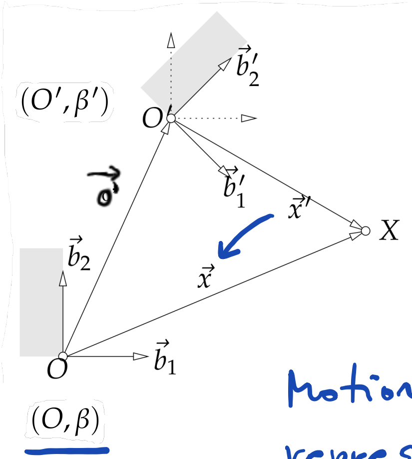
1 Motion

Let us introduce a mathematical model of rigid motion in three-dimensional Euclidean space. The important property of rigid motion is that it only relocates objects without changing their shape. Distances between points on rigidly moving objects remain unchanged. For brevity, we will use "motion" for "rigid motion".

1.1 Change of position vector coordinates induced by motion

§ 1 Alias representation of motion[1]. Figure 1.1(a) illustrates a model of motion using coordinate systems, points and their position vectors. A coordinate system (O, β) with origin O and basis β is attached to a moving rigid body. As the body moves to a new position, a new coordinate system (O', β') is constructed. Assume a point X in a general position w.r.t. the body, which is represented in the coordinate system (O, β) by its position vector $\vec{x}$. The same point X is represented in the coordinate system (O', β') by its position vector $\vec{x}'$. The motion induces a mapping $\vec{x}'_{\beta'} \mapsto \vec{x}_\beta$. Such a mapping also determines the motion itself and provides its convenient mathematical model.

Let us derive the formula for the mapping $\vec{x}'_{\beta'} \mapsto \vec{x}_\beta$ between the coordinates $\vec{x}'_{\beta'}$ of vector $\vec{x}'$ and coordinates $\vec{x}_\beta$ of vector $\vec{x}$. Consider the



Motion is represented by the map $\vec{x}'_{\beta'} \longmapsto \vec{x}_\beta$

[1] The terms *alias* and *alibi* were introduced in the classical monograph [?].

following equations:

$$\begin{aligned}
\vec{x} &= \vec{x}' + \vec{o}' && (1.1)\\
\vec{x}_\beta &= \vec{x}'_\beta + \vec{o}'_\beta && (1.2)\\
\vec{x}_\beta &= \begin{bmatrix} \vec{b}'_{1_\beta} & \vec{b}'_{2_\beta} & \vec{b}'_{3_\beta} \end{bmatrix} \vec{x}'_{\beta'} + \vec{o}'_\beta && (1.3)\\
\boxed{\vec{x}_\beta = \mathtt{R}\, \vec{x}'_{\beta'} + \vec{o}'_\beta} && && (1.4)
\end{aligned}$$

β

$\boxed{\vec{x}'_\beta = R\, \vec{x}'_{\beta'}}$

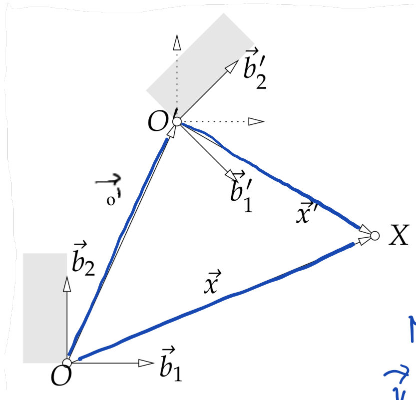
Vector $\vec{x}$ is the sum of vectors $\vec{x}'$ and $\vec{o}'$, Equation 1.1. We can express all vectors in (the same) basis β, Equation 1.2. To pass to the basis β' we introduce matrix $\mathtt{R} = \begin{bmatrix} \vec{b}'_{1_\beta} & \vec{b}'_{2_\beta} & \vec{b}'_{3_\beta} \end{bmatrix}$, which transforms the coordinates of vectors from β' to β, Equation 1.4. Columns of matrix $\mathtt{R}$ are coordinates $\vec{b}'_{1_\beta}, \vec{b}'_{2_\beta}, \vec{b}'_{3_\beta}$ of basic vectors $\vec{b}'_1, \vec{b}'_2, \vec{b}'_3$ of basis β' in basis β.

