivalent to finding polynomial equations that define the nullcone.

Example 11.11 (d=p=3)**.** Consider the 10-dimensional space $V = \mathrm{Sym}_3(K^3)$ of *ternary cubics*

$$\begin{aligned} f(\mathbf{u}) \;=\; & \quad x_1u_1^3 + x_2u_2^3 + x_3u_3^3 + x_4u_1^2u_2 + x_5u_1^2u_3 + \\ & x_6u_2^2u_1 + x_7u_2^2u_3 + x_8u_3^2u_1 + x_9u_3^2u_2 + x_0u_1u_2u_3. \end{aligned}$$

The group $G = \mathrm{SL}(3, K)$ acts on V by linear change of coordinates. The corresponding invariant ring is generated by two invariants I_4 and I_6 of degrees 4 and 6 respectively. In symbols, $K[\mathbf{x}]^G = K[I_4, I_6]$. The degree 4 invariant is the following sum of 25 monomials:

$$\begin{aligned} I_4 \quad = \quad & x_0^4 - 8x_0^2x_4x_9 - 8x_0^2x_5x_7 - 8x_0^2x_6x_8 - 216x_0x_1x_2x_3 + 24x_0x_1x_7x_9 \\ & +24x_0x_2x_5x_8 + 24x_0x_3x_4x_6 + 24x_0x_4x_7x_8 + 24x_0x_5x_6x_9 \\ & +144x_1x_2x_8x_9 + 144x_1x_3x_6x_7 - 48x_1x_6x_9^2 - 48x_1x_7^2x_8 + 144x_2x_3x_4x_5 \\ & -48x_2x_4x_8^2 - 48x_2x_5^2x_9 - 48x_3x_4^2x_7 - 48x_3x_5x_6^2 + 16x_4^2x_9^2 \\ & -16x_4x_5x_7x_9 - 16x_4x_6x_8x_9 + 16x_5^2x_7^2 - 16x_5x_6x_7x_8 + 16x_6^2x_8^2. \end{aligned}$$

The degree 6 invariant is also unique up to scaling. It is a sum of 103 monomials:

$$I_6 = x_0^6 - 12x_0^4x_4x_9 - 12x_0^4x_5x_7 - 12x_0^4x_6x_8 + 540x_0^3x_1x_2x_3 + \cdots + 96x_5x_6^2x_7x_8^2 - 64x_6^3x_8^3.$$

The invariant I_4 is the *Aronhold invariant*. This plays an important role in the theory of tensor decomposition. Indeed, we can regard f as a symmetric $3 \times 3 \times 3$-tensor. A random tensor f has rank 4. The Aronhold invariant f vanishes for those tensors of border rank ≤ 3. In other words, $I_4 = 0$ holds if and only if f is a sum of three cubes of linear forms, or can be approximated by a sequence of such. See the discussion of ranks of tensors in Chapter 9.

On the geometric side, we identify f with the cubic curve $V(f)$ it defines in the projective plane $\mathbb{P}^2$. To a number theorist, this is an *elliptic curve*. An important invariant of this curve is the *discriminant* Δ. This invariant has degree 12 and its explicit formula equals

$$\Delta \quad = \quad I_4^3 \; - \; I_6^2. \tag{11.11}$$

This expression vanishes if and only if the curve $V(f)$ has a singular point. Typically, this singularity is a *node*. In the special case when both I_4 and I_6 vanish, that singular point is a *cusp*. Thus, for ternary cubics, the nullcone $\mathcal{V}(\mathcal{I}_G)$ is given by plane cubics that have a cusp. The moduli space of elliptic curves is parametrized by the *j-invariant*, which equals I_4^3/Δ.

We now present a general-purpose algorithm, due to Harm Derksen, for computing the invariant ring of a reductive algebraic group G that acts polynomially on a vector space $V = K^n$. The group G can represented as an algebraic variety inside $\mathrm{GL}(n, K)$, that is, by polynomial equations in the entries of an unknown $n \times n$-matrix. This works for both finite groups and for polynomial representations of $\mathrm{SL}(d, K)$, such as the ones discussed about. As before, we use the notation $\sigma \mapsto \phi(\sigma)$ to write the representation of G on $V = K^n$ explicitly.

The product $G \times V \times V$ is an algebraic variety, with coordinates $(\sigma, \mathbf{x}, \mathbf{y})$. Inside its coordinate ring $K[\sigma, \mathbf{x}, \mathbf{y}]$, let $\mathcal{J}_G$ be the ideal generated by the n entries of the vector $\mathbf{y} - \phi(\sigma)\mathbf{x}$. This ideal is radical, and it is prime when

G is a connected group like $\mathrm{SL}(d, K)$. Its variety describes the action of the group. The elimination ideal $\mathcal{J}_G \cap K[\mathbf{x}, \mathbf{y}]$ is also radical (resp. prime). Its variety contains pairs of points in V that lie in the same G-orbit.

Theorem 11.12 (Derksen's Algorithm). *The ideal $\mathcal{I}_G$ of the nullcone is the image in $K[\mathbf{x}]$ of the elimination ideal $\mathcal{J}_G \cap K[\mathbf{x}, \mathbf{y}]$ under the substitution $\mathbf{y} = 0$. From any finite list of ideal generators of $\mathcal{I}_G$, algebra generators for the invariant ring $K[\mathbf{x}]^G$ are found via Corollary 11.8.*

Proof. Let I be any homogeneous invariant of positive degree. Then $I(\mathbf{x}) \equiv I(\phi(\sigma)\mathbf{x}) \equiv I(\mathbf{y})$ modulo the ideal $\mathcal{J}_G$ that defines the group action. Therefore, $I(\mathbf{x}) - I(\mathbf{y})$ lies in the elimination ideal $\mathcal{J}_G \cap K[\mathbf{x}, \mathbf{y}]$, and we find $I(\mathbf{x})$ in the ideal that is obtained by substituting $\mathbf{y} = 0$. This proves that $\mathcal{I}_G$ is contained in the ideal that is computed by Derksen's Algorithm. For the converse direction, we refer to the argument given in the proof of [**14**, Theorem 3.1]. □

Example 11.13 (p=d=2). Consider the 3-dimensional space $V = \mathrm{Sym}_2(K^2)$ of binary quadrics

$$f(u_1, u_2) \quad = \quad x_1 u_1^2 \,+\, x_2 u_1 u_2 \,+\, x_3 u_2^2.$$

The coordinate ring of the variety $\mathrm{SL}(2, K) \times V \times V$ is the polynomial ring

$$K[\sigma, \mathbf{x}, \mathbf{y}] \quad = \quad K\big[\, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22},\, x_1, x_2, x_3,\, y_1, y_2, y_3 \,\big]$$

modulo the principal ideal $\langle \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} - 1 \rangle$. Note that this ring has 10 generators. The ideal that encodes our action equals

$$\begin{aligned} \mathcal{J}_G \quad = \quad \big\langle\, & \sigma_{11}^2 x_1 + \sigma_{11}\sigma_{21} x_2 + \sigma_{21}^2 x_3 \,-\, y_1,\, \sigma_{12}^2 x_1 + \sigma_{12}\sigma_{22} x_2 \sigma_{22}^2 x_3 - y_3, \\ & 2\sigma_{11}\sigma_{12} x_1 + (\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}) x_2 + 2\sigma_{21}\sigma_{22} x_3 \,-\, y_2 \,\big\rangle \end{aligned}$$

Elimination of the four variables for the group elements yields the principal ideal

$$\mathcal{J}_G \cap K[\mathbf{x}, \mathbf{y}] \quad = \quad \langle\, 4x_1x_3 - x_2^2 \,-\, 4y_1y_3 + y_2^2 \,\rangle.$$

We now set $y_1 = y_2 = y_3 = 0$. The result is the familiar discriminant $\Delta = 4x_1x_3 - x_2^2$. In this manner, Derksen's Algorithm finds the invariant ring for binary quadrics $K[\mathbf{x}]^G = K[\Delta]$.

In Example 11.10, we determined the invariant ring for $\mathrm{SL}(2, K)$ acting on $2 \times 2 \times 2$ tensors that are symmetric. In what follows, we extend this computation to non-symmetric tensors. Thus, we present case study in invariant theory for $d = 2$ and $n = 8$. We identify K^8 with the space $(K^2)^{\otimes 3}$ of $2 \times 2 \times 2$-tensors. The corresponding polynomial ring is denoted by

$$K[\mathbf{x}] \;=\; K[x_{111}, x_{112}, x_{121}, x_{122}, x_{211}, x_{212}, x_{221}, x_{222}].$$

The group $G = \mathrm{SL}(2, K)$ acts on K^2 by matrix-vector multiplication. This action extends naturally to the triple tensor product of K^2. Explicitly, if

$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is a 2×2-matrix in G then σ acts by performing the following substitution in each polynomial on $K[\mathbf{x}]$:

$$x_{ijk} \ \mapsto \ \sum_{r=1}^{2}\sum_{s=1}^{2}\sum_{t=1}^{2} x_{rst}\sigma_{ri}\sigma_{sj}\sigma_{tk}. \tag{11.12}$$

Here are two nice polynomials that are invariant under this action:

Example 11.14. Up to scaling, there is a unique polynomial of degree 2 that is invariant under $G = \mathrm{SL}(2, K)$. That invariant is the following quadric, which we call the *hexagon invariant*:

$$\mathrm{Hex}(\mathbf{x}) \ = \ x_{112}x_{122} - x_{122}x_{121} + x_{121}x_{221} - x_{221}x_{211} + x_{211}x_{212} - x_{212}x_{112}.$$

Another nice invariant is homogeneous of degree four. This is the *hyperdeterminant*

$$\begin{aligned} \mathrm{Det}(\mathbf{x}) \ = \ & x_{221}^2x_{112}^2 + x_{211}^2x_{122}^2 + x_{121}^2x_{212}^2 + x_{111}^2x_{222}^2 \\ & +4x_{111}x_{221}x_{122}x_{212} + 4x_{121}x_{211}x_{112}x_{222} \\ & -2x_{211}x_{221}x_{112}x_{122} - 2x_{121}x_{221}x_{112}x_{212} - 2x_{121}x_{211}x_{122}x_{212} \\ & -2x_{111}x_{221}x_{112}x_{222} - 2x_{111}x_{211}x_{122}x_{222} - 2x_{111}x_{121}x_{212}x_{222}. \end{aligned}$$

One checks by computation that the substitution (11.12) maps the hexagon invariant $\mathrm{Hex}(\mathbf{x})$ to itself times the third power of $\det(\sigma) = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}$. Similary, the hyperdeterminant $\mathrm{Det}(\mathbf{x})$ transforms to itself times $\det(\sigma)^6$. Hence both are invariant when $\det(\sigma) = 1$.

