3D Computer Vision

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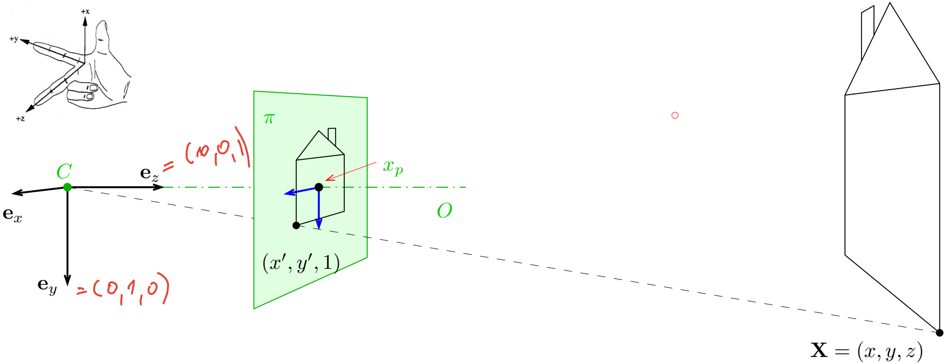
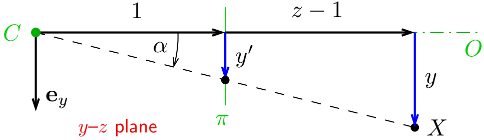
rev. October 3, 2023

Open Informatics Master's Course

▸Canonical Perspective Camera (Pinhole Camera, Camera Obscura)



1. in this picture we are looking ‘down the street’
2. right-handed canonical coordinate system (x, y, z) with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$
3. origin $=$ center of projection C
4. image plane π at unit distance from C
5. optical axis O is perpendicular to π
6. principal point x_p: intersection of O and π
7. perspective camera is given by C and π



projected point in the natural image coordinate system:

$$\tan\alpha = \frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

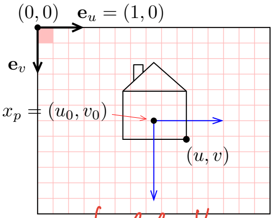
▸Natural and Canonical Image Coordinate Systems

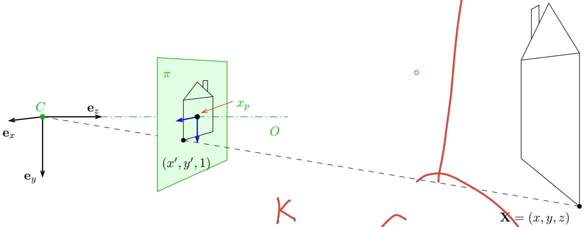
projected point **in canonical camera** $(z \neq 0)$

$$(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) = \frac{1}{z}(x, y, z) \simeq (x, y, z) \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0 = [\mathbf{I} \;\; \mathbf{0}]} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \underline{\mathbf{X}}$$

projected point **in scanned image**

scale by f and translate origin to image corner





focal length

$$u = f\,\frac{x}{z} + u_0$$

$$v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \overset{K}{\begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0\,\underline{\mathbf{X}} = \mathbf{P}\,\underline{\mathbf{X}} \simeq \underline{m}$$

- 'calibration' matrix $\mathbf{K}$ transforms canonical $\mathbf{P}_0$ to standard perspective camera $\mathbf{P}$

▶Computing with Perspective Camera Projection Matrix

Projection from world to image in standard camera $\mathbf{P}$:

$$\underbrace{\begin{bmatrix} f & 0 & u_0 & 0\\ 0 & f & v_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}_{\mathbf{P}} \begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix} = \begin{bmatrix} f x + u_0 z\\ f y + v_0 z\\ z\end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0\\ y + \frac{z}{f} v_0\\ \frac{z}{f}\end{bmatrix}}_{\text{(a)}} \simeq \begin{bmatrix} m_1\\ m_2\\ m_3\end{bmatrix} = \underline{\mathbf{m}}$$

cross-check: $\dfrac{m_1}{m_3} = \dfrac{f\,x}{z} + u_0 = u, \qquad \dfrac{m_2}{m_3} = \dfrac{f\,y}{z} + v_0 = v$ when $m_3 \neq 0$

f – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

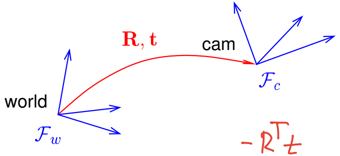
(u_0, v_0) – principal point in pixels

Perspective Camera:

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)

   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the u_0, v_0 in relative units
3. $(m_1, m_2, 0)$ represents points at infinity in image plane π — i.e. points with $z = 0$

▶Changing The Outer (World) Reference Frame



A transformation of a point from the world to camera coordinate system:

$$\boxed{\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}}$$

$-R^T t$

$\mathbf{R}$ – rotation matrix world orientation in the camera coordinate frame $\mathcal{F}_c$

$\mathbf{t}$ – translation vector world origin in the camera coordinate frame $\mathcal{F}_c$

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0\begin{bmatrix}\mathbf{X}_c\\ 1\end{bmatrix} = \mathbf{K}\mathbf{P}_0\begin{bmatrix}\mathbf{R}\mathbf{X}_w + \mathbf{t}\\ 1\end{bmatrix} = \mathbf{K}\underbrace{\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}}_{\mathbf{P}_0}\underbrace{\begin{bmatrix}\mathbf{R} & \mathbf{t}\\ \mathbf{0}^\top & 1\end{bmatrix}}_{\mathbf{T}\,\in\,\mathbb{R}^{4\times4}}\begin{bmatrix}\mathbf{X}_w\\ 1\end{bmatrix} = \mathbf{K}\overbrace{\underbrace{\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}}_{\mathbb{R}^{3\times4}}}^{P}\underline{\mathbf{X}}_w \simeq \underline{m} \qquad \mathbf{T}^{-1} = ?\begin{bmatrix}R^T & -R^T t\\ 0 & 1\end{bmatrix}$$

$SE(3)$

$\mathbf{P}_0$ (a 3×4 mtx) discards the last row of $\mathbf{T}$

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = +1$ $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \overbrace{\mathbf{K}\mathbf{R}}^{Q}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix} \qquad \text{i.e. } \mathbf{C} = -\mathbf{R}^\top\mathbf{t}$$

$\mathbf{C}$ – camera position in the world reference frame $\mathcal{F}_w$ $\mathbf{t} = -\mathbf{R}\mathbf{C}$

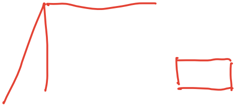
$\mathbf{r}_3^\top$ – optical axis in the world reference frame $\mathcal{F}_w$ cam: $\mathbf{o}_c = (1, 0, 0)$, world: $\mathbf{o}_w = -\mathbf{R}^\top\mathbf{o}_c = \mathbf{r}_3^\top$ third row of $\mathbf{R}$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}\underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

▶Changing the Inner (Image) Reference Frame

The general form of calibration matrix K includes

- skew angle θ of the digitization raster
- pixel aspect ratio a



$$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0 \\ 0 & f/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

units: $[f] = \text{px}$, $[u_0] = \text{px}$, $[v_0] = \text{px}$, $[a] = 1$

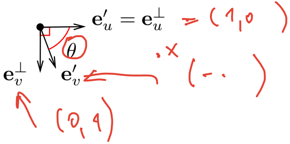
$$\begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & v_0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ? \quad \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

⊛ H1; 2pt: Give the parameters f, a, θ, u_0, v_0 a precise meaning by decomposing $\mathbf{K}$ to simple maps; deadline LD+2 wk

Hints:

1. image projects to orthogonal system $F^\perp$, then it maps by skew to F', then by scale $a\,f$, f to F'', then by translation by u_0, v_0 to F'''
2. Skew: Do not confuse it with the shear mapping. Express point $\mathbf{x}$ as

$$\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u^\perp\mathbf{e}_u^\perp + v^\perp\mathbf{e}_v^\perp\,, \qquad u, v \in \mathbb{R}$$



   $\mathbf{e}_:$ are unit-length basis vectors $\mathbf{e}_u^\perp = \mathbf{e}'_u = (1,0)$, $\mathbf{e}_v^\perp = (0,1), \ldots$

   consider their four pairwise dot-products $(\mathbf{e}'_u)^\top\mathbf{e}_u^\perp = 0$, $(\mathbf{e}'_u)^\top\mathbf{e}'_v = \cos(\theta), \ldots$

3. $\mathbf{K}$ maps from $F^\perp$ to F''' as

$$w'''\,[u''', v''', 1]^\top = \mathbf{K}[u^\perp, v^\perp, 1]^\top$$

►Summary: Projection Matrix of a General Finite Perspective Camera

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \simeq \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite perspective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left 3×3 submatrix $\mathbf{Q}$ non-singular.