§2 Alibi representation of motion. An alternative model of motion can be developed from the relationship between the points X and Y and their position vectors in Figure 1.1(b). The point Y is obtained by moving point X altogether with the moving object. It means that the coordinates $\vec{y}'_{\beta'}$ of the position vector $\vec{y}'$ of Y in the coordinate system (O', β') equal the coordinates $\vec{x}_\beta$ of the position vector $\vec{x}$ of X in the coordinate system (O, β), i.e.

$$\begin{aligned}
\vec{y}'_{\beta'} &= \vec{x}_\beta \\
\vec{y}_{\beta'} - \vec{o}'_{\beta'} &= \vec{x}_\beta \\
\mathtt{R}^{-1}\left(\vec{y}_\beta - \vec{o}'_\beta\right) &= \vec{x}_\beta \\
\boxed{\vec{y}_\beta = \mathtt{R}\, \vec{x}_\beta + \vec{o}'_\beta} && (1.5)
\end{aligned}$$

Equation 1.5 describes how is the point X moved to point Y w.r.t. the coordinate system (O, β).

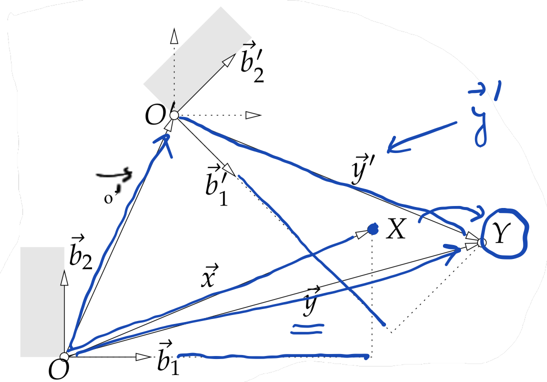
§1 Alias representation of motion[1].



Motion

$\vec{x}'_{\beta'} \mapsto \vec{x}_\beta$

§2 Alibi representation of motion.



Motion

$\vec{x}_\beta \mapsto \vec{y}_\beta$

1.2 Rotation matrix

Motion that leaves at least one point fixed is called rotation. Choosing such a fixed point as the origin leads to $O = O'$ and hence to $\vec{o} = \vec{0}$. The motion is then fully described by matrix R, which is called *rotation matrix*.

$$\vec{x}_\beta = \mathrm{R}\,\vec{x}'_{\beta'} + \vec{o}'_\beta$$

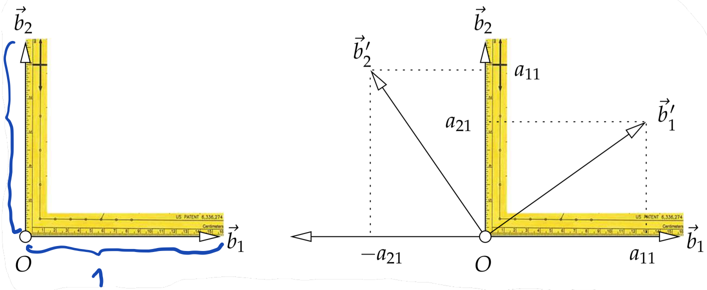
$$\mathrm{R} = \begin{bmatrix} \vec{b}'_{1_\beta} & \vec{b}'_{2_\beta} & \vec{b}'_{3_\beta} \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\,\vec{b}'_{1\beta} + y\,\vec{b}'_{2\beta} + z\,\vec{b}'_{3\beta}$$

$\mathrm{R} \in \mathbb{R}^{3\times 3}$ ... 3 dimensional

§ 1 Two-dimensional rotation. To understand the matrix R, we shall start with an experiment in two-dimensional plane. Imagine a right-angled triangle ruler as shown in Figure 1.2(a) with arms of equal length and let us define a coordinate system as in the figure. Next, rotate the triangle ruler around its tip, i.e. around the origin O of the coordinate system. We know, and we can verify it by direct physical measurement, that, thanks to the symmetry of the situation, the parallelograms through the tips of $\vec{b}'_1$ and $\vec{b}'_2$ and along $\vec{b}_1$ and $\vec{b}_2$ will be rotated by 90 degrees. We see that