Invariants can be used to test whether two tensors lie in the same orbit. Here is a concrete example. We write our $2 \times 2 \times 2$ tensors as vectors in $\mathbb{R}^8$ as follows: $\mathbf{c} = (c_{111}, c_{112}, c_{121}, c_{122}, c_{211}, c_{212}, c_{221}, c_{222})$. The following two tensors appear in the theory of signatures of paths. It is of interest to know whether their G-orbits agree up to scaling:

$$\mathbf{c}_{\text{axis}} \ = \ \left(\tfrac{1}{6}\,,\,\tfrac{1}{2}\,,\,0\,,\,\tfrac{1}{2}\,,\,0\,,\,0\,,\,0\,,\,\tfrac{1}{6}\right) \quad \text{and} \quad \mathbf{c}_{\text{mono}} \ = \ \left(\tfrac{1}{6}\,,\,\tfrac{1}{4}\,,\,\tfrac{1}{6}\,,\,\tfrac{4}{15}\,,\,\tfrac{1}{12}\,,\,\tfrac{2}{15}\,,\,\tfrac{1}{10}\,,\,\tfrac{1}{6}\right).$$

The two polynomials in Example 11.14 are relative invariants of the $\mathrm{GL}(2)$ action on the tensor space $\mathbb{R}^8$. The following rational function is an absolute invariant. It is homogeneous of degree zero, so it represents an invariant rational function on the projective space $\mathbb{P}^7$:

$$\frac{\mathrm{Hex}(\mathbf{x})^2}{\mathrm{Det}(\mathbf{x})}. \tag{11.13}$$

We find that the invariant (11.13) evaluates to 81 on $\mathbf{c}_{\text{axis}}$, and it evaluates to 45 on $\mathbf{c}_{\text{mono}}$. Hence the orbit closures of our two special core tensors of format $2 \times 2 \times 2$ are disjoint in $\mathbb{P}^7$.

We now come to determination of the full ring of invariants for the G-action on the space K^8 of $2 \times 2 \times 2$ tensors. Using Derksen's Algorithm, we derive:

Theorem 11.15. *The invariant ring $K[\mathbf{x}]^{\mathrm{SL}(2)}$ of $2 \times 2 \times 2$ tensors has Krull dimension five. It is minimally generated by 13 invariants, namely the hexagon invariant of degree two, eight invariants of degree four (including the hyperdeterminant), and four invariants of degree six.*

In addition to the hyperdeterminant, there are three additional invariants of degree four that deserve special attention. Each has 17 terms when expanded. One of these invariants is

$$\begin{aligned}(x_{111}x_{222} - x_{212}x_{121})^2 \ + x_{121}x_{222}x_{112}^2 + x_{111}x_{212}x_{122}^2 + x_{121}x_{222}x_{211}^2 \\ +x_{111}x_{212}x_{221}^2 - (x_{122} + x_{221})(x_{112} + x_{211})(x_{111}x_{222} + x_{212}x_{121}) \\ + 2x_{111}x_{122}x_{212}x_{221} + 2x_{112}x_{121}x_{211}x_{222}.\end{aligned} \tag{11.14}$$

The other two invariants in this family are obtained by permuting indices.

Corollary 11.16. *The three quartics in (11.14) together with* Hex *and* Det *form an algebraically independent system of five primary invariants. All other invariants in $K[\mathbf{x}]^{\mathrm{SL}(2)}$ are integral over the polynomial subring generated by these five. The five primary invariants cut out the null cone $\mathcal{V}\big(K[\mathbf{x}]_+^{\mathrm{SL}(2)}\big)$, which is a variety of dimension four and degree 12 in $\mathbb{P}^7$.*

It is instructive to restrict the 13 generating invariants in Theorem 11.15 to the 4-dimensional subspace $\mathrm{Sym}_3(K^2)$ of symmetric $2 \times 2 \times 2$ tensors, seen in Example 11.10. We do this by setting

$$x_{111} = x_1\,,\ \ x_{112} = x_{121} = x_{211} = \frac{1}{3}x_2\,,\ \ x_{122} = x_{212} = x_{221} = \frac{1}{3}x_3\,,\ \ x_{222} = x_4.$$

The resulting symmetric tensors correspond to binary cubics (11.8). The hyperdeterminant and five other generators of degree four specialize to the *discriminant* Δ of the binary cubic. The other eight generators of $K[\mathbf{x}]^{\mathrm{SL}(2)}$, including the hexagon invariant, specialize to zero. In this manner, the invariant ring in Theorem 11.15 maps onto the invariant ring of binary cubics.

Exercises

(1) Let G be the symmetry group of the square $[-1,1]^2$ in the plane $\mathbb{R}^2$. This is an order 8 subgroup in $\mathrm{GL}(2,\mathbb{R})$. List all eight matrices. Determine the invariant ring $\mathbb{R}[x_1,x_2]^G$.

(2) Let G be the symmetry group of the regular 3-cube, as a subgroup of $\mathrm{GL}(3,\mathbb{R})$. How many matrices are in G, and what are their characteristic polynomials? Determine the Molien series (11.7) of this group. What does it tell you about the invariant ring?

(3) Fix $n = 5$. Let $\psi(j)$ denote the number of monomials in the expansion of the power sum P_j in terms of the elementary symmetric functions E_1, E_2, E_3, E_4, E_5. Compute $\psi(j)$ for some small values, say $j \leq 20$. Guess a formula for $\psi(j)$. Can you prove it?

(4) Show that Noether's Degree Bound is always tight for finite cyclic groups.

(5) Find a subgroup of $\mathrm{GL}(4,K)$ that has order 15. Compute the invariant ring.

(6) Let T be the group of 3×3 diagonal matrices with determinant 1, acting on the space $V = \mathrm{Sym}_3(K^3)$ of ternary cubics. This group is the torus $T \simeq (K^*)^2$. Determine the invariant ring $K[V]^T$. Do you see any relationship to the invariants in Example 11.11?

(7) Let $G = A_n$ be the *alternating group* of order $n!/2$. Its elements are the even permutation matrices. Determine the invariant ring $K[\mathbf{x}]^G$.

(8) List all 103 monomials of the invariant I_6 of ternary cubics in Example 11.11. Give an explicit formula, in terms of $x_1, x_2, \ldots, x_9, x_0$, for the discriminant and the j-invariant.

(9) Consider the action of $\mathrm{SL}(3,K)$ on the space $\mathrm{Sym}_2(K^3) \simeq K^6$ of symmetric 3×3-matrices. The entries of the 6×6 matrix $\phi(\sigma)$ are quadratic forms in $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{33}$. Write this matrix explicitly, similarly to (11.9). What is the invariant ring?

(10) Using Derksen's Algorithm, determine the invariant ring for binary quartics ($d = 2, p = 4$). How many minimal generators does this ring have?

(11) The rotation group $\mathrm{SO}(2,\mathbb{R})$ acts by left multiplication on the space of 2×2-matrices. Determine the invariant ring.

(12) Is the invariant ring of every matrix group $G \subset \mathrm{GL}(n,K)$ finitely generated?

Chapter 12

Semidefinite Programming

The transition from linear algebra to nonlinear algebra has a natural counterpart in convex optimization, namely the passage from linear programming to semidefinite programming. This transition is the topic of this chapter. *Linear programming* concerns the solution of linear systems of inequalities, and the optimization of linear functions subject to linear constraints. The feasible region is a *convex polyhedron*, and the optimal solutions form a face of that polyhedron. In *semidefinite programming* we work in the space of symmetric $n \times n$-matrices. The inequality constraints now stipulate that some linear combination of matrices be positive semidefinite. The feasible region given by such constraints is a closed convex set, known as a *spectrahedron*. We again wish to optimize a linear function. The condition for a polynomial to be a sum of squares may be regarded as a semidefinite program. This furnishes a connection to the real Nullstellensatz (Chapter 6), thereby establishing semidefinite programming as a key tool for computing in real algebraic geometry.

12.1. Spectrahedra

In this chapter we work over the field $\mathbb{R}$ of real numbers. The Spectral Theorem in Linear Algebra states that all eigenvalues of a symmetric matrix $A \in \mathrm{Sym}_2(\mathbb{R}^n)$ are real. Moreover, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A. We say that the matrix A is *positive definite* if it satisfies the following conditions. It is a basic fact about quadratic forms that these three conditions are equivalent:

(1) All n eigenvalues of A are positive real numbers.

(2) All 2^n principal minors of A are positive real numbers.

(3) Every non-zero column vector $\mathbf{u} \in \mathbb{R}^n$ satisfies $\mathbf{u}^T A \mathbf{u} > 0$.

Here, by a *principal minor* we mean the determinant of any square submatrix of A whose set of column indices agree with its set of row indices. For the empty set, we get the 0×0 minor of A, which equals 1. Next there are the n diagonal entries of A, which are the 1×1 principal minors, and finally the determinant of A, which is the unique $n \times n$ principal minor. Each of the three conditions (1), (2) and (3) behaves as expected when we pass to the closure. This is not obvious because the closure of an open semialgebraic set $\{f > 0\}$, where $f \in \mathbb{R}[\mathbf{x}]$, is generally smaller than the corresponding closed semialgebraic set $\{f \geq 0\}$.

Example 12.1. Let $f = x^3 + x^2y + xy^2 + y^3 - x^2 - y^2$. The set $\{f > 0\}$ is the open halfplane above the line $x + y = 1$ in $\mathbb{R}^2$. The closure of the set $\{f > 0\}$ is the corresponding closed halfplane. It is properly contained in $\{f \geq 0\}$ which also contains the origin $(0, 0)$.

Luckily, no such thing happens with condition (2) for positive definite matrices.

Theorem 12.2. *For a symmetric $n \times n$ matrix A, the following three conditions are equivalent:*

(1') *All n eigenvalues of A are nonnegative real numbers.*

(2') *All 2^n principal minors of A are nonnegative real numbers.*

(3') *Every non-zero column vector $\mathbf{u} \in \mathbb{R}^n$ satisfies $\mathbf{u}^T A \mathbf{u} \geq 0$.*

If this holds then A is called positive semidefinite*. The semialgebraic set* PSD_n *of positive semidefinite $n \times n$ matrices is a full-dimensional closed convex cone in* $\mathrm{Sym}_2(\mathbb{R}^n)$*.*

We use the notation $X \succeq 0$ to express that a symmetric matrix X is positive semidefinite. A *spectrahedron* $\mathcal{S}$ is the intersection of the cone PSD_n with an affine-linear subspace $\mathcal{L}$ of the ambient space $\mathrm{Sym}_2(\mathbb{R}^n)$. Hence, spectrahedra are closed convex semialgebraic sets.

