Computational Game Theory

The Shapley Value

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How to divide the prey fairly?

Four animal friends hunt buffaloes together. The *pig* eats the dead prey and the *owl* can track down buffaloes. Only the *Komodo dragon* and the *tiger* can kill a buffalo.

$$v(A) = \begin{cases} 2 & A = N, 234 \\ 1 & A = 134, 34 \\ 0 & \text{otherwise} \end{cases}$$

What are the basic principles of fair allocation?

$$v(A) = \begin{cases} 2 & A = N, 234 \\ 1 & A = 134, 34 \\ 0 & \text{otherwise} \end{cases}$$

The same reward for the same working contribution

The dragon and the tiger should get an equal portion!

He who does not work, neither shall he eat

The pig should not get any portion of the prey!

John Banzhaf: "Weighted voting doesn't work"

Nassau County Board's voting system in 1967

Each town has the number of votes based on its population:

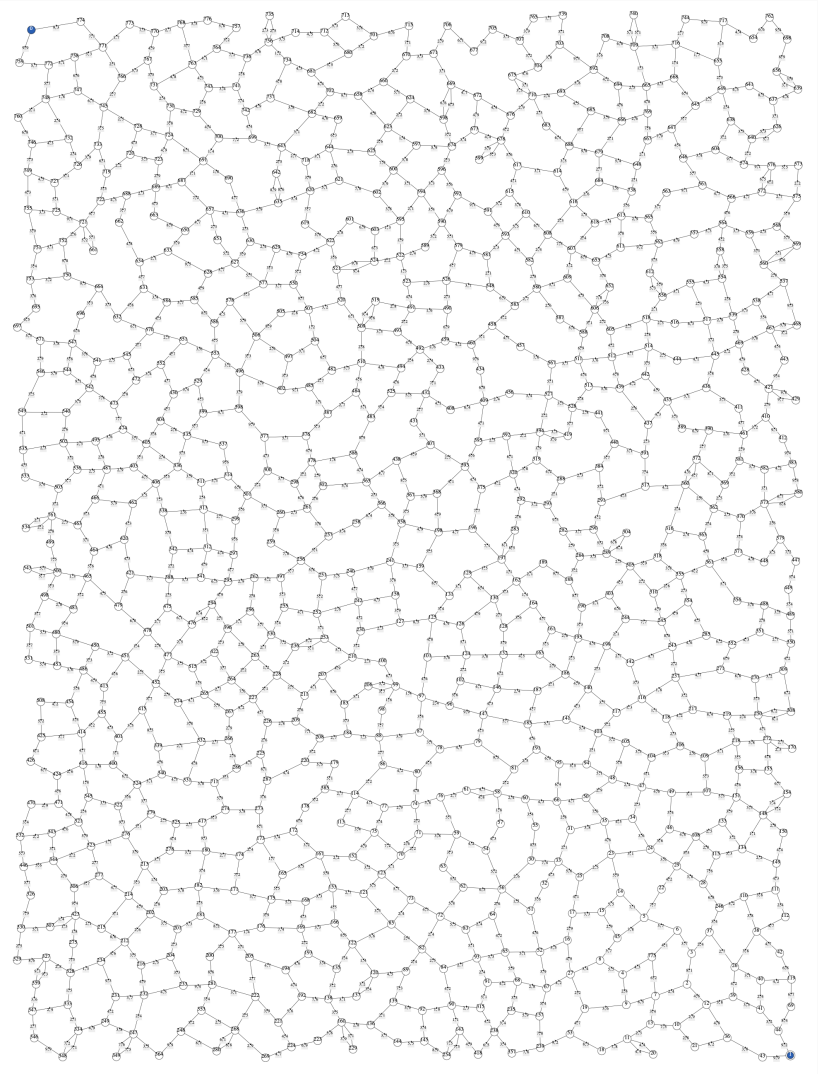
Town i	*Votes* w_i
1	31
2	31
3	28
4	21
5	2
6	2

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i \geq 58 \\ 0 & \text{otherwise} \end{cases}$$

Neither 5 nor 6 can overturn any decision!