1) 2D case



$$\vec{b}'_1 = a_{11}\,\vec{b}_1 + a_{21}\,\vec{b}_2 \tag{1.6}$$

$$\vec{b}'_2 = -a_{21}\,\vec{b}_1 + a_{11}\,\vec{b}_2 \tag{1.7}$$

for some real numbers a_{11} and a_{21}. By comparing it with Equation 1.3, we conclude that

$$\mathrm{R} = \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} \vec{b}'_{1\beta} & \vec{b}'_{2\beta} \end{bmatrix} \tag{1.8}$$

We immediately see that

$$\mathrm{R}^\top \mathrm{R} = \begin{bmatrix} a_{11} & a_{21} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{21}^2 & 0 \\ 0 & a_{11}^2 + a_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.9}$$

$= I$

$\mathrm{R}^{-1}\mathrm{R} = I = \mathrm{R}\,\mathrm{R}^{-1}$

since $(a_{11}^2 + a_{21}^2)$ is the squared length of the basic vector of b_1, which is one. We derived an interesting result

$$\boxed{\mathrm{R}^{-1} = \mathrm{R}^\top} \tag{1.10}$$

$$\mathrm{R} = \mathrm{R}^{-\top} \tag{1.11}$$

Next important observation is that for coordinates $\vec{x}_\beta$ and $\vec{x}'_{\beta'}$, related by a rotation, there holds true

$(x'\ y')\begin{pmatrix}x'\\ y'\end{pmatrix}$

$$(x')^2+(y')^2 = \vec{x}'^\top_{\beta'}\vec{x}'_{\beta'} = \left(\mathrm{R}\,\vec{x}_\beta\right)^\top \mathrm{R}\,\vec{x}_\beta = \vec{x}_\beta^\top \underbrace{\left(\mathrm{R}^\top\mathrm{R}\right)}_{I} \vec{x}_\beta = \vec{x}_\beta^\top\vec{x}_\beta = x^2+y^2 \quad (1.12)$$

→ Axiom

Now, if the basis β was constructed as in Figure 1.2, in which case it is called an *orthonormal basis*, then the parallelogram used to measure coordinates x, y of $\vec{x}$ is a rectangle, and hence x^2+y^2 is the squared length of $\vec{x}$ by the Pythagoras theorem. If β' is related by rotation ro β, then also $(x')^2+(y')^2$ is the squared length of $\vec{x}$, again thanks to the Pythagoras theorem.

We see that $\vec{x}_\beta^\top\vec{x}_\beta$ is the squared length of $\vec{x}$ when β is orthonormal and that this length is preserved by computing it in the same way from the new coordinates of $\vec{x}$ in the new coordinate system after motion. The change of coordinates induced by motion is modeled by rotation matrix R, which has the desired property $\mathrm{R}^\top\mathrm{R} = \mathrm{I}$ when the bases β, β' are both orthonormal.

2) 3D Case

§ 2 Three-dimensional rotation. Let us now consider three dimensions. It would be possible to generalize Figure 1.2 to three dimensions, construct orthonormal bases, and use rectangular parallelograms to establish the relationship between elements of R in three dimensions. However, the figure and the derivations would become much more complicated.

We shall follow a more intuitive path instead. Consider that we have found that with two-dimensional orthonormal bases, the lengths of vectors could be computed by the Pythagoras theorem since the parallelograms determining the coordinates were rectangular. To achieve this in three dimensions, we need (and can!) use bases consisting of three orthogonal vectors. Then, again, the parallelograms will be rectangular and hence the Pythagoras theorem for three dimensions can be used analogically as in two dimensions, Figure 1.3.

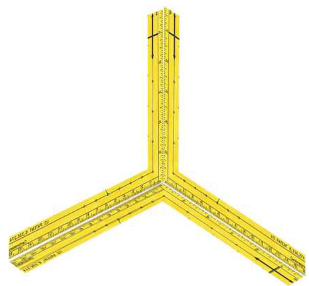


Figure 1.3: A three-dimensional coordinate system.