A subspace $\mathcal{L}$ of symmetric matrices is either given parametrically, or as the solution set to an inhomogeneous system of linear equations. In the equational representation, we write

(12.1)
$$\mathcal{L} \;=\; \bigl\{ X \in \mathrm{Sym}_2(\mathbb{R}^n) \,:\, \langle A_1, X\rangle = b_1,\, \langle A_2, X\rangle = b_2,\, \ldots,\, \langle A_s, X\rangle = b_s \bigr\}.$$

Here $A_1, A_2, \ldots, A_s \in \mathrm{Sym}_2(\mathbb{R}^n)$ and $b_1, b_2, \ldots, b_s \in \mathbb{R}$ are fixed. We employ the standard inner product in the space of square matrices, which is given by the trace of the matrix product:

$$\langle A, X\rangle \;:=\; \operatorname{trace}(AX) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij}. \tag{12.2}$$

The associated spectrahedron $\mathcal{S} = \mathcal{L} \cap \mathrm{PSD}_n$ consists of all positive semidefinite matrices that lie in the subspace $\mathcal{L}$. If the subspace is given by a parametric representation, say

$$\mathcal{L} \;=\; \bigl\{A_0 + x_1A_1 + \cdots + x_sA_s \,:\, (x_1,\ldots,x_s) \in \mathbb{R}^s\bigr\}, \tag{12.3}$$

then it is customary to identify the spectrahedron with its preimage in $\mathbb{R}^s$. Hence,

$$\mathcal{S} \;=\; \bigl\{(x_1,\ldots,x_s) \in \mathbb{R}^s \,:\, A_0 + x_1A_1 + \cdots + x_sA_s \succeq 0\bigr\}. \tag{12.4}$$

Proposition 12.3. *Every convex polyhedron is a spectrahedron. Convex polyhedra are precisely the spectrahedra that arise when the subspace $\mathcal{L}$ consists only of diagonal $n \times n$ matrices.*

Proof. Suppose that the matrices $A_0, A_1, \ldots, A_s$ are diagonal matrices. Then (12.4) is the solution set in $\mathbb{R}^s$ of a system of n inhomogeneous linear inequalities. Such a set is a convex polyhedron. Every convex polyhedron in $\mathbb{R}^s$ has such a representation. We simply write its defining linear inequalities as the diagonal entries of the matrix $A_0 + x_1A_1 + \cdots + x_sA_s$.

The formula $\mathcal{S} = \mathcal{L} \cap \mathrm{PSD}_n$ with $\mathcal{L}$ as in (12.1) corresponds to the standard representation of a convex polyhedron, as the set of non-negative points in an affine-linear space. Here the equations in (12.1) include those that require the off-diagonal entries of all matrices to be zero:

$$\langle X, E_{ij}\rangle \;=\; x_{ij} \;=\; 0 \quad \text{for } i \neq j.$$

In the other inequalities, the matrices A_i are diagonal and the b_i are typically nonzero. □

Example 12.4. Let $\mathcal{L}$ be the space of symmetric 3×3 matrices whose three diagonal entries are all equal to 1. This is an affine-linear subspace of dimension $s = 3$ in $\mathrm{Sym}_2(\mathbb{R}^3) \simeq \mathbb{R}^6$. The spectrahedron $\mathcal{S} = \mathcal{L} \cap \mathrm{SDP}_3$ is the yellow convex body seen in Chapter 1, Figure 1. To draw this spectrahedron in $\mathbb{R}^3$, one uses the representation (12.4), namely

$$\mathcal{S} \;=\; \left\{(x,y,z) \in \mathbb{R}^3 \,:\, \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0\right\}.$$

The boundary of $\mathcal{S}$ consists of all points (x,y,z) where the matrix has determinant zero and its nonzero eigenvalues are positive. The determinant is a polynomial of degree three in x, y, z, so the boundary lies in cubic surface in $\mathbb{R}^3$. This cubic surface also contains points where the three eigenvalues are positive, zero and negative. Such points are drawn in red in our picture 1 from Chapter 1. They lie in the Zariski closure of the yellow boundary points.

We next slice our 3-dimensional spectrahedron to get a picture in the plane.

Example 12.5. Suppose that $\mathcal{L} \subset \mathrm{Sym}_2(\mathbb{R}^3)$ is a general plane that intersects the cone PSD_3. The spectrahedron $\mathcal{S}$ is a planar convex body whose boundary is a smooth cubic curve, drawn in red in Figure 1. On that boundary, the 3×3 determinant vanishes and the other two eigenvalues are positive. For points $(x, y) \in \mathbb{R}^2 \backslash \mathcal{S}$, the matrix has at least one negative eigenvalue. The black curve lies in the Zariski closure of the red curve. It separates points in $\mathbb{R}^2 \backslash \mathcal{S}$ whose remaining two eigenvalues are positive from those with two negative eigenvalues.

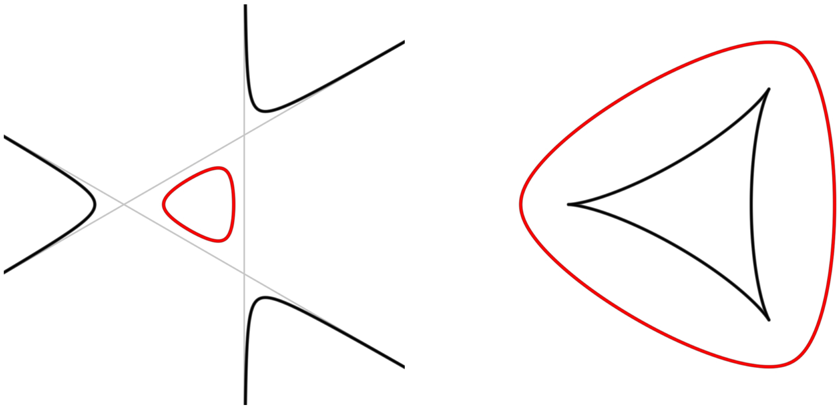


Figure 1. A plane curve of degree three (left) and its dual curve of degree six (right). The red part on the left bounds a spectrahedron while that on the right bounds its convex dual.

To be explicit, suppose that our planar cubic spectrahedron is defined as follows:

$$\mathcal{S} \;=\; \left\{ (x, y) \in \mathbb{R}^3 \,:\, \begin{pmatrix} 1 & x & x+y \\ x & 1 & y \\ x+y & y & 1 \end{pmatrix} \succeq 0 \right\}. \tag{12.5}$$

The cubic curve is the locus where the 3×3 matrix is singular. Its determinant is

$$f \;=\; 2x^2y + 2xy^2 - 2x^2 - 2xy - 2y^2 + 1. \tag{12.6}$$

The curve $\{f = 0\}$ has four connected components in $\mathbb{R}^2$, one in red and three in black, as shown in Figure 1 (left). The boundary of the cubic spectrahedron $\mathcal{S}$ is the convex part of the curve that is shown in red.

The picture on the right in Figure 1 shows the *dual curve.* This lives in the dual plane whose points (u, v) represent the lines $\ell = \{(x, y) : ux+vy = 1\}$ in $\mathbb{R}^2$. The points in the dual curve correspond to lines ℓ that are tangent to the original curve. The dual curve has degree six, and its equation is computed by the following ideal computation in $\mathbb{R}[x, y, u, v]$:

$$\begin{gathered}\langle\, f(x,y)\,,\, u\cdot x+v\cdot y-1\,,\ \partial f/\partial x\cdot v-\partial f/\partial y\cdot u\,\rangle\ \cap\ \mathbb{R}[u,v] \qquad = \\ \langle\, 8u^6-24u^5v+21u^4v^2-2u^3v^3+21u^2v^4-24uv^5+8v^6-24u^5+60u^4v \\ -24u^3v^2-24u^2v^3+60uv^4-24v^5+12u^4-24u^3v+36u^2v^2-24uv^3 \\ +12v^4+24u^3-36u^2v-36uv^2+24v^3-24u^2+24uv-24v^2+4\,\rangle.\end{gathered} \tag{12.7}$$

The black points on the sextic correspond to lines that are tangent at black points of the cubic, and similarly for the red points. Moreover, the convex set enclosed by the red sextic on the right in Figure 1 is dual, in the sense of convexity, to the spectahedron on the left.

The polynomials in (12.6) and (12.7) have degree three and six respectively, confirming what was asserted in the caption to Figure 1. A random line L will meet the curve in three (left) or six (right) complex points. Consider the point p on the other side that is dual to L, that is points of L correspond to lines through p. There are three (right) or six (left) complex lines through p that are tangent to the curve.

12.2. Optimization and Duality

We now finally come to *semidefinite programming* (SDP). This refers to the problem of maximizing or minimizing a linear function over a spectrahedron. Linear programming is the special case when the spectrahedron consists of diagonal matrices. If the spectrahedron is given in its standard form representation (12.1), then we get the SDP in its primal form:

$$\begin{gathered}\text{Minimize } \langle C, X\rangle \text{ subject to } \langle A_1, X\rangle = b_1, \\ \langle A_2, X\rangle = b_2,\, \ldots, \langle A_s, X\rangle = b_s \text{ and } X \succeq 0.\end{gathered} \tag{12.8}$$

Here $C = (c_{ij})$ is a matrix that represents the cost function. Every convex optimization problem has a dual problem. On first glance, it is not so easy to relate that duality to those for plane curves in Figure 1. The semidefinite problem dual to (12.8) takes the following form

$$\text{Maximize } b^T x \;=\; \sum_{i=1}^{s} b_i x_i \;\text{ subject to }\; C - x_1A_1 - x_2A_2 - \,\cdots\, - x_sA_s \;\succeq\; 0. \tag{12.9}$$

In this formulation, the spectrahedron of feasible points lives in $\mathbb{R}^s$, similarly to (12.4). We refer to either formulation (12.8) or (12.9) as a *semidefinite* *program*, also abbreviated SDP. Here the term "program" is simply an old-fashioned way of saying "optimization problem". The relationship between the primal and the dual SDP is given by the following theorem:

Theorem 12.6 (Weak Duality)**.** *If x is any feasible solution to (12.9) and X is any feasible solution to (12.8) then $b^T x \leq \langle C, X\rangle$. If the equality $b^T x = \langle C, X\rangle$ holds then both x and X are optimal.*

The term *feasible* means only that the point x resp. X satisfies the equations and inequalities that are required in (12.9) resp. (12.8). The point is *optimal* if it is feasible and it solves the program, i.e. it attains the minimum resp. maximum value for that optimization problem.

Proof. The inner product of two positive semidefinite matrices is a non-negative real number:
(12.10)

$$0 \;\leq\; \langle C - \sum_{i=1}^{s} x_i A_i, X\rangle \;=\; \langle C, X\rangle - \sum_{i=1}^{s} x_i \cdot \langle A_i, X\rangle \;=\; \langle C, X\rangle - b^T x.$$

This shows that the optimal value of the minimization problem (12.8) is an upper bound for the optimal value of the maximization problem (12.9). If the equality is attained by a pair (X, x) of feasible solutions then X must be optimal for (12.8) and x must be optimal for (12.9). □

There is also Strong Duality Theorem which states that, under suitable hypotheses, the *duality gap* $\langle C, X\rangle - b^T x$ must attain the value zero for some feasible pair (X, x). These hypotheses are always satisfied for diagonal matrices, and we recover the Duality Theorem for Linear Programming as a special case. Interior point methods for Linear Programming are numerical algorithms that start at an interior point of the feasible polyhedron and create a path from that point towards an optimal vertex. The same class of algorithms works for Semidefinite Programming. These run in polynomial time and are well-behaved in practice.

Semidefinite Programming has a much larger expressive power than Linear Programming. Many more problems can be phrased as an SDP. We illustrate this with a simple example.

Example 12.7 (The largest eigenvalue)**.** Let A be a real symmetric $n \times n$ matrix, and consider the problem of computing its largest eigenvalue $\lambda_{\max}(A)$. We would like to solve this without having to write down the characteristic polynomial and extract its roots. Let $C = \mathrm{Id}$ be the identity matrix and consider the SDP problems (12.8) and (12.9) with $s = 1$ and $b = 1$. They are

(12.8') Minimize trace(X) subject to $\langle A, X\rangle = 1$.

