3D Computer Vision

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Open Informatics Master's Course

Module II

Perspective Camera

covered by

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, Example: 2.19

▶Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	n	O
plane		π, φ

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^\top, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as $\mathbf{m} = (u, v)$, $\mathbf{X} = (x, y, z)$, etc.

- vectors are always meant to be columns $\mathbf{x} \in \mathbb{R}^{n \times 1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^\top, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^\top, \quad \underline{\mathbf{n}}$$

'in-line' forms: $\underline{\mathbf{m}} = (m_1, m_2, m_3)$, $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$, etc.

- matrices are $\mathbf{Q} \in \mathbb{R}^{m \times n}$, linear map of a $\mathbb{R}^{n \times 1}$ vector is $\mathbf{y} = \mathbf{Q}\mathbf{x}$
- j-th element of vector $\mathbf{m}_i$ is $(\mathbf{m}_i)_j$; element i, j of matrix $\mathbf{P}$ is $\mathbf{P}_{ij}$

▶Image Line (in 2D)

a finite line in the 2D (u, v) plane $\qquad (u, v) \in \mathbb{R}^2 \quad \text{s.t.} \quad a\,u + b\,v + c = 0$

has a parameter (homogeneous) vector $\qquad \underline{\mathbf{n}} \simeq (a,\, b,\, c)\,, \quad \|\underline{\mathbf{n}}\| \neq 0$

and there is an equivalence class for $\lambda \in \mathbb{R},\, \lambda \neq 0 \qquad (\lambda a,\, \lambda b,\, \lambda c) \simeq (a,\, b,\, c)$

'Finite' lines

- standard representative for finite $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ is $\lambda\underline{\mathbf{n}}$, where $\lambda = \frac{\mathbf{1}}{\sqrt{n_1^2+n_2^2}}$

  assuming $n_1^2 + n_2^2 \neq 0$; $\mathbf{1}$ is the unit, usually $\mathbf{1} = 1$

'Infinite' line

- we augment the set of lines for a special entity called the **line at infinity** (ideal line)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, \mathbf{1}) \qquad \text{(standard representative)}$$

- the set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0,0,0)$ forms the projective space $\mathbb{P}^2$ — a set of rays →21
- line at infinity is a proper member of $\mathbb{P}^2$
- I may sometimes wrongly use $=$ instead of $\simeq$, if you are in doubt, ask me

▶Image Point

Finite point $\mathbf{m} = (u, v)$ is incident on a finite line $\underline{\mathbf{n}} = (a, b, c)$ iff — iff = works either way!

$$a\,u + b\,v + c = 0$$

can be rewritten as (with scalar product): $(u, v, \mathbf{1}) \cdot (a, b, c) = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

'Finite' points

- a finite point is also represented by a homogeneous vector $\underline{\mathbf{m}} \simeq (u, v, \mathbf{1})$, $\|\underline{\mathbf{m}}\| \neq 0$
- the equivalence class for $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is $(m_1, m_2, m_3) = \lambda\,\underline{\mathbf{m}} \simeq \underline{\mathbf{m}}$
- the standard representative for finite point $\underline{\mathbf{m}}$ is $\lambda\,\underline{\mathbf{m}}$, where $\lambda = \frac{\mathbf{1}}{m_3}$ — assuming $m_3 \neq 0$
- when $\mathbf{1} = 1$ then units are pixels and $\lambda\underline{\mathbf{m}} = (u, v, 1)$
- when $\mathbf{1} = f$ then all elements have a similar magnitude, $f \sim$ image diagonal

use $\mathbf{1} = 1$ unless you know what you are doing;
all entities participating in a formula must be expressed in the same units

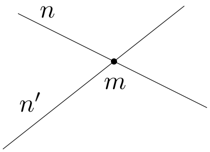
'Infinite' points

- we augment for **points at infinity** (ideal points) $\underline{\mathbf{m}}_\infty \simeq (m_1, m_2, 0)$ — proper members of $\mathbb{P}^2$
- all such points lie on the line at infinity (ideal line) $\underline{\mathbf{n}}_\infty \simeq (0, 0, 1)$, i.e. $\underline{\mathbf{m}}_\infty^\top \underline{\mathbf{n}}_\infty = 0$

▶Line Intersection and Point Join

The point of **intersection** m of image lines n and n', $n \not\simeq n'$ is

$$\underline{\mathbf{m}} \simeq \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$$

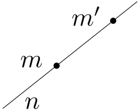


proof: If $\underline{\mathbf{m}} = \underline{\mathbf{n}} \times \underline{\mathbf{n}}'$ is the intersection point, it must be incident on both lines. Indeed, using known equivalences from vector algebra

$$\underline{\mathbf{n}}^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv \underline{\mathbf{n}}'^\top \underbrace{(\underline{\mathbf{n}} \times \underline{\mathbf{n}}')}_{\underline{\mathbf{m}}} \equiv 0$$

The **join** n of two image points m and m', $m \not\simeq m'$ is

$$\underline{\mathbf{n}} \simeq \underline{\mathbf{m}} \times \underline{\mathbf{m}}'$$