Considering orthonormal bases β, β', we require the following to hold true for all vectors $\vec{x}$ with $\vec{x}_\beta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ and $\vec{x}'_{\beta'} = \begin{bmatrix} x' & y' & z' \end{bmatrix}^\top$

Distances between points of rigidly moving objects remain unchanged.

Axiom :
$$
\begin{aligned}
(x')^2 + (y')^2 + (z')^2 &= x^2 + y^2 + z^2 \\
\vec{x}'^\top_{\beta'} \vec{x}'_{\beta'} &= \vec{x}^\top_\beta \vec{x}_\beta \\
(\mathtt{R}\,\vec{x}_\beta)^\top \mathtt{R}\,\vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta \\
\vec{x}^\top_\beta \left(\mathtt{R}^\top \mathtt{R}\right) \vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta \\
\vec{x}^\top_\beta \mathtt{C}\, \vec{x}_\beta &= \vec{x}^\top_\beta \vec{x}_\beta
\end{aligned}
\qquad \forall \vec{x}_\beta \in \mathbb{R}^3 \tag{1.13}
$$

$C = R^\top R$

Equation $\boxed{1.13}$ must hold true for all vectors $\vec{x}$ and hence also for special vectors such as those with coordinates ✓ ✓ ✓

$$
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{1.14}
$$

$+$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \checkmark$$

Let us see what that implies, e.g., for the first vector

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 1 & (1.15) \\
\boxed{c_{11} = 1} & & (1.16)
\end{aligned}
$$

$\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = c_{11}$

$c_{11} = c_{22} = c_{33} = 1$

Taking the second and the third vector leads similarly to $c_{22} = c_{33} = 1$. Now, let's try the fourth vector

$$
\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathtt{C} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= 2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & (1.17) \\
1 + c_{12} + c_{21} + 1 &= 2 & (1.18) \\
c_{12} + c_{21} &= 0 & (1.19)
\end{aligned}
$$

$c_{12} + c_{21} = 0$

$c_{12} = -c_{21}$

Again, taking the fifth and the sixth vector leads to $c_{13} + c_{31} = c_{23} + c_{32} = 0$. This brings us to the following form of C

$$\mathtt{C} = \begin{bmatrix} 1 & c_{12} & c_{13} \\ -c_{12} & 1 & c_{23} \\ -c_{13} & -c_{23} & 1 \end{bmatrix} \tag{1.20}$$

$\underbrace{c_{12} = c_{21} \;\&\; c_{12} = -c_{12}}$

$\begin{cases} c_{12} = 0 \\ c_{13} = 0 \\ c_{23} = 0 \end{cases}$

Moreover, we see that C is symmetric since

$$\mathtt{C}^\top = (\underbrace{\mathtt{R}^\top\mathtt{R}}_{\mathtt{C}})^\top = \mathtt{R}^\top\mathtt{R} = \mathtt{C} \tag{1.21}$$

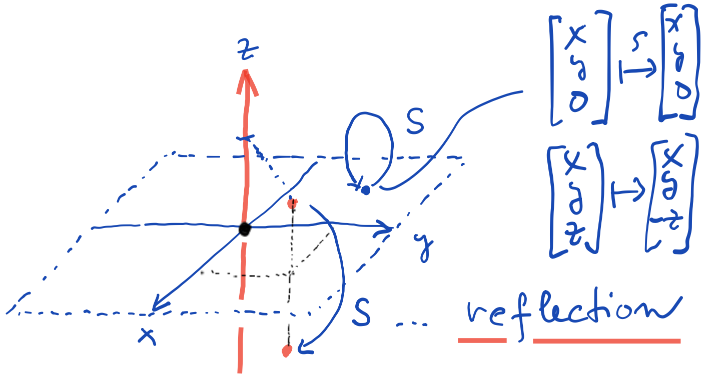
which leads to $-c_{12} = c_{12}$, $-c_{13} = c_{13}$ and $-c_{23} = c_{23}$, i.e. $c_{12} = c_{13} = c_{23} = 0$ and allows us to conclude that

$$\boxed{\mathtt{R}^\top\mathtt{R} = \mathtt{C} = \mathtt{I}} \tag{1.22}$$

Interestingly, not all matrices R satisfying Equation 1.22 represent motions in three-dimensional space.