(12.9') Maximize x subject to $\mathrm{Id} - xA \succeq 0$.

If x^* is the common optimal value of these two problems then $\lambda_{\max}(A) = 1/x^*$.

The inner product $\langle A, X\rangle = \mathrm{trace}(A \cdot X)$ of two positive semidefinite matrices A and X can only be zero when their matrix product $A \cdot X$ is zero. We record this for our situation:

Lemma 12.8. *If the expression in (12.10) is zero then* $(C - \sum_{i=1}^{s} x_i A_i) \cdot X$ *is the zero matrix.*

This lemma allows us to state the following algebraic reformulation of SDP:

Corollary 12.9. *Consider the following system of s linear equations and $\binom{n+1}{2}$ bilinear equations in the $\binom{n+1}{2} + s$ unknown coordinates of the pair (X, x):*

$$\langle A_1, X\rangle = b_1 \,,\ \ldots\ ,\ \langle A_s, X\rangle = b_s \quad \text{and} \quad (C - \sum_{i=1}^{s} x_i A_i) \cdot X = 0. \tag{12.11}$$

If $X \succeq 0$ and $C - \sum_{i=1}^{s} x_i A_i \succeq 0$ then X is optimal for (12.8) and x is optimal for (12.9).

The equations (12.11) are known as the *Karush-Kuhn-Tucker (KKT) equations*. These play a major role when one explores semidefinite programming from an algebraic perspective. In particular, they allow us to study the nature of the optimal solution as function of the data. A key feature of the KKT system is that the two optimal matrices have complementary ranks. This follows from the *complementary slackness* condition on the right of (12.11):

$$\mathrm{rank}\big(C - \sum_{i=1}^{s} x_i A_i\big) \, + \, \mathrm{rank}(X) \, \le \, n.$$

In particular, if X is known to be nonzero then the determinant of $C - \sum_{i=1}^{s} x_i A_i$ vanishes. For instance, for the eigenvalue problem in Example 12.7, we have $(\mathrm{Id} - xA) \cdot X = 0$ and $\langle A, X\rangle = 1$. This implies $\det(\mathrm{Id} - xA) = 0$, so $1/x$ is a root of the characteristic polynomial.

Example 12.10. Consider the problem of maximizing a linear function $\ell(x, y) = ux + vy$ over the spectrahedron $\mathcal{S}$ in (12.5). This is the primal SDP (12.8) with $s = 2$ and $b = (u, v)$ and

$$A_1 \,=\, -\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 \,=\, -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The KKT system (12.11) consists of eight equations in eight unknowns, with two parameters:

$$2x_{12} + 2x_{13} + u = 2x_{13} + 2x_{23} + v = 0 \quad \text{and}$$
$$\begin{pmatrix} 1 & x & x{+}y \\ x & 1 & y \\ x{+}y & y & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By eliminating the variables x_{ij} we obtain an ideal I in $\mathbb{Q}[u,v,x,y]$ that characterizes the optimal solution (x^*, y^*) to our SDP as an algebraic function of (u,v). Let ℓ^* now be a new unknown, and consider the elimination ideal $\big(I + \langle ux + vy - \ell^* \rangle\big) \cap \mathbb{Q}[u,v,\ell^*]$. Its generator is a ternary sextic in u, v, ℓ^*. This is precisely the homogenization of the dual sextic in (12.7). It expresses the optimal value ℓ^* as an algebraic function of degree six in the cost (u,v).

This relationship between the dual hypersurface and the optimal value function generalizes to arbitrary polynomial optimization problems, including semidefinite programs. This is the content of [**4**, Theorem 5.23]. We refer to the book [**4**], and especially Chapter 5, for further reading on spectrahedra, semidefinite programming, and the relevant duality theory.

A fundamental task in Convex Algebraic Geometry [**4**] is the computation of the convex hull of a given algebraic variety or semialgebraic set. Recall that the *convex hull* of a set is the smallest convex set containing the given set. Spectrahedra or their linear projections, known as *spectrahedral shadows*, can be used for this task. This matters for optimization since minimizing a linear function over a set is equivalent to minimizing over its convex hull.

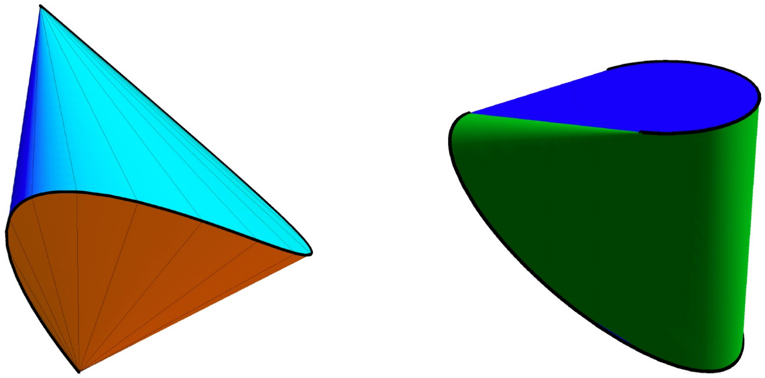


Figure 2. Toeplitz spectrahedron and its dual convex body.

Example 12.11 (Toeplitz Spectrahedron)**.** Consider the convex body

$$K = \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, \begin{bmatrix} 1 & x & y & z \\ x & 1 & x & y \\ y & x & 1 & x \\ z & y & x & 1 \end{bmatrix} \succeq 0 \right\}. \tag{12.12}$$

The determinant of the given *Toeplitz matrix* of size 4 × 4 factors as

$$(x^2 + 2xy + y^2 - xz - x - z - 1)(x^2 - 2xy + y^2 - xz + x + z - 1).$$

The *Toeplitz spectrahedron* (12.12) is the convex hull of the *cosine moment curve*

$$\big\{ \big(\cos(\theta), \cos(2\theta), \cos(3\theta)\big) \,:\, \theta \in [0, \pi] \,\big\}.$$

The curve and its convex hull are shown on the left in Figure 2. The two endpoints, $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (-1, 1, -1)$, correspond to rank 1 matrices. All other points on the curve have rank 2. To construct the Toeplitz spectrahedron geometrically, we form the cone from each endpoint over the cosine curve, and we intersect these two quadratic cones. The two cones intersect along this curve and the line through the endpoints of the cosine curve.

Shown on the right in Figure 2 is the convex body K^* dual to the Toeplitz spectrahedron K. It is the set of trigonometric polynomials $1 + a_1 \cos(\theta) + a_2 \cos(2\theta) + a_3 \cos(3\theta)$ that are nonnegative on $[0, \pi]$. This convex body K^* is not a spectrahedron because it has a non-exposed edge, that is an edge which is not an intersection of K^* with a kernel of a linear form that is nonnegative on K^* (cf. [**4**, Exercise 6.13]).

12.3. Sums of Squares

Semidefinite programming can be used to model and solve arbitrary polynomial optimization problems. The key to this is the representation of nonnegative polynomials in terms of sums of squares, or, more generally, the Real Nullstellensatz (cf. Chapter 6)). We explain this for the simplest scenario, namely the problem of unconstrained polynomial optimization.

Let $f(x_1, \ldots, x_n)$ be a polynomial of even degree $2p$, and suppose that f attains a minimal real value f^* on $\mathbb{R}^n$. Our goal is to compute f^* and a point $\mathbf{u}^* \in \mathbb{R}^n$ such that $f(\mathbf{u}^*) = f^*$. Minimizing a function is equivalent to finding the best possible lower bound λ for that function. Our goal is therefore equivalent to solving the following optimization problem:

$$\text{Maximize } \lambda \text{ such that } \quad f(\mathbf{x}) - \lambda \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{12.13}$$

This is a difficult problem. Instead, we consider the following relaxation:

$$\text{Maximize } \lambda \text{ such that } \quad f(\mathbf{x}) - \lambda \quad \text{is a sum of squares in } \mathbb{R}[\mathbf{x}]. \tag{12.14}$$

Here *relaxation* means that we restricted the set of feasible solutions. Indeed, every sum of squares is nonnegative, but not every nonnegative polynomial is a sum of squares of polynomials. For instance, the Motzkin polynomial $x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is nonnegative but it is not a sum of squares of polynomials. For that reason, the optimal value of (12.14) is always a lower bound for the optimal value of (12.13), but the two values can be different in some cases. However, here is the good news:

Proposition 12.12. *The optimization problem (12.14) is a semidefinite program.*

Proof. Let $\mathbf{x}^{[p]}$ be the column vector whose entries are all monomials in $x_1, \ldots, x_n$ of degree $\leq p$. Thus $\mathbf{x}^{[p]}$ has length $\binom{n+p}{n}$. Let $G = (g_{ij})$ be a symmetric $\binom{n+p}{n} \times \binom{n+p}{n}$ matrix with unknown entries. Then $(\mathbf{x}^{[p]})^T \cdot G \cdot \mathbf{x}^{[p]}$ is a polynomial of degree $d = 2p$ in $x_1, \ldots, x_n$. We set

$$f(\mathbf{x}) - \lambda \quad = \quad (\mathbf{x}^{[p]})^T \cdot G \cdot \mathbf{x}^{[p]}. \tag{12.15}$$

By collecting coefficients of the $\mathbf{x}$-monomials, this gives a system of $\binom{2p+n}{n}$ linear equations in the unknowns g_{ij} and λ. The number of unknowns is $\binom{\binom{n+p}{n}+1}{2} + 1$.

Suppose the linear system (12.15) has a solution (G, λ) such that G is positive semidefinite. Then we can write $G = H^T H$ where H is a real matrix with r rows and $\binom{p+n}{n}$ columns. (This is known as a *Cholesky factorization* of H.) The polynomial in (12.15) then equals

$$f(\mathbf{x}) - \lambda \quad = \quad (H\mathbf{x}^{[p]})^T \cdot (H\mathbf{x}^{[p]}). \tag{12.16}$$

This is the scalar product of a vector of length r with itself. Hence $f(\mathbf{x}) - \lambda$ is a sum of squares. Conversely, every representation of $f(\mathbf{x}) - \lambda$ as a sum of squares of polynomials uses polynomials of degree $\leq p$, and it can hence be written in the form as in (12.16).

Our argument shows that the optimization problem (12.14) is equivalent to

$$\begin{array}{c}\text{Maximize } \lambda \text{ subject to } (G, \lambda) \text{ satisfying} \\ \text{the linear equations (12.15) and } G \succeq 0.\end{array} \tag{12.17}$$

This is a semidefinite programming problem, and so the proof is complete. □

If $n = 1$ or $d = 2$ or ($n = 2$ and $d = 4$) then every nonnegative polynomial is a sum of squares. In those special cases, problems (12.13) and (12.17) are equivalent.