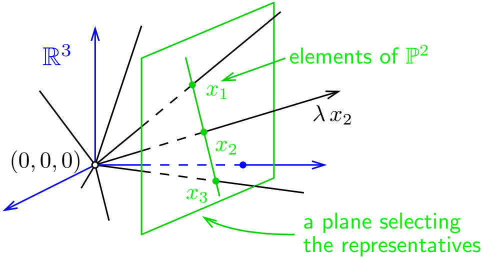


Paralel lines intersect (somewhere) on the line at infinity $\underline{\mathbf{n}}_\infty \simeq (0,0,1)$:

$$a\,u + b\,v + c = 0,$$
$$a\,u + b\,v + d = 0, \qquad d \neq c$$
$$(a,b,c) \times (a,b,d) \simeq (b,-a,0)$$

- all such intersections lie on $\underline{\mathbf{n}}_\infty$
- line at infinity therefore represents the set of (unoriented) directions in the plane
- Matlab: `m = cross(n, n_prime);`

▶Homography in $\mathbb{P}^2$



Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$, factorized to linear equivalence classes ('rays'), $\underline{\mathbf{x}} \simeq \lambda \underline{\mathbf{x}}$, $\lambda \neq 0$ including 'points at infinity'

we call $\underline{\mathbf{x}} \in \mathbb{P}^2$ 'points'

Homography in $\mathbb{P}^2$: Non-singular linear mapping in $\mathbb{P}^2$ an analogic definition for $\mathbb{P}^3$

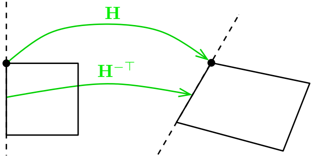
$$\underline{\mathbf{x}}' \simeq \mathbf{H}\,\underline{\mathbf{x}}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

Defining properties

- collinear points are mapped to collinear points lines of points are mapped to lines of points
- concurrent lines are mapped to concurrent lines concurrent = intersecting at a point
- and point-line incidence is preserved e.g. line intersection points mapped to line intersection points

- $\mathbf{H}$ is a 3×3 non-singular matrix, $\lambda\,\mathbf{H} \simeq \mathbf{H}$ equivalence class, 8 degrees of freedom
- homogeneous matrix representative: $\det \mathbf{H} = 1$ $\mathbf{H} \in \mathrm{SL}(3)$
- what we call homography here is often called 'projective collineation' in mathematics

▶Mapping 2D Points and Lines by Homography



$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}}$	(image) point	
$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}}$	(image) line	$\mathbf{H}^{-\top} = (\mathbf{H}^{-1})^\top = (\mathbf{H}^\top)^{-1}$

- incidence is preserved: $(\underline{\mathbf{m}}')^\top \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^\top \mathbf{H}^\top \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^\top \underline{\mathbf{n}} = 0$

Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$

1. extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
2. map by homography, $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
3. if $m_3' \neq 0$ convert the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ back to Cartesian coordinates (pixels),

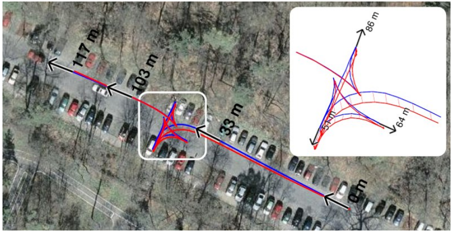
$$u' = \frac{m_1'}{m_3'}\,\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\,\mathbf{1}$$

- note that, typically, $m_3' \neq 1$ — $m_3' = 1$ when $\mathbf{H}$ is affine
- an infinite point $\underline{\mathbf{m}} = (u, v, 0)$ maps the same way

Rectification of camera rotation: →59 (geometry), →129 (homography estimation)

$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}$ maps from image plane to facade plane

Homographic Mouse for Visual Odometry: [Mallis 2007]



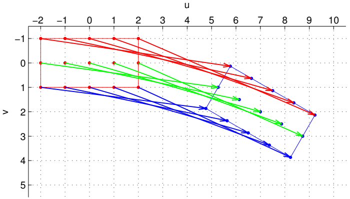
illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

$$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\top}{d}\right)\mathbf{K}^{-1}$$ maps from plane to translated plane [H&Z, p. 327]

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathrm{SE}(2)$$

- note: action $H(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{t} \colon \mathbb{R}^2 \to \mathbb{R}^2$, not commutative



rotation by 30°, then translation by $(7,\ 2)$

EM = The most general homography preserving

1. **lengths**: Let $\mathbf{x}_i' = H(\mathbf{x}_i)$. Then

$$\|\mathbf{x}_2' - \mathbf{x}_1'\| = \|H(\mathbf{x}_2) - H(\mathbf{x}_1)\| \overset{\circledast\ \text{P1; 1pt}}{=} \cdots = \|\mathbf{x}_2 - \mathbf{x}_1\|$$

2. **angles** — check the dot-product of normalized differences from a point $(\mathbf{x}-\mathbf{z})^\top(\mathbf{y}-\mathbf{z})$ (Cartesian(!))
3. **areas**: $\det \mathbf{H} = 1 \ \Rightarrow$ unit determinant of the action's Jacobian $\mathbf{J}$ — it follows from: $\mathbf{J} = \mathbf{R}$, $\det \mathbf{R} = 1$