Voting power is the ability of a legislator to cast a decisive vote.

Why is the Shapley value important for computer science?



Quantify the effect of individual components on the performance of the entire system.

More applications

- Google Analytics
- Explainable AI algorithms
- Computational biology
- Who has the power in EU
- Centralities in networks

The Shapley value: From Axioms to the Formula

How to associate a unique allocation with any game?

Definition

Let Γ be the linear space of all games over the player set N.
A value (or an allocation rule) is a mapping

$$\varphi \colon \Gamma \to \mathbb{R}^n, \qquad \varphi = (\varphi_1, \ldots, \varphi_n).$$

The vector $\varphi(v) = (\varphi_1(v), \ldots, \varphi_n(v))$

- provides a unique outcome of any game v
- can be interpreted as the allocation to the players decided by an external arbitrator or by an authority

Any value φ must reflect basic axioms of fairness.

Formulating basic axioms of fairness (1)

A player $i \in N$ is called a null player in a game v if

$$v(A \cup i) = v(A), \qquad \text{for all } A \subseteq N \setminus i.$$

He who does not work, neither shall he eat

Value φ satisfies the null player property if the following implication holds for each game v and each player $i \in N$:

$$i \text{ is a null player in } v \qquad \Longrightarrow \qquad \varphi_i(v) = 0$$

Formulating basic axioms of fairness (2)

Players $i, j \in N$ are symmetric in a game v if

$$v(A \cup i) = v(A \cup j), \qquad \text{for each coalition } A \subseteq N \setminus ij.$$

The same reward for the same working contribution

Value φ is symmetric if the following implication holds for each game v and all players $i, j \in N$:

$$i \text{ and } j \text{ are symmetric} \qquad \Longrightarrow \qquad \varphi_i(v) = \varphi_j(v).$$

Additional axioms of value

Efficiency

$$\varphi_1(v) + \cdots + \varphi_n(v) = v(N) \qquad \text{for every } v \in \Gamma$$

Additivity

$$\boldsymbol{\varphi}(u + v) = \boldsymbol{\varphi}(u) + \boldsymbol{\varphi}(v) \qquad \text{for every } u, v \in \Gamma$$

Linearity

$$\boldsymbol{\varphi}(\alpha u + \beta v) = \alpha \boldsymbol{\varphi}(u) + \beta \boldsymbol{\varphi}(v) \qquad \text{for every } u, v \in \Gamma \text{ and all } \alpha, \beta \in \mathbb{R}$$

The axioms determine the Shapley value for all games!

The Shapley value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{A \subseteq N \setminus i} \frac{|A|!(n - |A| - 1)!}{n!} \cdot \big(v(A \cup i) - v(A)\big).$$

Theorem (Shapley, 1953)

The Shapley value φ^S is the unique value $\Gamma \to \mathbb{R}^n$ that satisfies efficiency, additivity, symmetry, and the null player property. Moreover, none of the axioms is superfluous.

The axioms determine the Shapley value uniquely: Why?

For each coalition $A \neq \emptyset$, define a game $u_A(B) := \begin{cases} 1 & A \subseteq B \\ 0 & \text{otherwise} \end{cases}$

Fact 1

$\{u_A \mid \emptyset \neq A \subseteq N\}$ is a basis of the vector space Γ

Fact 2

If a value φ is efficient, symmetric, and has the null player property, then, for every $\alpha \in \mathbb{R}$ and each coalition $A \neq \emptyset$,

$$\varphi_i(\alpha \cdot u_A) = \begin{cases} \frac{\alpha}{|A|} & i \in A \\ 0 & \text{otherwise} \end{cases}$$

The Shapley value is the expectation of marginal contributions

$$\varphi_i^S(v) = \sum_{A \subseteq N \setminus i} \underbrace{\frac{|A|!(n-|A|-1)!}{n!}}_{p_i(A):=} \cdot \underbrace{\big(v(A \cup i) - v(A)\big)}_{\text{marginal contribution of } i \text{ to } A}$$