Example 12.13 ($n = 1, p = 2, d = 4$)**.** Suppose we seek to find the minimum of the degree 4 polynomial $f(x) = 3x^4 + 4x^3 - 12x^2$. Of course, we know how to do this using Calculus. However, we here present the SDP approach. The linear equations (12.15) have a one-dimensional space of solutions. Introducing a parameter μ for that line, the solutions can be written as

$$(12.18) \qquad f(x)-\lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu-6 \\ 2 & -2\mu & 0 \\ \mu-6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}.$$

Consider the set of all pairs (λ,μ) such that the 3×3 matrix in (12.18) is positive semidefinite. This set is a cubic spectrahedron in the plane $\mathbb{R}^2$, just like that shown on the left in (1). We seek to maximize λ over all points in that cubic spectrahedron. The optimal point equals $(\lambda^*,\mu^*)=(-32,-2)$. Substituting this into the matrix in (12.18) we obtain a positive definite matrix of rank 2. This can be factored as $G=H^TH$, where H has format 2×3. The resulting representation (12.16) as a sum of two squares equals

$$f(x)-\lambda^* \;=\; f(x)+32 \;=\; \big((\sqrt{3}x-\frac{4}{\sqrt{3}})\cdot(x+2)\big)^2 \;+\; \frac{8}{3}(x+2)^2.$$

The right hand side is nonnegative for all x. It takes the value 0 only when $x=-2$.

Any polynomial optimization problem can be translated into a relaxation that is a semidefinite programming problem. If we are minimizing $f(\mathbf{x})$ subject to some polynomial constraints, then we seek a certificate for $f(\mathbf{x})-\lambda<0$ to have no solution. This certificate is promised by the Real Nullstellensatz or Postitivstellensatz. If we fix a degree bound then the existence of a certificate translates into a semidefinite program, and so does the additional requirement for λ to be minimal. This relaxation may or may not give the correct solution for some fixed degree bound. However, if one increases the degree bound then the SDP formulation is more likely to succeed, albeit at the expense of having to solve a much larger problem. This is a powerful and widely used approach to polynomial optimization, known as *SOS programming.* The term *Lasserre hierarchy* refers to varying the degree bounds.

Every spectrahedron $\mathcal{S}=\mathcal{L}\cap\mathrm{PSD}_n$ has a special point in its relative interior. This point, defined as the unique matrix in $\mathcal{S}$ whose determinant is maximal, is known as *analytic center.* Finding the analytic center of $\mathcal{S}$ is a convex optimization problem, since the function $X\mapsto\log\det(X)$ is strictly convex on the cone of positive definite matrices X. The analytic center is important for semidefinite programming because it serves as the starting point for interior point methods. Indeed, the *central path* of an SDP starts at the analytic center and runs to the optimal face. It is computed by a sequence of numerical approximations.

Example 12.14. The determinant function takes on all values between 0 and 1 on the spectrahedron $\mathcal{S}$ in (12.5). The value 1 is attained only by the identity matrix, for $(x, y) = (0, 0)$. This point is therefore the analytic center of $\mathcal{S}$.

We close by relating spectrahedra and their analytic centers to statistics. Every positive definite $n \times n$ matrix $\Sigma = (\sigma_{ij})$ is the *covariance matrix* of a multivariate normal distribution. Its inverse Σ^{-1} is the *concentration matrix* of that distribution.

A *Gaussian graphical model* is specified by requiring that some off-diagonal entries of Σ^{-1} are zero. These entries correspond to the non-edges of the graph. Maximum likelihood estimation for this graphical model translates into a matrix completion problem. Suppose that S is the sample covariance matrix of a given data set. We regard S as a partial matrix, with visible entries only on the diagonal and on the edges of the graph. One considers the set of all completions of the non-edge entries that make the matrix S positive definite. The set of all these completions is a spectrahedron. Maximum likelihood estimate for the data S in the graphical model amounts to maximizing the logarithm of the determinant. We hence seek to compute the analytic center of the spectrahedron of all completions.

Example 12.15 (Positive definite matrix completion)**.** Suppose that the eight entries σ_{ij} in the following symmetric 4×4-matrix are visible, but the entries x and y are unknown:

$$(12.19) \qquad \Sigma \quad = \quad \begin{pmatrix} \sigma_{11} & \sigma_{12} & x & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & y \\ x & \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{14} & y & \sigma_{34} & \sigma_{44} \end{pmatrix}.$$

This corresponds to the graphical model of the four-cycle $12, 23, 34, 41$. Given visible entries σ_{ij}, we consider the set of pairs (x, y) that make Σ positive definite. This is the interior of a planar spectrahedron $\mathcal{S}_\sigma$ bounded by a quartic curve. The MLE is the analytic center of $\mathcal{S}_\sigma$.

One is also interested in conditions on the σ_{ij} such that $\mathrm{int}(\mathcal{S}_\sigma)$ is nonempty. When can we find (x, y) that make Σ positive definite? A necessary condition is that the diagonal entries σ_{ii} and the four visible principal 2×2-minors are positive:

$$(12.20) \qquad \sigma_{11}\sigma_{22} > \sigma_{12}^2\,,\ \ \sigma_{22}\sigma_{33} > \sigma_{23}^2\,,\ \ \sigma_{33}\sigma_{44} > \sigma_{34}^2\,,\ \ \sigma_{11}\sigma_{44} > \sigma_{14}^2.$$

But this is not sufficient. The answer is a cone that is bounded by the hypersurface

$$\begin{gathered}\sigma_{33}^2\sigma_{44}^2\sigma_{12}^4 - 2\sigma_{22}\sigma_{33}^2\sigma_{44}\sigma_{12}^2\sigma_{14}^2 - 2\sigma_{11}\sigma_{33}\sigma_{44}^2\sigma_{12}^2\sigma_{23}^2 - 2\sigma_{11}\sigma_{22}\sigma_{33}\sigma_{44}\sigma_{14}^2\sigma_{23}^2 \\ +4\sigma_{33}\sigma_{44}\sigma_{12}^2\sigma_{14}^2\sigma_{23}^2 + \sigma_{11}^2\sigma_{44}^2\sigma_{23}^4 + 8\sigma_{11}\sigma_{22}\sigma_{33}\sigma_{44}\sigma_{12}\sigma_{14}\sigma_{23}\sigma_{34} - 4\sigma_{33}\sigma_{44}\sigma_{12}^3\sigma_{14}\sigma_{23}\sigma_{34} \\ -4\sigma_{22}\sigma_{33}\sigma_{12}\sigma_{14}^3\sigma_{23}\sigma_{34} + \sigma_{22}^2\sigma_{33}^2\sigma_{14}^4 - 4\sigma_{11}\sigma_{44}\sigma_{12}\sigma_{14}\sigma_{23}^3\sigma_{34} - 2\sigma_{11}\sigma_{22}\sigma_{33}\sigma_{44}\sigma_{12}^2\sigma_{34}^2 \\ -2\sigma_{11}\sigma_{22}^2\sigma_{33}\sigma_{14}^2\sigma_{34}^2 + 4\sigma_{22}\sigma_{33}\sigma_{12}^2\sigma_{14}^2\sigma_{34}^2 - 2\sigma_{11}^2\sigma_{22}\sigma_{44}\sigma_{23}^2\sigma_{34}^2 + 4\sigma_{11}\sigma_{44}\sigma_{12}^2\sigma_{23}^2\sigma_{34}^2 \\ +4\sigma_{11}\sigma_{22}\sigma_{14}^2\sigma_{23}^2\sigma_{34}^2 - 4\sigma_{11}\sigma_{22}\sigma_{12}\sigma_{14}\sigma_{23}\sigma_{34}^3 + \sigma_{11}^2\sigma_{22}^2\sigma_{34}^4.\end{gathered}$$

This polynomial of degree eight is found by eliminating x and y from the determinant and its partial derivatives with respect to x and y, after saturating by the ideal of 3×3-minors. For more details on this example see to [**55**, Theorem 4.8].

Exercises

(1) Prove Theorem 12.2.

(2) Show that a real symmetric matrix G is positive semidefinite if and only if it admits a Cholesky factorization $G = H^T H$ over the real numbers, with H upper triangular.

(3) What is the largest eigenvalue of any of the 3×3 matrices in the set $\mathcal{S}$ in (12.5)?

(4) Maximize and minimize the linear function $13x + 17y + 23z$ over the spectrahedron $\mathcal{S}$ in Example 12.4. Use SDP software if you can.

(5) Maximize and minimize the linear function $13x + 17y + 23z$ over the Toeplitz spectrahedron in Example 12.11. Use SDP software if you can.

(6) Write the dual SDP and solve the KKT system for the previous two problems.

(7) Determine the convex body dual to the spectrahedron $\mathcal{S}$ in Example 12.4.

(8) Consider the problem of minimizing the univariate polynomial $x^6+5x^3+7x^2+x$. Express this problem as a semidefinite program.

(9) In the partial matrix (12.19) set $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 5$, $\sigma_{12} = \sigma_{23} = \sigma_{34} = 1$ and $\sigma_{14} = 2$. Compute the spectrahedron $\mathcal{S}_\sigma$, draw a picture, and find the analytic center.

(10) Find numerical values for the eight entries σ_{ij} in (12.19) that satisfy (12.20) but $\mathrm{int}(\mathcal{S}_\sigma) = \emptyset$.

Chapter 13

Combinatorics

"*Combinatorics is the nanotechnology of mathematics.*"
Sara Billey

Combinatorics interacts in many fruitful ways with algebra and geometry, for instance in the interplay between convex polytopes and toric varieties. This chapter offers *a pinch of combinatorics in a vast sea of algebra.* We present topics that are important for nonlinear algebra. The first such topic is matroid theory. Matroids encode independence, just like groups encode symmetry. The theory of matroids has many connections to toric geometry and we will present a few of them. One of our main aims is to highlight connections between Grassmannians (Chapter 5), toric varieties (Chapter 8) and matroids. In all topics the *lattice polytopes* will play a prominent role. We will finish by presenting a snapshot of *generating functions.* Their role as *Hilbert series* brings us back to the two key invariants of a variety: dimension and degree. We would like to stress the fact that our emphasis is not on combinatorics itself, but rather on its *interactions* with algebra and geometry.

13.1. Matroids

In this section we give an introduction to the theory of *matroids.* The name matroids suggests that these should be regarded as generalizations of matrices. Indeed, as we will soon see every matrix defines a matroid. We fix a finite set E, which we will refer to as the ground set of a matroid. We would like to distinguish a family of subsets of E that we could call *independent.* Thus a matroid M will be a family $\mathfrak{I} \subset 2^E$ of subsets E that we refer to as independent sets. These are of course assumed to satisfy certain axioms.

A first observation is that whenever we have an independent set $I \subset E$, it is reasonable to assume that every subset of I is also independent. We obtain the first axiom of a matroid for the family $\mathfrak{I}$:

1. If $I \in \mathfrak{I}$ and $J \subset I$, then $J \in \mathfrak{I}$.

What we defined so far is a very important object in mathematics: *simplicial complex*. Another observation is that we would like $\mathfrak{I}$ to be nonempty, or equivalently we want the empty set to be independent:

2. We have $\emptyset \in \mathfrak{I}$.

It turns out that to obtain a matroid we need just one more axiom. To motivate it we make a following observation. Whenever we have finite linearly independent subsets $I, J \subset V$ of a vector space V, if $|I| < |J|$, then we may extend I by an element of $j \in J$, in such a way that $I \cup \{j\}$ is still linearly independent. This simple observation is precisely what we need to get the last axiom for the family $\mathfrak{I}$:

3. If $I, J \in \mathfrak{I}$ and $|I| < |J|$ then there exists $j \in J$ such that $I \cup \{j\} \in \mathfrak{I}$.

Definition 13.1. A *matroid* is a family of subsets $\mathfrak{I}$ satisfying Axioms 1,2,3.