Consider, e.g., matrix

$$\mathtt{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad S^T S = I \tag{1.23}$$



Matrix S does not correspond to any rotation of the space since it keeps the plane xy fixed and reflects all other points w.r.t. this xy plane. We see that some matrices satisfying Equation 1.22 are rotations but there are also some such matrices that are not rotations. Can we somehow distinguish them?

Notice that $|\mathtt{S}| = -1$ while $|\mathtt{I}| = 1$. It might be therefore interesting to study the determinant of C in general. Consider that

$|S| = 1 \cdot 1 \cdot (-1) = -1$

$\left|\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\right| = 1 \cdot 1 \cdot 1 = 1$

$$1 = |\mathtt{I}| = |(\mathtt{R}^\top\mathtt{R})| = |\mathtt{R}^\top|\,|\mathtt{R}| = |\mathtt{R}|\,|\mathtt{R}| = (|\mathtt{R}|)^2 \tag{1.24}$$

+1

−1

which gives that $|\mathtt{R}| = \pm 1$. We see that the sign of the determinant splits all matrices satisfying Equation 1.22 into two groups – rotations, which have a positive determinant, and reflections, which have a negative determinant. The product of any two rotations will again be a rotation, the product of a rotation and a reflection will be a reflection and the product of two reflections will be a rotation.

To summarize, rotation in three-dimensional space is represented by a 3×3 matrix $\mathtt{R}$ with $\mathtt{R}^\top \mathtt{R} = \mathtt{I}$ and $|\mathtt{R}| = 1$. The set of all such matrices, and at the same time also the corresponding rotations, will be called $SO(3)$, for *special orthonormal three-dimensional group*. Two-dimensional rotations will be analogically denoted as $SO(2)$.

1.3 Coordinate vectors

We see that the matrix $\mathtt{R}$ induced by motion has the property that coordinates and the basic vectors are transformed in the same way. This is particularly useful observation when β is formed by the standard basis, i.e.

$$\beta = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \tag{1.25}$$

For a rotation matrix $\mathtt{R}$, Equation **??** becomes

$$\begin{bmatrix} \vec{b}_1' \\ \vec{b}_2' \\ \vec{b}_3' \end{bmatrix} = \mathtt{R} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 \\ r_{21}\,\vec{b}_1 + r_{22}\,\vec{b}_2 + r_{23}\,\vec{b}_3 \\ r_{31}\,\vec{b}_1 + r_{32}\,\vec{b}_2 + r_{33}\,\vec{b}_3 \end{bmatrix} \tag{1.26}$$

and hence

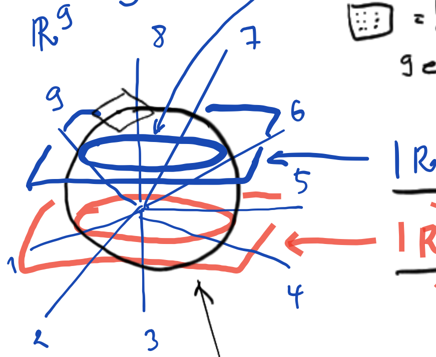
$$\vec{b}_1' = r_{11}\,\vec{b}_1 + r_{12}\,\vec{b}_2 + r_{13}\,\vec{b}_3 = r_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_{12} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_{13} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} \tag{1.27}$$

$R \in \mathbb{R}^{3\times 3} \sim \mathbb{R}^9$ $\}3$

$3\ \square\ 3$

$R^T R = I$ & $|R| = 1$

9 eqns → 9 poly in r_{ij}



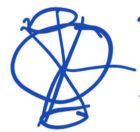
$|R| = 1$ poly eq.

$|R| \cancel{=} -1$

$R^T R = I$

$R \in \mathbb{R}^{3\times 3} \equiv \mathbb{R}^9 <$ $R^T R = I$, $|R| = 1$

Algebraic variety



$f(x,y,z) = 1$ … surface

$f_1(x_1, \dots, x_9) = 1$ …

⋮

$f_k(\qquad) = 0$

2 Rotation

2.1 Properties of rotation matrix

Let us study additional properties of the rotation matrix in three-dimensional space.