- eigenvalues $\left(1,\ e^{-i\phi},\ e^{i\phi}\right)$
- eigenvectors when $\phi \neq k\pi$, $k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

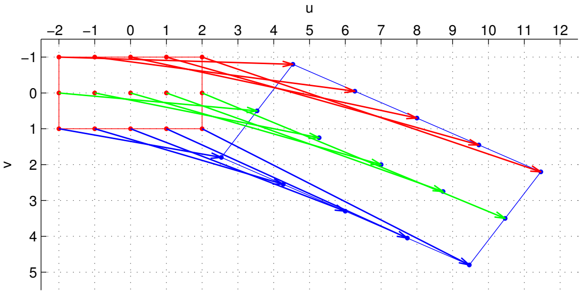
$\mathbf{e}_2$, $\mathbf{e}_3$ – circular points, i – imaginary unit

4. **circular points**: complex points at infinity $(i, 1, 0)$, $(-i, 1, 0)$ (preserved even by similarity)

- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

▶Homography Subgroups: Affine Mapping (Affinity)

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



rotation by 30°
then scaling by $\mathrm{diag}(1,\ 1.5,\ 1)$
then translation by $(7,\ 2)$

Affinity = The most general homography preserving

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints, centers of gravity)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

observe $\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\mathbf{n}}_\infty \quad \Rightarrow \quad \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$

does not preserve

- lengths
- angles
- areas
- circular points

▶Homography Subgroups: General Homography

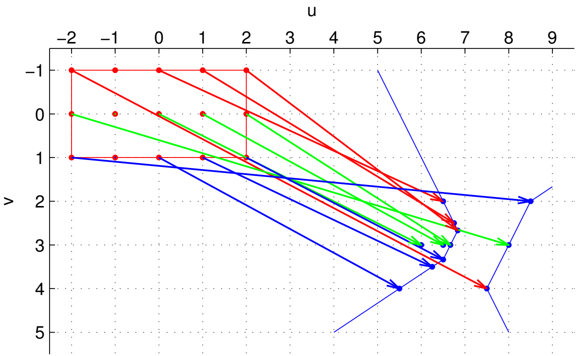
$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \qquad \mathbf{H} \in \mathrm{SL}(3)$$

preserves only

- incidence and concurrency
- collinearity
- cross-ratio (ratio of ratios) on the line →46

does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$

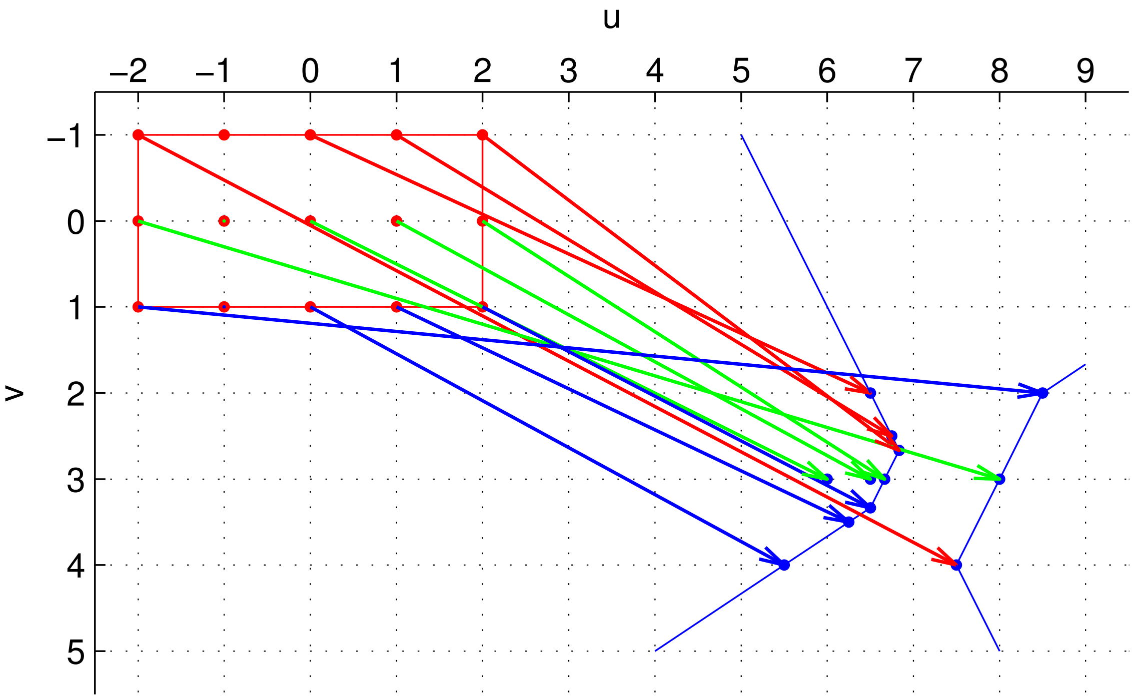


$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

(where in the picture is the line n?)

Thank You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