Exercise 1 asks to prove that the following structures are matroids:

Example 13.2.

- (Representable/Realizable matroid) Let V be a vector space over an arbitrary field F. Let $E \subset V$ be a nonempty, finite subset. We define $\mathfrak{I}$ to be the family of subsets of E that are linearly independent. We say that the matroid is *representable* over F. In coordinates, $V \simeq F^n$ and we may identify the set E as an $|E| \times n$ matrix.
- (Graphic matroid) Let G be a graph with edge set E. Let $\mathfrak{I}$ be the family of those subsets of E that do not contain a cycle. Equivalently $\mathfrak{I}$ is the family of forests in G.
- (Algebraic matroid) Let $F \subset K$ be an arbitrary field extension. Let E be a finite subset of K. Let $\mathfrak{I}$ be the family of subsets of E that are algebraically independent over F.
- (Uniform matroid) Let E be a finite set and $k \leq |E|$. Let $\mathfrak{I}$ be the family of subsets of cardinality at most k. This matroid is denoted by $U_{k,E}$ or $U_{k,|E|}$.

Matroids are known for having many equivalent definitions, depending on the point of view on the matroid. For example, due to the first axiom to determine a matroid we do not have to know all independent sets, just those that are inclusion maximal. By analogy to linear algebra, the inclusion maximal independent sets are called *basis*. It turns out - as the reader is asked to prove in Exercise 2 - that a nonempty family $\mathfrak{B} \subset 2^E$ of subsets of E is a family of basis of some matroid if and only if the following axiom is satisfied:

- For all $B_1, B_2 \in \mathfrak{B}$, $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\} \in \mathfrak{B}$.

The seemingly weak axiom on $\mathfrak{B}$ in fact implies the following two statements:

- For all $B_1, B_2 \in \mathfrak{B}$, $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that both $(B_1 \setminus \{b_1\}) \cup \{b_2\}, (B_2 \setminus \{b_2\}) \cup \{b_1\} \in \mathfrak{B}$.
- For all $B_1, B_2 \in \mathfrak{B}$ and any subset $A_2 \subset B_2 \setminus B_1$ there exists a subset $A_1 \subset B_1 \setminus B_2$ such that both $(B_1 \setminus A_1) \cup A_2, (B_2 \setminus A_2) \cup A_1 \in \mathfrak{B}$.

The first point is known as the *symmetric exchange property* and the second one as the *multiple symmetric exchange property*. The facts that both exchange properties hold is nontrivial - we refer the reader to the proofs in [**7, 58**]. We will soon see the algebraic meaning of the exchange properties.

Exercise 3 states that all basis of a matroid have the same cardinality. The cardinality of a basis is known as the *rank* of a matroid. More generally for a matorid on a ground set E we may define the rank of any subset $A \subset E$.

Definition 13.3. For a matroid on a ground set E and independent sets $\mathfrak{I} \subset 2^E$ we define the *rank* function:

$$r : 2^E \ni A \to \max_{I \in \mathfrak{I}} \{|I \cap A|\} \in \mathbb{Z}.$$

Equivalently, the rank of a set is the cardinality of a largest independent set contained in it.

We note that for a representable matroid the rank is simply the dimension of the vector subspace spanned by the given vectors. Clearly, for any matroid the rank function r satisfies the following:

- $0 \leq r(A)$ for all $A \subset E$ and $r(\emptyset) = 0$.
- $r(A) \leq r(A \cup \{b\}) \leq r(A) + 1$ for all $A \subset E$, $x \in E$.

Further, the rank function has one more property known as *submodularity*:

- for all $A, B \subset E$ we have $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

In Exercise 7 the reader is asked to prove that any function $r : 2^E \to \mathbb{Z}$ satisfying the three axioms above is a rank function of a matroid. The independent sets can be reconstructed as those $I \subset E$ for which $r(I) = |I|$. This gives us another possible definition of a matroid.

13.2. Lattice Polytopes

In this section we discuss the interplay between toric geometry and matroids. It is not possible to even state all of the interesting results ; we refer to [**19, 28, 20, 41**].

We start by recalling the definition of a lattice polytope.

Definition 13.4 (Lattice polytope)**.** Let $\mathbb{R}^n$ be a real vector space. A polytope P is the convex hull of a finite set of points $p_1, \dots, p_k \in \mathbb{R}^n$:

$$P := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{k} \lambda_i p_i \text{ for some real } \lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = 1\}.$$

We say that P is a *lattice polytope* if we may find $p_1, \dots, p_k \in \mathbb{Z}^n \subset \mathbb{R}^n$.

For each polytope P there is an inclusion minimal set of p_i's of which it is a convex hull. We call these p_i's the vertices of P.

To pass from a combinatorial object, like a matroid, to a polytope, we apply the following 'standard' construction. Consider a vector space $\mathbb{R}^{|E|}$ with basis elements b_e corresponding to the elements $e \in E$. Any subset $A \subset E$ can be identified with a point $p_A := \sum_{e \in A} b_e \in \mathbb{R}^{|E|}$. In this way a family of subsets may be identified with a set of points.

Definition 13.5 (Matroid basis polytope)**.** Let M be a matroid on the ground set E and basis set $\mathfrak{B}$. We use the notation introduced above. We define the *matroid basis polytope* $P_M \subset \mathbb{R}^{|E|}$ as the convex hull of the points $p_B := \sum_{e \in B} b_e \in \mathbb{R}^{|E|}$, where we take all $B \in \mathfrak{B}$.

Clearly P_M is a lattice polytope, hence we may consider the toric variety associated to it. Precisely, it is the closed image of the map given by monomials, in variables corresponding to elements of E, that are products of elements in a basis.

Example 13.6. Consider the rank two uniform matroid on the set $E = \{1, 2, 3\}$. Its set of bases is

$$\mathfrak{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

We consider $\mathbb{R}^3$. The three basis correspond, in the given order, to the three points

$$(1, 1, 0), (1, 0, 1), (0, 1, 1) \in \mathbb{R}^3.$$

Hence, the matroid basis polytope is a triangle. The polynomial map is:

$$(\mathbb{C}^*)^3 \ni (x_1, x_2, x_3) \to (x_1x_2, x_1x_3, x_2x_3) \in \mathbb{P}^2.$$

The closure of the image is the whole $\mathbb{P}^2$, which is the associated toric variety.

The combinatorial statement equivalent to the proposition presented below was proved by White [**57**].

Proposition 13.7. *A matroid basis polytope is normal in the lattice it spans.*

In order to present the proof we state one of the most useful theorems about matroids.

Theorem 13.8 (The matroid union theorem)**.** *Let $M_1, \dots, M_k$ be matroids on the same ground set E with respective families of independent sets $\mathfrak{I}_1, \dots, \mathfrak{I}_k$ and rank functions $r_1, \dots, r_k$. Let*

$$\mathfrak{I} := \{I \subset E : I = \bigcup_{i=1}^{k} I_i \text{ for } I_i \in \mathfrak{I}_i\}.$$

Then $\mathfrak{I}$ is also a family of independent sets for a matroid, known as the union of $M_1, \dots, M_k$. Further, the rank of any set $A \subset E$ for the union matroid is given by:

$$r(A) = \min_{B \subset A}\{|A \setminus B| + \sum_{i=1}^{k} r_i(B)\}.$$

For the proof we refer to [**41**, 12.3.1]. As a corollary of the matroid union theorem we obtain the following theorem due to Edmonds.

Theorem 13.9. *Let M be a matroid on a ground set E with rank function r. E can be partitioned into k independent sets if and only if $|A| \leq k \cdot r(A)$ for all subsets $A \subset E$.*

Proof. The implication ⇒ is straightforward.

For the other implication consider the union U of M with itself k times. We apply the matroid union theorem to compute the rank of E:

$$r_U(E) = \min\{|E| - |B| + k \cdot r_M(B)\}.$$

Clearly by assumption $|E| - |B| + k \cdot r_M(B) \geq |E|$ and equality holds for $B = \emptyset$. Hence, $r_U(E) = |E|$. This means that E is an independent set in U, and hence by definition it is a union of k independent sets of M. □

Definition 13.10. Let M be a matroid on a ground set E with the family of independent sets $\mathfrak{I}$. Let $E' \subset E$. The *restriction* of M to E' is a matroid where $A \subset E'$ is independent if and only if $A \in \mathfrak{I}$.

Proof of Proposition 13.7. Let M be a matroid on the ground set $\{1, \dots, n\}$. Let $p \in kP_M$. We know that $p = \sum_{b \in \mathfrak{B}} \lambda_B p_B$ with $\sum \lambda_B = k$ and $0 \leq \lambda_B \in \mathbb{Q}$. After clearing the denominators we have:

$$dp = \sum \lambda'_B p_b,$$

where $\sum \lambda'_B = dk$ and $0 \leq \lambda_B \in \mathbb{Z}$.

By restricting the matroid M we may assume that all coordinates of $p = (p_1, \dots, p_n)$ are nonzero.

We define two matroids. The first matroid N is on the ground set $E_N := \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq p_i\}$. In other words, we replace a point i in the original matroid by p_i equivalent points. A subset $\{(i_1,j_1), \dots, (i_s,j_s)\} \subset E_N$ is independent if only if:

- all i_q's are distinct,
- $\{i_1, \dots, i_s\}$ is an independent set in M.

We note that a basis of N maps naturally to a basis of M. Also the rank function for N is the same as the one for M if we forget the second coordinates. The point p has a decomposition as a sum of k points corresponding to basis of M if and only if the matroid N is covered by k basis (i.e. the ground set is a union of k basis). Hence, by Theorem 13.9 our aim is to prove the following statement:

For any $A \subset E_N$ we have $|A| \leq kr_N(A)$.

The second matroid N' is on the ground set $E_{N'} := \{(i,j,l) : 1 \leq i \leq n, 1 \leq j \leq p_i, 1 \leq l \leq d.\}$. In other words we replace any point of N by d equivalent points. A subset $\{(i_1,j_1,l_1), \dots, (i_s,j_s,l_s)\} \subset E_N$ is independent if only if:

- all i_q's are distinct,
- $\{i_1, \dots, i_s\}$ is an independent set in M.

We have a natural projection $\pi : E_{N'} \to E_N$ given by forgetting the last coordinate. We note that $r_{N'}(\pi^{-1}(A)) = r_N(A)$. As the point dp is decomposable we know that the matroid N' can be covered by kd basis. Hence, for any $B \subset E_{N'}$ we have: $|B| \leq dk \cdot r_{N'}(B)$. Applying this to $\pi^{-1}(A)$ we obtain:

$$k|A| = |\pi^{-1}(A)| \leq dk \cdot r_{N'}(\pi^{-1}(A)) = dk \cdot r_N(A).$$

This is equivalent to the statement we wanted to prove! □

Our next aim is to relate matroids with the geometry of special subvarieties of Grassmannians. We recall that one of the possible definitions of a Grassmannian $G(k,n)$ is an orbit of $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ under the action of the group of $n \times n$ invertible matrices $GL(n)$. While the Grassmannian is an orbit of the big group $GL(n)$, we may ask how smaller groups act on $G(k,n)$. In particular, consider the torus $T := (\mathbb{C}^*)^n$ of diagonal matrices. This torus acts on $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ and on $G(k,n)$. However, in general $G(k,n)$ is not an orbit of T or even a closure of an orbit of T. Indeed, we already know that $G(k,n)$ has dimension $k(n-k)$ which may be much larger than n. Let us fix a point $p \in G(k,n)$. The questions that motivate us are:

- What is the T-orbit of p?
- What is the closure of this orbit?
- How can we describe this variety?