2.1.1 Inverse of R

$r_i \in \mathbb{R}^3 \quad i = 1, 2, 3$

Let

$$\mathrm{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} \tag{2.1}$$

be a rotation matrix with columns $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. We can find the inverse of R by evaluating its adjugate matrix [?] and use $\mathrm{R}^{-1} = \mathrm{R}^\top$ and $|\mathrm{R}| = 1$

[The text]

$$\begin{aligned} \mathrm{R}^{-1} &= \frac{1}{|\mathrm{R}|}\,\mathtt{Adj}(\mathrm{R}) && (2.2) \\ \mathrm{R}^\top &= \mathtt{Adj}(\mathrm{R}) && (2.3) \\ &= \begin{bmatrix} \mathbf{r}_2 \times \mathbf{r}_3 & \mathbf{r}_3 \times \mathbf{r}_1 & \mathbf{r}_1 \times \mathbf{r}_2 \end{bmatrix}^\top && (2.4) \\ &= \begin{bmatrix} r_{22}r_{33} - r_{23}r_{32} & r_{13}r_{32} - r_{12}r_{33} & r_{12}r_{23} - r_{13}r_{22} \\ r_{23}r_{31} - r_{21}r_{33} & r_{11}r_{33} - r_{13}r_{31} & r_{13}r_{21} - r_{11}r_{23} \\ r_{21}r_{32} - r_{22}r_{31} & r_{12}r_{31} - r_{11}r_{32} & r_{11}r_{22} - r_{12}r_{21} \end{bmatrix} && (2.5) \end{aligned}$$

Eigenvalues & Eigenvectors

$A \in \mathbb{R}^{n \times n} \quad A\vec{x} = \lambda \vec{x}$

$\vec{x} \neq \vec{0}, \quad \lambda \in \mathbb{R} \to \mathbb{C}$

$$\boxed{\begin{gathered} R \in \mathbb{R}^{3\times 3} \\ R^T R = I \ \& \ |R| = 1 \end{gathered}}$$

$R^T R = I \quad \begin{bmatrix} -r_1^T- \\ -r_2^T- \\ -r_3^T- \end{bmatrix} \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix} =$

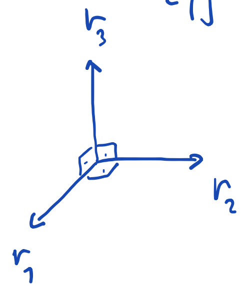
$r_i^T r_i = 1, \quad r_i^T r_j = 0$

$i, j \quad i \neq j$

$r_1 \perp r_2$

$r_2 \perp r_3$

$r_3 \perp r_1$



$$\left\{ \begin{aligned} r_3 &= r_1 \times r_2 \\ r_2 &= r_3 \times r_1 \\ r_1 &= r_2 \times r_3 \end{aligned} \right.$$

which also gives an alternative expression of

$$\mathtt{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} r_{22}\,r_{33} - r_{23}\,r_{32} & r_{23}\,r_{31} - r_{21}\,r_{33} & r_{21}\,r_{32} - r_{22}\,r_{31} \\ r_{13}\,r_{32} - r_{12}\,r_{33} & r_{11}\,r_{33} - r_{13}\,r_{31} & r_{12}\,r_{31} - r_{11}\,r_{32} \\ r_{12}\,r_{23} - r_{13}\,r_{22} & r_{13}\,r_{21} - r_{11}\,r_{23} & r_{11}\,r_{22} - r_{12}\,r_{21} \end{bmatrix} \tag{2.6}$$

$$\mathtt{R}^\top = \begin{bmatrix} r_{22}\,r_{33} - r_{23}\,r_{32} & r_{13}\,r_{32} - r_{12}\,r_{33} & r_{12}\,r_{23} - r_{13}\,r_{22} \\ r_{23}\,r_{31} - r_{21}\,r_{33} & r_{11}\,r_{33} - r_{13}\,r_{31} & r_{13}\,r_{21} - r_{11}\,r_{23} \\ r_{21}\,r_{32} - r_{22}\,r_{31} & r_{12}\,r_{31} - r_{11}\,r_{32} & r_{11}\,r_{22} - r_{12}\,r_{21} \end{bmatrix}$$