Then

$$\sum_{A \subseteq N \setminus i} p_i(A) = 1$$

and the probability distribution $p_i \colon \mathcal{P}(N \setminus i) \to [0,1]$ corresponds to the following random scheme:

- Players in N are ordered randomly and uniformly
- $p_i(A)$ is the probability that i is preceded by A:

$$A,\ i,\ N \setminus (A \cup i)$$

Example: The Shapley value in 3-player games

$N = \{1, 2, 3\}$

The Shapley value of player 2 is determined as follows:

Coalition A	**Permutations**	$p_2(A)$
$\emptyset$	213, 231	$1/3$
1	123	$1/6$
3	321	$1/6$
13	132, 312	$1/3$

$$\varphi_2(v) = \tfrac{1}{3}v(2) + \tfrac{1}{6}\big(v(12) - v(1)\big) + \tfrac{1}{6}\big(v(23) - v(3)\big) + \tfrac{1}{3}\big(v(123) - v(13)\big)$$

How to divide the prey fairly – the Shapley value

$$v(A) = \begin{cases} 2 & A = N, 234 \\ 1 & A = 134, 34 \\ 0 & \text{otherwise} \end{cases}$$

The Shapley values of animals

- $\varphi_3^S(v) = \varphi_4^S(v)$ by symmetry of players 3 and 4
- $\varphi_1^S(v) = 0$ by null player property
- $\varphi_2^S(v) + 2\varphi_3^S(v) = 2$ by efficiency

$$\varphi_2^S(v) = \tfrac{1}{12} + \tfrac{1}{4} = \tfrac{1}{3} \qquad \Longrightarrow \qquad \varphi^S(v) = \left(0, \tfrac{2}{6}, \tfrac{5}{6}, \tfrac{5}{6}\right)$$

Shapley value as the expectation of marginal vectors

Let Π be the set of all permutations of N. Then

$$\varphi_i^S(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \underbrace{(v(A_i^\pi \cup i) - v(A_i^\pi))}_{x_i^\pi} \qquad \text{for all } i \in N$$

In case $n = 3$ we obtain $\boldsymbol{\varphi}^S(v) = \frac{1}{6}(\boldsymbol{x}^{\pi_1} + \cdots + \boldsymbol{x}^{\pi_6})$.

π_k	$x_1^{\pi_k}$	$x_2^{\pi_k}$	$x_3^{\pi_k}$
123	$v(1)$	$v(12) - v(1)$	$v(N) - v(12)$
132	$v(1)$	$v(N) - v(13)$	$v(13) - v(1)$
213	$v(12) - v(2)$	$v(2)$	$v(N) - v(12)$
231	$v(N) - v(23)$	$v(2)$	$v(23) - v(2)$
312	$v(13) - v(3)$	$v(N) - v(13)$	$v(3)$
321	$v(N) - v(23)$	$v(23) - v(3)$	$v(3)$

Beyond the Shapley value

Why not different probabilities (1)?

- We can replace the probabilities in the Shapley value formula

$$\varphi_i^S(v) = \sum_{A \subseteq N \setminus \{i\}} \frac{|A|!(n - |A| - 1)!}{n!} \cdot \big(v(A \cup \{i\}) - v(A)\big)$$

  with any probability distribution $p \colon \mathcal{P}(N \setminus i) \to [0, 1]$

- This choice defines the so-called probabilistic value

$$\varphi_i^p(v) := \sum_{A \subseteq N \setminus \{i\}} p(A) \cdot \big(v(A \cup \{i\}) - v(A)\big),$$

  which is linear and has the null-player property

- In general, φ_i^p lacks symmetry and efficiency

Why not different probabilities (2)?