A beautiful answer was provided by Gelfand, Goresky, MacPherson and Serganova [**23**]. The point $p = [v_1 \wedge \cdots \wedge v_k] \in G(k,n)$ represents a k-dimensional subspace $V = \langle v_1, \ldots, v_k \rangle$ in $\mathbb{C}^n$. We may present the vectors $v_1, \ldots, v_k$ as a $k \times n$ matrix N_p. From Chapter 5 we know that the coordinates of $p \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$, are given by maximal minors of N_p. How does a point $t = (t_1, \ldots, t_n) \in T$ act on p? In general, t acts on the coordinate indexed by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ rescaling it by $t_{i_1} \cdots t_{i_k}$. Hence, the orbit of p is the image of the map:

$$T \ni (t_1, \ldots, t_n) \to (t_{i_1} \cdots t_{i_k} \det((N_p)_{i_1,\ldots,i_k}))_{1 \leq i_1 < \cdots < i_k \leq n} \in \mathbb{P}(\bigwedge^k \mathbb{C}^n),$$

where $(N_P)_{i_1,\ldots,i_k}$ denotes the $k \times k$ submatrix of N_p with the chosen columns indexed by $i_1, \ldots, i_k$.

Example 13.11. Consider the two dimensional subspace of the four dimensional space spanned by the rows of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

In the coordinates of the Grassmannian we have the associated point:

$$(e_1+e_2+e_3+e_4)\wedge(e_1+2e_2+3e_3+4e_4) = e_1\wedge e_2+2e_1\wedge e_3+3e_1\wedge e_4+e_2\wedge e_3+2e_2\wedge e_4+e_3\wedge e_4.$$

The orbit in the coordinates above is parameterized as follows:

$$(t_1, t_2, t_3, t_4) \to (t_1t_2, 2t_1t_3, 3t_1t_4, t_2t_3, 2t_2t_4, t_3t_4).$$

This is almost a monomial map! Indeed, the only thing that changes are the constants given by minors of the matrix N_p. However, these constants do not depend on $t \in T$ and hence we may define an automorphism of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ that turns the orbit to an image of a monomial map, by simply rescaling the coordinates.

At this point one could have a false impression that the orbit is isomorphic to the image of a monomial map defined by all squarefree monomials of degree k. This is not the case, as some of the monomials may not appear at all. This happens if the corresponding minor was equal to zero - then we cannot rescale it.

Example 13.12. (1) First we continue Example 13.11. The polytope associated to the toric variety has the following vertices:

$$(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1).$$

This is the hypersimplex $\Delta_{2,4}$. The associated projective toric variety is three dimensional. It represents the closure of the T-orbit of a general point in the Grassmannian $G(2,4)$. The associated matroid is the uniform rank two matroid on four elements.

(2) Let us now consider a different point of $G(2,4)$ given by the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

The orbit is parameterized as follows:

$$(t_1,t_2,t_3,t_4) \to (2t_1t_2, 3t_1t_3, 4t_1t_4, 0, 0, 0).$$

The polytope representing the toric variety has the following vertices:

$$(1,1,0,0),(1,0,1,0),(1,0,0,1).$$

It is isomorphic to a two dimensional simplex, hence the closure of the orbit is a $\mathbb{P}^2$, as can be seen directly from the parameterization.

Which monomials are thus left? Exactly those for which the corresponding minor of N_p was not zero.

Let us consider the representable matroid M_p of n points in $\mathbb{C}^k$, defined by the columns of the matrix N_p. Clearly a set of points is a basis of M_p if and only if the corresponding minor of N_p is nonzero. We have proved the following proposition.

Proposition 13.13. *The closure of the T-orbit of any point $p = [v_1 \wedge \cdots \wedge v_k]$ in a Grassmannian $G(k,n)$ is the toric variety represented by the matroid base polytope, for the representable matroid defined by columns of the $k \times n$ matrix N_p with i-th row equal to v_i.*

The results of Chapter 8 combined with Proposition 13.7 show the following.

Proposition 13.14. *Any torus orbit closure in any Grassmannian in the Plücker embedding is projectively normal.*

The proposition also holds, if we reembed the Grassmannian by a Veronese map, and for more general manifolds, known as flag varieties.

We now turn to the interpretation of basis exchange properties in terms of algebraic geometry. Consider a matroid with basis polytope P. We recall that:

- the ideal of the associated toric variety is generated by binomials,
- every binomial in the ideal corresponds to an integral relation among lattice points of P.

How do these statements specialize in the case of matroids? A lattice point of P is the characteristic function of a basis. A sum of lattice points is the sum of these characteristic functions. This corresponds to taking a sum of basis as *multisets.*

Example 13.15. Consider the rank two uniform matroid on four elements $\{p_1, p_2, p_3, p_4\}$. An integral relation among the vertices of the matroid polytope is:

$$(1,1,0,0)+(0,0,1,1)=(1,0,1,0)+(0,1,0,1).$$

As a sum of basis elements this corresponds to:

$$\{p_1, p_2\} \cup \{p_3, p_4\} = \{p_1, p_3\} \cup \{p_2, p_4\}.$$

It is a degree two binomial in the ideal of the associated toric variety.

Hence, we say that two multisets of basis are *compatible* if their union (as multisets) is the same. Equivalently, every element of the base set belongs to the same number of basis in the first and second multiset of basis. Thus, the binomials in the ideal of the toric variety represented by matroid base polytope are in bijection with pairs of compatible multisets of basis.

What are the quadrics in such an ideal? Equivalently, when $\{B_1, B_2\}$ is equivalent to $\{B_3, B_4\}$? This is if and only if $B_1 \cup B_2 = B_3 \cup B_4$. In other words, this is if and only if we change:

- B_1 by subtracting from it a set $A_1 \subset B_1 \setminus B_2$ and adding to it $A_2 \subset B_2 \setminus B_1$ and
- B_2 by adding to it A_1 and subtracting A_2.

We see that quadrics in the ideal correspond to multiple symmetric exchanges. It follows that symmetric basis exchanges form a distinguished set of quadrics in the ideal. The following four conjectures are due to White.

Conjecture 13.16.
- *Representable case: The ideal of any torus orbit closure in any Grassmannian is:*
  1. *generated by quadrics,*
  2. *generated by quadrics corresponding to symmetric basis exchanges.*
- *General case: For any matroid M any two finite multisets of basis (B_i), (B_j) such that $\bigcup B_i = \bigcup B_j$ can be transformed to one another in a finite number of such steps that:*
  1. *we replace two basis B, B' in one multiset, by two basis $\tilde{B}, \tilde{B}'$ obtained by multiple symmetric exchange (i.e. $B \cup B' = \tilde{B} \cup \tilde{B}'$),*

(2) *we replace two basis B, B' in one multiset, by two basis $\tilde{B}, \tilde{B}'$ obtained by a symmetric exchange (i.e. $B = \tilde{B} \cup \{b_1\} \setminus \{b_2\}$ and $B' = \tilde{B}' \cup \{b_2\} \setminus \{b_1\}$).*

One can show that the general case implies the representable case.

13.3. Generating Functions

In this section we introduce multivariate generating functions that are given by rational functions. The key example is multigraded Hilbert series. We discuss methods for computing them, and we explore connections to regular triangulations. In particular, we discuss the Ehrhart series.

We start with the familiar example of the polynomial ring $K[\mathbf{x}]$. From Chapter 8 we know we may interpret it as a semigroup algebra $K[\mathbb{Z}^n_{\geq 0}]$. How to interpret the Hilbert function $h(q)$ introduced in Chapter 1? As the monomials of degree d form a basis of $K[\mathbb{Z}^n_{\geq 0}]_d$ the $h(d)$ equals the number of lattice points in the semigroup that belong to the hyperplane $H_d \subset \mathbb{R}^n$ defined by $\sum_{i=1}^n y_i = d$. Let Δ be the standard simplex, i.e. the convex hull of basis vectors. We note that the Hilbert function counts the number of lattice points in dilations of Δ, precisely $h(d) = |d\Delta \cap \mathbb{Z}^n|$. Further, for the Hilbert series we obtain

$$HS(z) = \sum_{q=0}^{\infty} |q\Delta \cap \mathbb{Z}^n| z^q = \frac{1}{(1-z)^n},$$

as in Example 1.22. Our next aim is to refine our counting. So far we have treated all monomials of the same degree on an equal footing. What happens if we try to remember each monomial, not only its degree?

Definition 13.17. Let $C \subset \mathbb{R}^n$ be a rational pointed polyhedral cone. We define the associated *multigraded Hilbert series* as a formal power series:

$$\mathrm{MHS}_C(\mathbf{x}) = \sum_{c \in C \cap \mathbb{Z}^n} \mathbf{x}^c.$$

If $C \subset \mathbb{R}^n_{\geq 0}$ then we may reconstruct the Hilbert series of $\mathbb{C}[C]$ from the multigraded Hilbert series, by setting $x_1 = \cdots = x_n = z$. However, the multigraded Hilbert series remembers much more information: all lattice points of the cone. Our next aim is to represent MHS as a rational function, just as we did with the Hilbert series. A cone generated by linearly independent vectors is called *simplicial*.

Lemma 13.18. *Let C be a simplicial cone with ray generators $c_1, \ldots, c_d \in \mathbb{Z}^n$. Then*

$$\mathrm{MHS}_C(\mathbf{x}) = \frac{\kappa_C(\mathbf{x})}{\prod_{i=1}^d (1 - \mathbf{x}^{\mathbf{c_i}})},$$

where $\kappa_C(\mathbf{x})$ is a Laurent polynomial with non-negative coefficients.

Proof. Consider the following half-open parallelepiped:

$$P := \{x \in C : x = \sum_{i=1}^{d} \lambda_i c_i, 0 \leq \lambda_1, \ldots, \lambda_d < 1\}.$$

As c_i are linearly independent and generate C as a cone, every lattice point $c \in C$ has a unique representation $c = p + \sum_{i=1}^{d} s_i c_i$, where $p \in P$ is a lattice point and s_i are non-negative integers. We obtain:

$$\mathrm{MHS}_C(\mathbf{x}) = \sum_{c \in C \cap \mathbb{Z}^n} \mathbf{x}^c = \sum_{p \in P \cap \mathbb{Z}^n} \mathbf{x}^p \left(\prod_{i=1}^{d} \left(\sum_{s_i=0}^{\infty} \mathbf{x}^{s_i c_i} \right) \right) =$$

$$\sum_{p \in P \cap \mathbb{Z}^n} \mathbf{x}^p \left(\prod_{i=1}^{d} \frac{1}{1 - \mathbf{x}^{c_i}} \right) = \frac{\sum_{p \in P \cap \mathbb{Z}^n} \mathbf{x}^p}{\prod_{i=1}^{d} (1 - \mathbf{x}^{c_i})}.$$

□

Remark 13.19. We could freely manipulate with infinite summations in Lemma 13.18 as the assumption that C is pointed assures that the series is absolutely convergent in some neighborhood.