2.1.2 Eigenvalues of R

Let R be a rotation matrix. Then for every $\vec{v} \in \mathbb{C}^3$

$$(\mathtt{R}\,\vec{v})^\dagger \mathtt{R}\,\vec{v} = \vec{v}^\dagger \mathtt{R}^\top \mathtt{R}\,\vec{v} = \vec{v}^\dagger (\mathtt{R}^\top \mathtt{R})\,\vec{v} = \vec{v}^\dagger \vec{v} \tag{2.7}$$

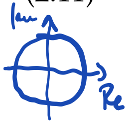
where † is the conjugate transpose[1]. We see that for all $\vec{v} \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}$ such that

$$\mathtt{R}\,\vec{v} = \lambda\,\vec{v} \tag{2.8}$$

there holds true

$$\begin{aligned} (\lambda\,\vec{v})^\dagger(\lambda\vec{v}) &= (\vec{v}^\dagger\vec{v}) & (2.9)\\ \bar{\lambda}\,\lambda\,(\vec{v}^\dagger\vec{v}) &= (\vec{v}^\dagger\vec{v}) & (2.10)\\ |\lambda|^2(\vec{v}^\dagger\vec{v}) &= (\vec{v}^\dagger\vec{v}) & (2.11)\end{aligned}$$

$\vec{v} \neq 0 \Downarrow |\lambda|^2 = 1$



Eigenvalues are solutions to algebraic equations of deg = 3 ⇒ We need to work in $\mathbb{C}$.

The standard scalar product in $\mathbb{C}^n$ is a generalization of the standard scalar product in $\mathbb{R}$

Thus: $\forall\, \vec{u}, \vec{v} \in \mathbb{C}^n : \vec{u}\cdot\vec{v} \in \mathbb{R}$

But, e.g. in $\mathbb{C}^2$:

$\vec{u}\cdot\vec{u} = \vec{u}^\top\vec{u} = (a+bi,\ c+di)\begin{pmatrix} a+bi \\ c+di \end{pmatrix} =$

$(a+bi)^2 + (c+di)^2 = a^2 + (a+b)i - b^2 + c^2 + (c+d)i + d^2 \notin \mathbb{R}$

Hence: $\vec{u}\cdot\vec{v} = \vec{u}^\dagger\,\vec{v} = (\bar{\vec{u}})^\top\vec{v}$

[1] *Conjugate transpose* [?] on vectors with complex coordinates means, e.g., that

$$\begin{bmatrix} a_{11} + b_{11}\,i & a_{12} + b_{12}\,i \\ a_{21} + b_{21}\,i & a_{22} + b_{22}\,i \end{bmatrix}^\dagger = \begin{bmatrix} a_{11} - b_{11}\,i & a_{21} - b_{21}\,i \\ a_{12} - b_{12}\,i & a_{22} - b_{22}\,i \end{bmatrix}$$

for all $a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$. Also recall [?] that $\overline{a\,b} = \bar{a}\,\bar{b}$ for all $a, b \in \mathbb{C}$, † becomes ⊤ for real matrices and $\lambda^\dagger = \bar{\lambda}$ for scalar $\lambda \in \mathbb{C}$. Conjugate transpose is a natural generalization of the Euclidean scalar product in real vector spaces to complex vector spaces. As $\vec{x}^\top\vec{x} = \|\vec{x}\|^2$ gives the squared Euclidean norm for real vectors, $\vec{x}^\dagger\vec{x} = \|\vec{x}\|^2$ gives the squared "Euclidean" norm for complex vectors. It therefore also makes a good sense to extend the notion of angle between complex vectors to $\vec{x}, \vec{y}$ as $\cos\angle(\vec{x},\vec{y}) = \frac{Re(\vec{x}^\dagger\vec{y})}{\sqrt{\vec{x}^\dagger\vec{x}}\sqrt{\vec{y}^\dagger\vec{y}}}$.

and hence $|\lambda|^2 = 1$ for all $\vec{v} \neq \vec{0}$. We conclude that the absolute value of eigenvalues of R is one.