random finite camera: `Q = rand(3,3); while det(Q)==0, Q = rand(3,3); end, P = [Q, rand(3,1)];`

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$ full rank (if finite perspective camera; see [H&Z, Sec. 6.3] for cameras at infinity)
$\mathbf{K} \in \mathbb{R}^{3,3}$ upper triangular with positive diagonal elements
$\mathbf{R} \in \mathbb{R}^{3,3}$ rotation mtx: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \begin{bmatrix}\mathbf{KR} & \mathbf{Kt}\end{bmatrix}$ also →35
2. RQ decomposition of $\mathbf{Q} = \mathbf{KR}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = (\mathbf{Q}\,\underbrace{\mathbf{R}_{32})\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}} \qquad \mathbf{QR}_{32} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \underset{\uparrow 3,2}{0} & \cdot \end{bmatrix}, \ \mathbf{QR}_{32}\mathbf{R}_{31} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \underset{\uparrow 3,1}{0} & 0 & \cdot \end{bmatrix}, \ \mathbf{QR}_{32}\mathbf{R}_{31}\mathbf{R}_{21} = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g.
(see the next slide for derivation details)

$$(3,2) \neq 0 \qquad \mathbf{QR}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \text{ gives } \begin{matrix} c^2 + s^2 = 1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33} \end{matrix} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{KR} = \mathbf{KT}^{-1}\mathbf{TR}$, where $\mathbf{T} = \operatorname{diag}(-1,-1,1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive 'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array[q_{#1,#2} &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q.R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\,q_{1,2} + s\,q_{1,3} & -s\,q_{1,2} + c\,q_{1,3} \\ q_{2,1} & c\,q_{2,2} + s\,q_{2,3} & -s\,q_{2,2} + c\,q_{2,3} \\ q_{3,1} & c\,q_{3,2} + s\,q_{3,3} & -s\,q_{3,2} + c\,q_{3,3} \end{pmatrix}$$

```
s1 = Solve[{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}}\right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\,q_{3,2}+q_{1,2}\,q_{3,3}}{\sqrt{q_{3,2}^2+q_{3,3}^2}} & \frac{q_{1,2}\,q_{3,2}+q_{1,3}\,q_{3,3}}{\sqrt{q_{3,2}^2+q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\,q_{3,2}+q_{2,2}\,q_{3,3}}{\sqrt{q_{3,2}^2+q_{3,3}^2}} & \frac{q_{2,2}\,q_{3,2}+q_{2,3}\,q_{3,3}}{\sqrt{q_{3,2}^2+q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

▶Center of Projection (Optical Center)

Observation: finite $\mathbf{P}$ has a non-trivial right null-space $\quad \underline{c} = \begin{bmatrix} C \\ 1 \end{bmatrix}$ rank 3 but 4 columns

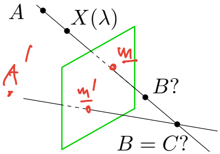
Theorem

Let $\mathbf{P}$ be a camera and let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equivalent to the projection center $\underline{\mathbf{C}}$ (homogeneous, in world coordinate frame).

Proof.

1. Let AB be a spatial line (B given from $\mathbf{P}\underline{\mathbf{B}} = \mathbf{0}$, $A \neq B$). Then

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R} \qquad \text{(world frame)}$$



2. It projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + \cancel{(1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}}} \simeq \mathbf{P}\,\underline{\mathbf{A}} \simeq \underline{m}$$

- the entire line projects to a single point ⇒ it must pass through the projection center of $\mathbf{P}$
- this holds for any choice of $A \neq B$ ⇒ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$ □

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C} \\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \;\Rightarrow\; \boxed{\mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}}$$

⊛ verify from →30

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. Consider the following spatial line (world coordinate frame)

   $\mathbf{d} \in \mathbb{R}^3$ line direction vector, $\|\mathbf{d}\| = 1$, $\underline{\lambda \in \mathbb{R}}$, Cartesian representation

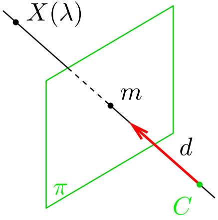