Proposition 13.20. *Let C be a pointed, rational polyhedral cone with ray generators $c_1, \ldots, c_d \in \mathbb{Z}^n$. Then*

$$\mathrm{MHS}_C(\mathbf{x}) = \frac{\kappa_C(\mathbf{x})}{\prod_{i=1}^{d} (1 - \mathbf{x}_{\mathbf{i}}^{\mathbf{c}})},$$

where $\kappa_C(\mathbf{x})$ is a Laurent polynomial with integral coefficients.

Proof. We may *triangulate* the cone C, i.e. present it as a union of simplicial cones which rays are rays of C and which intersect only in lower dimensional simplicial cones. This may be done e.g. by induction on the number of rays.

Lemma 13.18, together with induction on dimension of C and inclusion-exclusion allow us to conclude. □

Remark 13.21. We note that the proofs of Proposition 13.20 and Lemma 13.18 give us an algorithm to compute the multigraded Hilbert series, as well as combinatorial interpretation of the numerator.

The case when C is a cone over a lattice polytope is particularly nice. Let P be a lattice polytope in $\mathbb{R}^n$. We may regard it as a polytope $P \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$. Let $C \subset \mathbb{R}^{n+1}$ be the cone over P, i.e. the smallest cone that contains $P \times \{1\}$.

Proposition 13.22. *Let C and P be as defined above. The Hilbert function for C with respect to the grading induced by the last variable is given by:*

$$h(q) = |qP \cap \mathbb{Z}^{n+1}|.$$

It counts the number of lattice points in dilations of P. It is known as the Ehrhart polynomial *of P and indeed it coincides with a polynomial for $q \in \mathbb{Z}_{\geq 0}$.*

Proof. The only nontrivial statement is that $h(q)$ is equal to a polynomial for *all* positive integers q. By induction on $\dim P$ and by triangulating P it is enough to prove the statement when P is a simplex with vertices $v_1, \ldots, v_d$. Let $f(q) := \binom{d+q-1}{q} = \binom{d+q-1}{d-1}$ for $q \geq 0$ and $f(q) = 0$ for $q < 0$. As in the proof of Lemma 13.18 we have:

$$h(q) = \sum_{i=0}^{d-1} a_i f(q-i),$$

where a_i is the number of lattice points with the last coordinate i in the set

$$\{x : x = \sum_{i=1}^{d} \lambda_i (v_i, 1), 0 \leq \lambda_1, \ldots, \lambda_d < 1\}.$$

We only have to consider a_i for $i < d$, as there are no lattice points in this parallelepiped with last coordinate greater or equal to d.

We note that f is *not* a polynomial if we consider the negative arguments. The punchline is that the polynomial $g(q) := (d+q-1)(d+q-2)\cdots q/(d-1)!$ equals f also for negative, integral q, as long as $q \geq -d+1$. Hence, for $q \in \mathbb{Z}_{\geq 0}$ we have:

$$h(q) = \sum_{i=0}^{d-1} a_i g(q-i).$$

□

Given a lattice polytope P with N lattice points, we may associate to it a projective toric variety in $\mathbb{P}^{N-1}$ as in Chapter 8. Hence, we obtain a binomial ideal $I_P \subset K[\mathbf{x}] = K[x_1, \ldots, x_N]$. Let us fix a term order $\prec$ which induces the initial ideal $\mathrm{in}_\prec(I_P)$. The latter is a monomial ideal. Thus the radical $\mathrm{rad}\,\mathrm{in}_\prec(I_P)$ has the following property:

- if a product m of distinct variables does not belong to $\mathrm{rad}\,\mathrm{in}_\prec(I_P)$, then neither does any monomial that divides m.

This may be restated as:

- subsets S of variables such that $\prod_{x \in S} x \notin \mathrm{rad}\,\mathrm{in}_\prec(I_P)$ form a simplicial complex.

Our aim is to obtain a nice, geometric description of this simplicial complex. The main idea is that the variables in the ring are in bijection with lattice points of P. Let Δ be a subdivision of P into polytopes P_i that are convex hulls of sets of points S, such that the product of variables corresponding to S is not in $\operatorname{rad}\operatorname{in}_{\prec}(I_P)$.

Example 13.23. Let P be the square $\operatorname{conv}((0,0,1),(0,1,1),(1,0,1),(1,1,1))$. The associated projective variety is a surface in $\mathbb{P}^3$ defined by $x_0x_3 - x_1x_2$. We fix a term order for which x_0x_3 is the leading term. The triangulation Δ of P contains two triangles: $\operatorname{conv}((0,0,1),(0,1,1),(1,0,1))$ and $\operatorname{conv}((0,1,1),(1,0,1),(1,1,1))$. The minimal nonface is the pair of vertices $(0,0,1),(1,1,1)$ corresponding to the unique generator of the initial ideal.

If we change the term order so that x_1x_2 becomes the leading term we obtain a different triangulation of P, given by the other diagonal.

The following proposition relates Gröbner bases to triangulations.

Proposition 13.24. *Using the notation introduced above Δ is a triangulation of P. The minimal non-faces of Δ correspond to (radicals of) generators of* $\operatorname{in}_{\prec}(I_P)$.

Definition 13.25. The triangulations of the form Δ, induced by any term order, are called *regular*. There exist triangulations that are not regular.

The story that we are telling has in fact three sides: combinatorial, algebraic and geometric. From the combinatorial point of view, as described above, we are triangulating a lattice polytope P into simplices. The algebraic part is the finest: we *degenerate* a toric, i.e. binomial, prime ideal I_P to a monomial ideal that shares with I_P all the most important invariants, like dimension and degree.

Let us now describe the geometry in this picture. Here we degenerate the variety $\mathcal{V}(I_P)$ to $\mathcal{V}(\operatorname{in}_{\prec}(I_P))$. One of the problems is that $\operatorname{in}_{\prec}(I_P)$ maybe not radical, thus we may loose some information, however let us ignore this for a moment.

What is $\mathcal{V}(\operatorname{in}_{\prec}(I_P))$? As $\mathcal{V}(\operatorname{in}_{\prec}(I_P)) = \mathcal{V}(\operatorname{rad}(\operatorname{in}_{\prec}(I_P)))$ the question is what is the set of solutions of a squarefree monomial ideal. Let us state the solution to Exercise 12 in Chapter 2:

- the variety $\mathcal{V}(\operatorname{rad}(\operatorname{in}_{\prec}(I_P)))$ is the union of (coordinate) vector subspaces. Each subspace is spanned by basis vectors $(e_i)_{i\in S}$ such that $\prod_{i\in S} x_i \notin \operatorname{rad}(\operatorname{in}_{\prec}(I_P))$.

In particular: the simplices in the induced triangulation of P correspond naturally to components of $\mathcal{V}(\operatorname{in}_{\prec}(I_P))$ - as the triangulation breaks the polytope into simple pieces, our variety breaks into simple components.

We note that the idea of computing the dimension and degree of an ideal, by passing to the initial ideal is equivalent to the idea of computing the Hilbert series of a cone, by subdividing it into simplicial cones.

We know that the dimension of the (projective) toric variety associated to a polytope P equals the dimension of P. What about the degree?

Proposition 13.26. *Let P be a d dimensional lattice polytope, whose lattice points generate the lattice $\mathbb{Z}^d$. The degree of the ideal I_P equals the Euclidean volume of P times $d!$.*

Sketch of the proof.

(1) The degree times the factorial of the dimension is the leading coefficient of the the Ehrhart polynomial h. Thus it is enough to show that for any $\epsilon > 0$ there exists a constant C such that:

$$\frac{\operatorname{vol} P - \epsilon}{d!} q^d - C \leq h_P(q) \leq \frac{\operatorname{vol} P + \epsilon}{d!} q^d + C, \tag{13.1}$$

for any positive integer q, where $h_P(q)$ is the number of lattice points in qP.

(2) It is easy to prove inequality 13.1 for *rational* polytopes that are products of intervals.

(3) Point (2) implies point (1), by covering P with small products of intervals, according to the definition of the Lebesgue measure.

□

Example 13.27. Let P be a d-dimensional simplex, given as the convex hull of 0 and d basis vectors. The Ehrhart polynomial is given by $h(q) = \binom{d+q}{d} = \frac{1}{d!}q^d +$ lower order terms. Indeed $\operatorname{vol} P = \frac{1}{d!}$ and $\dim P = d$.

The usual Euclidean volume multiplied by $d!$ is known as the *normalized volume*. The simplex that is the convex hull of 0 and all standard basis vectors has normalized volume equal to one. Every lattice polytope has normalized volume that is a positive integer, equal to the degree of the associated variety, if one works in the correct lattice.

How is the triangulation compatible with the degree computation? Clearly the volume of the polytope is equal to the sum of volumes of (maximal) simplices in its triangulation.

Theorem 13.28. *Let P be a d dimensional lattice polytope whose lattice points generate the lattice $\mathbb{Z}^d$. Let I_P be the associated toric ideal. Let $\prec$ be a term order and Δ the associated triangulation of P.*

(1) *The minimal primes of $\operatorname{in}_\prec(I_P)$ are in bijection with the maximal simplices in the triangulation Δ.*

(2) *The (unique) primary ideal corresponding to a minimal prime of* $\mathrm{in}_{\prec}(I_P)$ *has degree equal to the normalized volume of the associated simplex in* Δ.

Exercises

(1) Show that Example 13.2 presents matroids.

(2) (a) Fix a family of independent sets $\mathfrak{I}$ for a matroid M. Prove that the inclusion maximal elements in $\mathfrak{I}$ satisfy the axiom for the basis of a matroid.

(b) Fix a nonempty set $\mathfrak{B} \subset 2^E$ satisfying the axiom for basis of a matroid. Prove that $\mathfrak{I} := \{I \subset E : \exists_{B \in \mathfrak{B}} : I \subset B\}$ satisfies the axioms for the independent sets.

(3) Prove that all basis in a matroid have the same cardinality.

(4) Prove that the points p_B in Definition 13.5 are vertices of the polytope P_M. Prove that these are the only lattice points of P_M.

(5) In this exercise we examine matroid duality.

(a) Let $\mathfrak{B} \subset 2^E$ be a set of basis of a matroid M. Let $\mathfrak{B}^* := \{B \subset E : E \setminus B \in \mathfrak{B}\}$. Prove that $\mathfrak{B}^*$ is a set of basis of a matroid M^*. The matroid M^* is known as the dual matroid (of M).

(b) Prove that a dual of a representable matroid is repesentable.

(6) Prove that for any matroid the rank function is submodular.

(7) Prove that any function $2^E \to \mathbb{Z}$ satisfying the three axioms of the rank function is indeed a rank function of some matroid.

(8) How many distinct torus orbit closures are there in $G(2,4)$? How many up to isomorphism (of algebraic varieties)?

(9) Prove White's conjectures for uniform matroids.

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