- We replace the probabilities in the 2nd Shapley value formula

$$\varphi_i^S(v) = \sum_{\pi \in \Pi} \frac{1}{n!} \cdot (v(A_i^\pi \cup i) - v(A_i^\pi))$$

  with any probability distribution $p \colon \Pi \to [0, 1]$

- This choice defines the so-called random-order value

$$\bar{\varphi}_i^p(v) := \sum_{\pi \in \Pi} p(\pi) \cdot (v(A_i^\pi \cup i) - v(A_i^\pi)),$$

  which is always an efficient probabilistic value

- In general, $\bar{\varphi}_i^p$ lacks symmetry

Example: Symmetric probabilistic value

Definition

The Banzhaf value of player i in game v is

$$\varphi_i^B(v) := \sum_{A \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} \cdot \big(v(A \cup \{i\}) - v(A)\big)$$

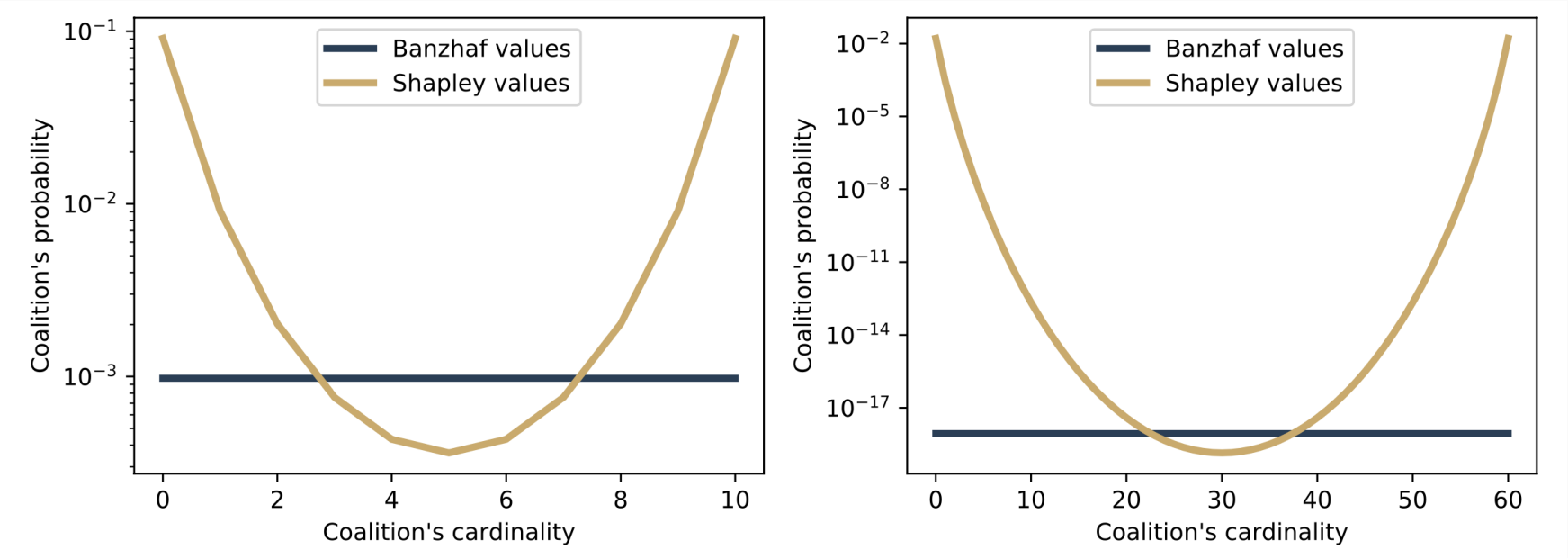
- Banzhaf value lacks efficiency,

  $$\varphi_1^B(v) + \cdots + \varphi_n^B(v) \neq v(N),$$

  so it *shouldn't* be applied to games in which the exact worth of grand coalition N has to be allocated to players
- It is *not* a random-order value

Computing values (1)

- The computation of values involves exponentially many terms
- The probabilities of coalitions are very small for large n:



Computing values (2)

Sampling algorithm for the Shapley value

Input: Game v over n players and a selected player i

1. Pick the size of the random sample $m \ll n!$
2. Sample with replacement permutations $(\pi_1, \ldots, \pi_m)$ with uniform probability $\frac{1}{n!}$
3. Estimate the Shapley value of player i by

$$\frac{1}{m} \sum_{k=1}^{m} x_i^{\pi_k}$$