$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\,\mathbf{d}$$

2. The projection of the (finite) point $X(\lambda)$ is

$QC + q = 0 \rightarrow 35$

$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X}(\lambda)\\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda\,\underbrace{\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}}_{P} \begin{bmatrix}\mathbf{d}\\ 0\end{bmatrix} \leftarrow X_\infty$$



. . . which is also the image of a point at infinity in $\mathbb{P}^3$

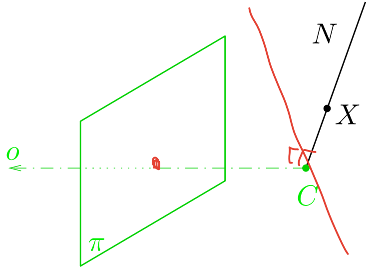
- optical ray line corresponding to image point m is the set

$$\boxed{\mathbf{X}(\mu) = \mathbf{C} + \mu\,\mathbf{Q}^{-1}\underline{\mathbf{m}}, \qquad \mu \in \mathbb{R} \qquad (\mu = 1/\lambda)}$$

- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in the world coordinate frame

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π



1. points X on a given line N parallel to π project to a point at infinity $(u, v, 0)$ in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

2. therefore the set of points X is parallel to π iff

$q_{31} \cdot x + q_{32} \cdot y + q_{33} \cdot z$

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane equation with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

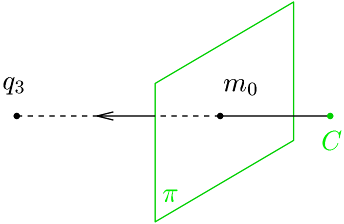
$$\boxed{\mathbf{o} = \det(\mathbf{Q})\, \mathbf{q}_3}$$

if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\, \mathbf{q}_3$, hence $\mathbf{o} \mapsto \mathbf{o} \cdot \lambda^4 = \partial$ $\lambda \neq \pm 1$ [H&Z, p. 161]

- the axis is expressed in the world coordinate frame

►Principal Point

Principal point: The intersection of image plane and the optical axis



1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to the principal point m_0
3. then

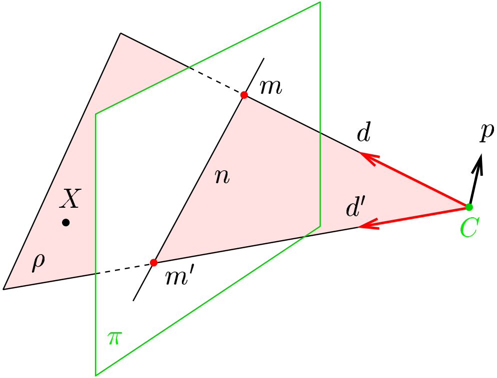
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\boxed{\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3}$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p containing the projection center C and a given image line n.



optical ray given by m $\quad \mathbf{d} \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$\mathbf{p} \simeq \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') \overset{\circledast 1}{=} \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

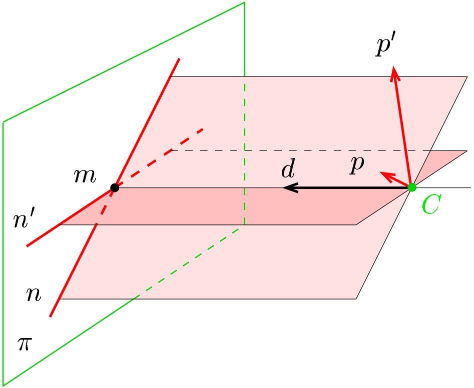
- note the way **Q** factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top} \underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\to 30} = \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \boxed{\underline{\boldsymbol{\rho}} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}}$

$$\rho_1\,x + \rho_2\,y + \rho_3\,z + \rho_4 = 0$$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad$ $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad$ $\mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1} (\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1} \underline{\mathbf{m}}$$

▶Summary: Projection Center; Optical Ray, Axis, Plane

General (finite) camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14}\\ \mathbf{q}_2^\top & q_{24}\\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P}), \quad \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$	projection center (world coords.) →35
$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$	optical ray direction (world coords.) →36
$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$	outward optical axis (world coords.) →37
$\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$	principal point (in image plane) →38
$\underline{\boldsymbol{\rho}} = \mathbf{P}^\top\,\underline{\mathbf{n}}$	optical plane (world coords.) →39
$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0\\ 0 & f/\sin\theta & v_0\\ 0 & 0 & 1\end{bmatrix}$	camera (calibration) matrix (f, u_0, v_0 in pixels) →31
$\mathbf{R}$	rotation matrix (cam coords.) →30
$\mathbf{t}$	translation vector (cam coords.) →30

Thank You