$\Rightarrow \quad |\lambda| = 1$

Next, by looking at the characteristic polynomial of R

$$
\begin{aligned}
p(\lambda) &= |(\lambda\, \mathtt{I} - \mathtt{R})| = \left|\left(\begin{bmatrix} \lambda - r_{11} & -r_{12} & -r_{13} \\ -r_{21} & \lambda - r_{22} & -r_{23} \\ -r_{31} & -r_{32} & \lambda - r_{33} \end{bmatrix}\right)\right| && (2.12)\\
&= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 \\
&\quad + (r_{11}\,r_{22} - r_{21}\,r_{12} + r_{11}\,r_{33} - r_{31}\,r_{13} + r_{22}\,r_{33} - r_{23}\,r_{32})\,\lambda && (2.13)\\
&\quad + r_{11}\,(r_{23}\,r_{32} - r_{22}\,r_{33}) - r_{21}\,(r_{32}\,r_{13} - r_{12}\,r_{33}) + r_{31}\,(r_{13}\,r_{22} - r_{12}\,r_{23}) \\
&= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 + (r_{33} + r_{22} + r_{11})\,\lambda - |\mathtt{R}| && (2.14)\\
&= \lambda^3 - \operatorname{trace}\mathtt{R}\,(\lambda^2 - \lambda) - 1 && (2.15)\\
&= \boxed{(\lambda - 1)}\,(\lambda^2 + (1 - \operatorname{trace}\mathtt{R})\,\lambda + 1) && (2.16)
\end{aligned}
$$

$$
\mathtt{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} r_{22}\,r_{33} - r_{23}\,r_{32} & r_{23}\,r_{31} - r_{21}\,r_{33} & r_{21}\,r_{32} - r_{22}\,r_{31} \\ r_{13}\,r_{32} - r_{12}\,r_{33} & r_{11}\,r_{33} - r_{13}\,r_{31} & r_{12}\,r_{31} - r_{11}\,r_{32} \\ r_{12}\,r_{23} - r_{13}\,r_{22} & r_{13}\,r_{21} - r_{11}\,r_{23} & r_{11}\,r_{22} - r_{12}\,r_{21} \end{bmatrix} \quad (2.6)
$$

we conclude that 1 is always an eigenvalue of R. Notice that we have used identities in Equation 2.6 to pass from Equation 2.13 to Equation 2.14[2].

Let us denote the eigenvalues as $\lambda_1 = 1$, $\lambda_2 = x + yi$ and $\lambda_3 = x - yi$ with real x, y. It follows from the above that $x^2 + y^2 = 1$. We see that there is either one real or three real solutions since if $y = 0$, then $x^2 = 1$ and hence $\lambda_2 = \lambda_3 = \pm 1$. We conclude that we encounter only two situations when all eigenvalues are real. Either $\lambda_1 = \lambda_2 = \lambda_3 = 1$, or $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$.

✓ $\lambda_1 = 1$

$\begin{cases} \lambda_2 = x + yi \\ \lambda_3 = x - yi \end{cases}$ $\quad |\lambda_2| = |\lambda_3| = 1$

with $x^2 + y^2 = 1$

A x' are $\in \mathbb{R}$

$y = 0 \Rightarrow \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 1 \Rightarrow \mathtt{I} \\ \lambda_1 = 1 \\ \lambda_2 = \lambda_3 = -1 \end{cases}$ e.g. $\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$

[2] Alternatively, it follows from the Fundamental theorem of algebra [?] the $p(\lambda) = 0$ has always a solution in $\mathbb{C}$ and since coefficients of $p(\lambda)$ are all real, the solutions must come in complex conjugated pairs. The degree of $p(\lambda)$ is three and thus at least one solution must be real and hence equal to ± 1. Now, since $p(0) = -|(\mathtt{R})| = -1$, $\lim_{\lambda\to\infty} p(\lambda) = \infty$, and $p(\lambda)$ is a continuous function, it must (by the mean value theorem [?]) cross the positive side of the real axis and hence one of its eigenvalues has to be equal to one.