GRAPHICAL MARKOV MODELS
EXAM WS2019 (27P)

Assignment 1. (3p)
A homogeneous Markov model defined on a chain has the following matrix P of transition probabilities

$$P = \begin{pmatrix} 0 & .4 & 0 \\ 1 & .2 & 1 \\ 0 & .4 & 0 \end{pmatrix}$$

a) Find a stationary distribution for the states. Is it unique?
b) Prove that all states of the model are a-periodic.

Assignment 2. (6p)
Consider the following standard Markov chain model for sequences $s = (s_1, \dots, s_n)$ of length n with states $s_i \in K$ given by:

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1}).$$

The conditional probabilities $p(s_i \mid s_{i-1})$ and the marginal probability $p(s_1)$ for the first element are assumed to be known.

The state space K is partitioned into two sets A, B, i.e. $A \cup B = K$ and $A \cap B = \varnothing$. We consider transitions from A to B, i.e. $s_i \in A$ and $s_{i+1} \in B$ and want to know how often such transitions happen on average. Propose an efficient algorithm for computing this average.

Assignment 3. (7p)
Suppose you want to learn the transition probabilities of a homogeneous Markov chain model

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})$$

with discrete valued states $s_i \in K$. For this you are given an i.i.d. sample $\mathcal{T}$ of sequences s as training data. However, some of the sequences have missing elements in some positions. These positions are known and may differ from sequence to sequence. Propose a learning approach based on maximum likelihood.

Assignment 4. (3p)
Prove that the function $f \colon \mathbb{Z}^2 \to \mathbb{R}$ defined by $f(x, y) = |x - y|$ is submodular.

Assignment 5. (8p)
Consider a (Min,+)-problem for K valued labellings $s \colon V \to K$ of a graph (V, E)

$$F(s) = \sum_{i \in V} u_i(s_i) + \sum_{ij \in E} u_{ij}(s_i, s_i) \to \min_{s \in K^V} .$$

Suppose we want to extend the objective function to $G(s) = F(s) + \alpha N(s)$, where $N(s)$ "counts" the number of labels appearing in the labelling s, i.e.

$$N(s) = |\{k \in K \mid \exists i \in V \colon s_i = k\}|.$$

The extended objective function is not any more a sum of functions of arity two or less.

a) Find an equivalent representation of the extended optimisation task, such that the objective function is a sum of functions with arity at most two, by introducing auxiliary nodes (variables) and edges. (Hint: the label set for the auxiliary variables can be different from K).

b) Suppose that the initial (Min,+)-problem was submodular w.r.t. some ordering of K. Is the extended (Min,+)-problem constructed by you still submodular?